Dark Matter

David G. Cerdeño
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Question 1 (Weighing the Universe)

We can estimate the total amount of luminous matter in the Universe, using information about the number density of galaxies and the mass-to light ratio, Υ, for each type of galaxy. The mass-to-light ratio is defined, in Solar units, as $\Upsilon = (M/M_\odot)/(L/L_\odot)$. The mass-to-light ratio is well known for stars in the main branch and stellar evolution inside galaxies can be used to determine this quantity for a whole galaxy. The exact value of Υ depends on the type of galaxy (e.g., for elliptical galaxies $\Upsilon \approx 6.5$, for spiral galaxies $\Upsilon \approx 1-5$, and irregular galaxies have $\Upsilon \approx 1$).

The number of galaxies per volume and per unit luminosity (a quantity referred to as "galaxy luminosity density function", $\Phi(L)$) can be measured on small scales and extrapolated to the whole Universe. An analytical fitting function (Schechter function) can be found that reads

$$\Phi(L) = \frac{\Phi_*}{L_*}\left(\frac{L}{L_*}\right)^\alpha e^{-L/L_*} \,,$$

with parameters $L_* = 2.53 \times 10^{10}\ L_\odot$, $\Phi_* = 4.1 \times 10^{-3}$ Mpc^{-3}, and $\alpha = -1.25$.

How much mass is concentrated in luminous galaxies (compared to the critical density)? Hint: estimate the mean luminosity density as $\mathcal{L} = \int L\Phi(L)$ and then the matter density multiplying by the mass-to-light ratio $\rho = \mathcal{L}\Upsilon$.

Use $M_\odot = 1.116 \times 10^{57}$ GeV $\rho_c = 8.0992 \times h^2 \times 10^{-47}$ GeV^{-4}, and 1 Mpc=1.5637×10^{38} GeV^{-1}.

Question 2 (Boltzman Equation 1)

In the derivation of Boltzmann's equation for the evolution of the number density of massive species inthe Early Universe we expanded the Liouville operator, $\hat{L}$, that acts on the density distribution function. Prove that for a FRW universe the following relation holds

$$\begin{aligned} \hat{L} &= p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\sigma\rho} p^\sigma p^\rho \frac{\partial}{\partial p^\mu} \\ &= E\frac{\partial}{\partial t} - H|\vec{p}|^2 \frac{\partial}{\partial E} \end{aligned} \tag{1}$$

Question 3 (Boltzman Equation 2)

Later, we performed an integral in momentum space that yielded

$$\frac{g}{(2\pi)^3}\int \frac{d^3\vec{p}}{E}\left[E\frac{\partial f}{\partial t} - H|\vec{p}|^2 \frac{\partial f}{\partial E}\right] = \frac{dn}{dt} + 3Hn \tag{2}$$

Check this result.

Question 4 (electron chemical potential in the Early Universe)

In this exercise we are going to show that the chemical potential for electrons can be neglected in the Early Universe.

- We will start by computing the difference between the number densities of electrons and positrons. Show that in the relativistic limit, this can be expressed as

$$n_{e^-} - n_{e^+} \approx \frac{gT^3}{6\pi^2}\left[\pi^2\left(\frac{\mu_e}{T}\right) + \left(\frac{\mu_e}{T}\right)^3\right] \; . \tag{3}$$

- In epochs in which the only charge carriers are protons and electrons, the universe neutrality implies that the number densities of electrons and protons are equal. Using the baryon-to-photon ratio, η, to express n_B, show that we obtain

$$\pi^2\left(\frac{\mu_e}{T}\right) + \left(\frac{\mu_e}{T}\right)^3 = 6\eta\zeta(3) \; . \tag{4}$$

- Solve to show that $\mu_e \ll T$ and therefore argue that the chemical potential for electrons can be neglected in the Early Universe.

Question 5 (Yield at freeze-out)

It is easy to estimate the value of the Yield that we need in order to reproduce the correct DM relic abundance, $\Omega h^2 \approx 0.1$, since

$$\Omega h^2 = \frac{\rho_\chi}{\rho_c} h^2 = \frac{m_\chi n \chi h^2}{\rho_c} = \frac{m_\chi Y_\infty s_0 h^2}{\rho_c} \; , \tag{5}$$

where Y_∞ corresponds to the DM Yield today and s_0 is todayÕs entropy density. We can assume that the Yield did not change since DM freeze-out and therefore

$$\Omega h^2 = \frac{m_\chi Y_f s_0 h^2}{\rho_c} \; . \tag{6}$$

Using the measured value $s_0 = 2970$ cm^{-3}, and the value of the critical density $\rho_c = 1.054 \times 10^{-5}\, h^2$ GeV cm^{-3}, as well as PlanckÕs result on the DM relic abundance, estimate the correct range of values of the yield at freeze-out and the approximate values of x_f.

