INTEGRAL TRANSFORMS

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PHYS3591

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CONTENTS

Disclaimer: These notes may (and most likely will) contain typographical erorrs and must be used with care. They are solely meant as a guideline of the materials that will be covered in the class but by no means can substitute the basic references.

In these lectures we will introduce the concept of integral transforms by means of the following transformation

$$g(w) = \int_a^b K(w,x) f(x) dx \ , \tag{0.1}$$

where we will refer to $g(w)$ as the integral transform of the function $f(x)$ by the kernel $K(w,x)$, a function of two variables. This function is sometimes referred to as propagator. The integration limits a and b depend on the specific problem, they can be infinity (most often they are infinity). It is customary to represent this operation by means of an operator T

$$g(w) = T f(x) \ . \tag{0.2}$$

We can understand T as a map in the set of all functions.

$$\begin{array}{rccc} T: & F & \to & F \\ & f & \to & T(f) \end{array}$$

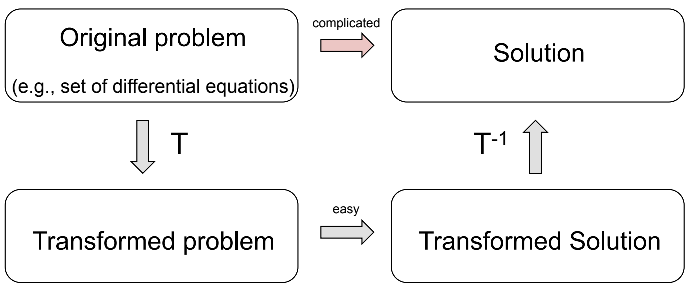
This operation is linear, as it satisfies

$$T\left[\alpha_1 f_1(x) + \alpha_2 f_2(x)\right] = \alpha_1 T f_1(x) + \alpha_2 T f_2(x) = \alpha_1 g_1(w) + \alpha_2 g_2(w) \ . \tag{0.3}$$

We will also expect these transformations to be invertible, such that

$$f(x) = T^{-1} g(w) \ . \tag{0.4}$$

Integral transforms are very useful tools to simplify certain problems which are difficult to solve in their original representation.



The specific problem will determine the best choice of integral transform to be used. For example, in Chapter 1, we will study Fourier Transforms, defined as

$$\mathfrak{F}\left[f(x)\right] \equiv \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{iwx} dx,$$

and we will show that they are particularly well suited to solve certain kind of differential equations. Also, they are generally used in electronics, for instance in digital signal processing and filtering. Notice in this sense that wen applied to a generic function of time, the Fourier transform returns an expansion in frequencies. Moreover, the three dimensional for of the Fourier transform is used in quantum mechanics to go from position to momentum space, since the kernel can be interpreted as an expansion in plane waves.

Likewise, Laplace transforms will be the subject of Chapter 2 and are defined as

$$\mathfrak{L}\left[f(t)\right](s) \equiv \bar{f}(s) = \int_0^\infty f(t)e^{-ts}dt. \tag{0.5}$$

Laplace transforms are extremely useful to solve differential equations and also in electronics.

CHAPTER 1

FOURIER TRANSFORMS

1.1 Fourier Transforms

Given a real function of a real variable $f(x)$, we define the Fourier transform as

$$\mathfrak{F}\left[f(x)\right] \equiv \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{iwx} dx \tag{1.1}$$

In general the Fourier transform returns a complex function. Not every function has an integral transform, as the above integral has to converge. A sufficient condition is that the function $f(x)$ is square-integrable (this is, such that $\int_{\infty}^{\infty} |f(x)|^2 dx$ is finite). If in addition to this, $f(x)$ is continuous, then the Fourier transform is invertible, and its inverse is the integral transform

$$\mathfrak{F}^{-1}\left[\hat{f}(w)\right] = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(w) e^{-iwx} dw \tag{1.2}$$

In the case of discontinuous functions, the inverse transform can still be defined but one has to treat carefully the limit in the discontinuities. Notice that the numerical prefactor in equation (1.1) is arbitrary, we have chosen $1/\sqrt{2\pi}$ so that the definition of the inverse transform (1.2) contains the same prefactor. This notation coincides with the one used in Ref. [1], but other sources prefer to associate a prefactor $1/2\pi$ to either the Fourier transform or its inverse. Similarly, some authors exchange the sign in the complex exponential factors of both the Fourier transform and its inverse.

This definition can be trivially extended to the three-dimensional space, where

$$\mathfrak{F}\left[f(\mathbf{x})\right] \equiv \hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{x} \tag{1.3}$$

$$\mathfrak{F}^{-1}\left[\hat{f}(\mathbf{k})\right] = f(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \hat{f}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{1.4}$$

The second equality can be verified by using the three-dimensional delta function. We will come back to this in Section 1.5.

The Fourier transform is intimately linked to physical properties. For example, when applied to a time-dependent electrical signal, $f(t)$, the transformed function $\hat{f}(w)$ corresponds to an expansion in frequencies (the frequency spectrum). Similarly, the three-dimensional form of the transform is widely used in particle physics to go from the position space to momentum space, where both quantities refer to the properties of plane waves.