Question 6 (Freeze out of DM particles)

Using Botlzmann equation, expressed in terms of the yield $Y = n/s$, which reads

$$\frac{dY}{dx} = -\frac{\lambda\langle\sigma v\rangle}{x^2}\left(Y^2 - Y_{eq}^2\right) \; , \tag{7}$$

define the quantity $\Delta_Y \equiv Y - Y_{eq}$ and show that, for non-relativistic particles, the solution can be approximated as

$$\Delta_Y = -\frac{\frac{dY_{eq}}{dx}}{Y_{eq}} \frac{x^2}{2\lambda\langle\sigma v\rangle}, \qquad 1 < x \ll x_f \tag{8}$$

$$\Delta_{Y_\infty} = Y_\infty = \frac{x_f}{\lambda\left(a + \frac{b}{2\,x_f}\right)}, \qquad x \gg x_f \tag{9}$$

Question 7 (Dark Matter relic density 1)

Consider a simple model in which the Dark Matter is a Dirac fermion, χ, which only couples to the Standard Model sector through the exchange of the a pseudoscalar particle A. The pseudoscalar A has couplings g_χ to the dark matter and g_b to b quarks as described by the Lagrangian

$$\mathcal{L} = i\left(g_\chi \bar{\chi}\gamma^5\chi + g_b \bar{b}\gamma^5 b\right) A$$

- Draw the Feynman diagram that corresponds to the pair-annihilation of two dark matter particles into $b\bar{b}$.
- Considering only Dark Matter annihilation into $b\bar{b}$, the annihilation cross section in the Early Universe can be expanded in plane waves as $\langle\sigma v\rangle \approx a_{b\bar{b}} + b_{b\bar{b}}\, x$, with (see e.g, Ref.[**?**])

  $$a_{b\bar{b}} = \frac{1}{m_\chi^2}\left(\frac{N_c}{32\pi}\left(1 - \frac{4m_b^2}{s}\right)^{1/2}\frac{1}{2}\int_{-1}^{1} d\cos\theta_{CM}|\mathcal{M}_{\chi\chi\to bb}|^2\right)_{s=4m_\chi^2}$$

  Show that to leading order in velocity (i.e., $x = 0$)

  $$\langle\sigma v\rangle \approx \frac{3}{2\pi}\frac{(g_\chi g_b)^2 m_\chi^2\sqrt{1 - m_b^2/m_\chi^2}}{(4m_\chi^2 - m_A^2)^2 + m_A^2\Gamma_A^2}$$

  Remember to average over initial spins and sum over final ones. You will also need the following trace, $\mathrm{Tr}\left[(\not{p}_1 - m_\chi)\gamma^5(\not{p}_2 + m_\chi)\gamma^5\right] = 4(-p_1\cdot p_2 - m_\chi^2)$.
- Show that if the mediator is a scalar particle instead of a pseudoscalar then $a_{b\bar{b}} = 0$.
- Given a dark matter mass $m_\chi = 100$ GeV and a pseudoscalar mass $m_A = 1000$ GeV, estimate the value of the coupling $g_\chi g_b$ for which the correct relic density is obtained. Neglect the pseudoscalar decay width, Γ_A and use that

  $$\Omega_\chi h^2 \approx \frac{3\times 10^{-10}\ \mathrm{GeV}^{-2}}{\langle\sigma v\rangle}$$

Question 8 (Dark Matter detection 1)

Consider now a scalar Dark Matter model, ϕ, which only couples to the Standard Model sector through the exchange of the Higgs boson. The coupling, $C_{\phi\phi H^0_{SM}}$ (which can be understood as coming from a quartic term $\phi\phi H^0_{SM}H^0_{SM}$) is fixed by imposing that the relic density is correct, $\Omega_\phi h^2 \approx 0.1$, obtaining $C_{\phi\phi H^0_{SM}} \approx 20GeV$. Compute the prediction for the spin-independent scattering cross-section off protons, $\sigma^{SI}_{\phi-p}$, and compare it with current experimental constraints from LUX and SuperCDMS. Is this cadidate viable or is it excluded if it has a mass $m_\phi < 20$ GeV?