A simple way to understand the Fourier transform is in terms of the limit of the Fourier series for a periodic function. Consider the function $f(x)$, which is periodic in the interval $[-L, L]$ so that it can be expanded in terms of a Fourier series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{1.5}$$

with

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(t) dt, \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} dt, \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} dt. \end{aligned} \tag{1.6}$$

Substituting these expressions in equation (1.5), and using the Euler formula to express the trigonometric functions in terms of exponentials, we obtain

$$\begin{aligned} f(x) &= \frac{1}{2L} \int_{-L}^{L} f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \int_{-L}^{L} f(t) \left(\cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} dt + \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} \right) dt \\ &= \frac{1}{2L} \int_{-L}^{L} f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \int_{-L}^{L} f(t) \cos \frac{n\pi(t-x)}{L} dt \\ &= \frac{1}{2L} \int_{-L}^{L} f(t) dt + \frac{1}{2L} \sum_{n=1}^{\infty} \int_{-L}^{L} f(t) \left(e^{i\frac{n\pi(t-x)}{L}} + e^{-i\frac{n\pi(t-x)}{L}} \right) dt \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(t) e^{i\frac{n\pi(t-x)}{L}} dt. \end{aligned}$$

We can define $w_n \equiv \frac{n\pi}{L}$ in such a way that $\Delta w \equiv w_{n+1} - w_n = \frac{\pi}{L}$ and rewrite the expressions above as

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \frac{\Delta w}{2\pi} \int_{-L}^{L} f(t) e^{iw_n t} e^{-iw_n x} dt, \\ &\xrightarrow{L\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{iwt} dt \right) e^{-iwx} dw \end{aligned}$$

EXAMPLE 1.1

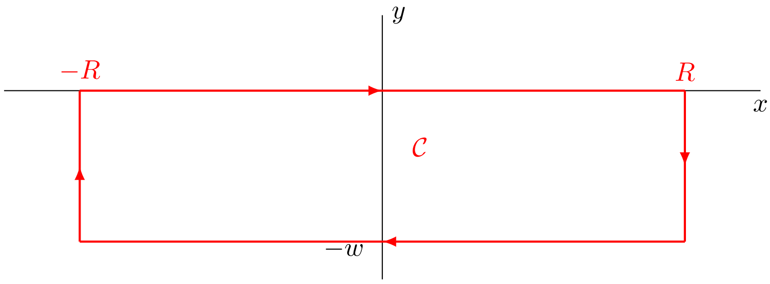
Consider the Gaussian function

$$f(x) = e^{-\frac{1}{2}x^2} ,$$

whose Fourier transform reads

$$\begin{aligned} \mathfrak{F}[f](w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} e^{iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-iw)^2 - \frac{w^2}{2}} dx \\ &= \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-iw)^2} dx \end{aligned}$$

Notice that in this integral we cannot simply shift $x \to x + iw$ since the exponent is a complex quantity. Instead we can try using Cauchy's theorem on a contour integration to obtain the result.



Consider the following integral in the complex plane ($z = x + iy$), which vanishes due to Cauchy's theorem,

$$\oint_{\mathcal{C}} e^{-\frac{1}{2}z^2} dz = 0$$

It can be separated in four integrals along each of the sides, with

$$\begin{aligned} \int_{\mathcal{C}_1} e^{-\frac{1}{2}z^2} dz &= \int_{-R}^{R} e^{-\frac{1}{2}x^2} dx \\ \int_{\mathcal{C}_2} e^{-\frac{1}{2}z^2} dz &= \int_{0}^{-w} e^{-\frac{1}{2}(R+iy)^2} dy \\ \int_{\mathcal{C}_3} e^{-\frac{1}{2}z^2} dz &= \int_{R}^{-R} e^{-\frac{1}{2}(x-iw)^2} dx \\ \int_{\mathcal{C}_4} e^{-\frac{1}{2}z^2} dz &= \int_{-w}^{0} e^{-\frac{1}{2}(R+iy)^2} dy \end{aligned}$$

We are interested in the limit $R \to \infty$ and it can be easily shown that

$$\left| \int_{\mathcal{C}_2} e^{-\frac{1}{2}z^2} dz \right| \leq e^{-\frac{1}{2}R^2} \int_{0}^{-w} e^{-\frac{1}{2}y^2} dy \xrightarrow{R\to\infty} 0$$

$$\left| \int_{\mathcal{C}_4} e^{-\frac{1}{2}z^2} dz \right| \leq e^{-\frac{1}{2}R^2} \int_{-w}^{0} e^{-\frac{1}{2}y^2} dy \xrightarrow{R\to\infty} 0$$

Thus from the integrals in $\mathcal{C}_1$ and $\mathcal{C}_3$ it follows that

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x-iw)^2} dx = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$$

From where we finally obtain

$$\mathfrak{F}\left[f\right](w) = e^{-\frac{w^2}{2}}$$

The Fourier transform of a Gaussian is a Gaussian

PROBLEMS

1.1 Calculate the Fourier transform of a sawtooth pulse, given by the following function

$$f(x) = \begin{cases} 0 & x < -1 \\ x & -1 \leq x < 1 \\ 0 & 1 \leq x \end{cases}$$

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x e^{iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{iw} e^{iwx} \Big|_{-1}^{1} - \frac{1}{iw} \int_{-1}^{1} e^{iwx} dx \right) \\ &= i \sqrt{\frac{2}{\pi}} \frac{(-w \cos w + \sin w)}{w^2} \end{aligned}$$

1.2 Prove that the Fourier transform of a real and symmetric function is a pure real function, whereas the Fourier transform of a real and antisymmetric function is a pure imaginary one.