To do this,

- Write down the effective Lagrangian that describes the elastic scattering of ϕ with quarks and express the interaction strength, α_q, in terms of the fundamental coupling $C_{\phi\phi H^0_{SM}}$.
- Assume that the scattering off protons can be computed assuming that the contribution of s quarks is dominant.
- The expression for the scattering cross-section of scalar dark matter can be found, e.g., in Section 3.4 of Ref. [?].
$$\sigma^{SI}_{\phi-p} = \frac{f_p m_p^2}{4\pi(m_\phi + m_p)^2} \ , \tag{10}$$
where
$$\frac{f_p}{m_p} = \sum_{q_i=u,d,s} f^p_{T_{q_i}} \frac{\alpha_{q_i}}{m_{q_i}} + \frac{2}{27} f^p_{TG} \sum_{q_i=c,b,t} \frac{\alpha_{q_i}}{m_{q_i}} \ . \tag{11}$$
We can consider for simplicity that the s quark contribution dominates, and use $f_{T_{q_s}} = 0.229$.

Question 9 (Dark Matter detection 2)

In the previous question we noticed that the predictions for $\sigma^{SI}_{\chi-p}$ exceed the current experimental limits from the direct detection experiments LUX and SuperCDMS. Is there any way in which we can "fix" this model?

- Think about why the annihilation cross section and the scattering cross section are related in the example above. How can we break this relation?
- Consider enlarging the "exotic" sector by including more particles.

Question 10 (Neutrino decoupling 1)

In the Early Universe, neutrinos remain in equilibrium through the process $e^+ + e^- \longleftrightarrow \nu_e + \bar{\nu}_e$. Using that both the electron-positron and neutrino populations are relativistic and therefore their number density scales as $n \sim T^3$, the decoupling temperature of neutrinos can be roughly estimated by equating the annihilation rate $\Gamma = n\langle\sigma v\rangle$ and the Hubble expansion rate $H = \sqrt{8\pi G\rho/3}$. The energy density of the Universe scales as $\rho \sim T^4$. Show that neutrinos decouple at approximately $T \sim 1$ MeV.

Question 11 (Neutrino decoupling 2)

From the question above, we know that when neutrinos decouple, they are still relativistic. The other relativistic species in the thermal bath are electrons, positrons, photons and the three neutrinos and antineutrinos. With this information the relic density of neutrinos in the Universe today can be estimated as a function of the neutrino mass.

To do that, remember that for relativistic species the Yield at equilibrium can be written as $y_{eq} = \frac{45}{2\pi^4}\zeta(3)\frac{g_{eff}}{g_{*s}} \approx 0.278\frac{g_{eff}}{g_{*s}}$.

Question 12 (Freeze-in Dark Matter)

Following the notation of arXiv:0911.1120 ”Freeze-In Production of FIMP Dark Matter” consider a Freeze-in dark matter model in which an axion-like dark matter candidate, ϕ, couples to the SM through the mediation of the SM Higgs boson. The interaction terms in the Lagrangian can be written as

$$\frac{\lambda_{h\phi}}{4}\, H^2\phi^2 + \frac{\lambda_{h\phi v}}{\sqrt{2}} H\phi^2 \tag{12}$$

where v is the vacuum expectation value of the Higgs and $\lambda_{h\phi}$ is a small coupling constant.

- Write the Feynman diagrams for the different production channels of the DM particle.
- For (very) small values of the coupling, DM can be produced via a freeze-in mechanism in which the main channel is the decay of the Higgs boson into two DM particles. Write down the Boltzmann equation that describes this process.
- Solve the resulting differential equation and compute the relic abundance (leave it as a function of the Higgs decay width).
- Discuss how, for large values of the coupling, this model can also explain the observed DM relic abundance through a freeze-out mechanism.

References

[1] R. Kolb and M. Turner, “The Early Universe”

[2] T. Nihei, L. Roszkowski and R. Ruiz de Austri, JHEP **0203** (2002) 031 [hep-ph/0202009].

[3] M. Ishitsuka [Super-Kamiokande Collaboration], hep-ex/0406076.

[4] Y. Fukuda *et al.* [Super-Kamiokande Collaboration], Phys. Rev. Lett. **81** (1998) 1562 [hep-ex/9807003].

[5] J. R. Ellis, A. Ferstl and K. A. Olive, Phys. Lett. B **481** (2000) 304 [hep-ph/0001005].

[6] D. G. Cerdeno, M. Peiro and S. Robles, JCAP **1408** (2014) 005 [arXiv:1404.2572 [hep-ph]].