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\cos(wx) + i \sin(wx) \right) dx \\ &= \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos(wx) dx & \text{if} \quad f(x) = f(-x) \\ \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \sin(wx) dx & \text{if} \quad f(x) = -f(-x) \end{cases} \end{aligned}$$

1.3 Find the Fourier transform of the Cauchy distribution, given by

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\pi(1 + x^2)} e^{iwx} dx$$

This is an integral that we can evaluate by going to the complex plane and using the residue theorem. Notice that the sign of the complex exponent is important to close the contour in the upper or lower plane (for convergence).

• If $w < 0$ we close the contour in the lower plane and find

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} (-2\pi i) \text{Res} \left(\frac{e^{iwx}}{\pi(1 + x^2)} \right)_{x=-i} = \frac{e^w}{\sqrt{2\pi}}$$

• If $w \geq 0$ we close the contour in the upper plane and find

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}(2\pi i)\text{Res}\left(\frac{e^{iwx}}{\pi(1+x^2)}\right)_{x=i} = \frac{e^{-w}}{\sqrt{2\pi}}$$

Combining these results

$$\hat{f}(w) \quad = \quad \frac{e^{-|w|}}{\sqrt{2\pi}}$$

1.4 Find the Fourier transform of the function

$$f(x) = \frac{1}{x}$$

$$\hat{f}(w) \quad = \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{x}e^{iwx}dx$$

We perform the integral in the complex plane as in the previous exercise
• If $w < 0$ we close the contour in the lower plane and find

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}(-\pi i)\text{Res}\left(\frac{e^{iwx}}{x}\right)_{x=o} = -i\sqrt{\frac{\pi}{2}}$$

• If $w \geq 0$ we close the contour in the upper plane and find

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}(\pi i)\text{Res}\left(\frac{e^{iwx}}{x}\right)_{x=o} = i\sqrt{\frac{\pi}{2}}$$

Combining these results we obtain a rescaled sign function,

$$\hat{f}(w) \quad = \quad i\sqrt{\frac{\pi}{2}}\text{sgn}(w)$$

1.5 Using quantum electrodynamics and working in momentum space, it is easy to describe the scattering of distinguishable fermions in the non-relativistic limit in terms of a Feynman diagram in which a particle ϕ is exchanged. This results in the following potential

$$\hat{V}(\mathbf{k}) = \frac{-1}{(2\pi)^{3/2}}\frac{g^2}{|\mathbf{k}|^2 + m_\phi^2}$$

where m_ϕ is the mass of the mediator particle, g is the charge of the fermions involved and $\mathbf{k}$ is the three-dimensional momentum. Compute the inverse Fourier transform to find the expression of the potential $V(\mathbf{r})$.

$$\begin{aligned}
V(\mathbf{r}) &= -\frac{1}{(2\pi)^{3/2}} \int \hat{V}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k} \\
&= -\frac{g^2}{(2\pi)^3} \int \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^2 + m_\phi^2} d^3\mathbf{k} \\
&= -\frac{g^2}{(2\pi)^3} \int \frac{e^{-ikr\cos\theta}}{k^2 + m_\phi^2} k^2 d\varphi\, d\cos\theta\, dk \\
&= -\frac{g^2}{(2\pi)^2} \int_0^\infty \frac{e^{ikr} - e^{-ikr}}{ikr} \frac{k^2}{k^2 + m_\phi^2} dk \\
&= -\frac{g^2}{ir(2\pi)^2} \int_{-\infty}^{\infty} \frac{k\, e^{ikr}}{k^2 + m_\phi^2} dk
\end{aligned}$$

This integral can be solved by closing the contour in the upper half of the complex plane, and yields

$$\begin{aligned}
V(\mathbf{r}) &= -\frac{g^2}{ir(2\pi)^2} 2\pi i \mathrm{Res} \left(\frac{k\, e^{ikr}}{k^2 + m_\phi^2} \right)_{k = i m_\phi} \\
&= -\frac{g^2}{4\pi r} e^{-m_\phi r}
\end{aligned}$$

We obtain the Yukawa potential, and, in the limit in which the mediator is massless (e.g., a photon), the electromagnetic potential is recovered.

1.2 Properties of the Fourier transform

1.2.1 Fourier transform of a derivative

The Fourier transform is a very useful tool to solve partial differential equations, due to the fact that the Fourier transform of the derivative of a function $f(x)$ can be expressed in terms of the Fourier transform of the function

$$\mathfrak{F}\left[\frac{\partial f}{\partial x}\right] = -iw\mathfrak{F}\left[f\right] \tag{1.7}$$

To prove this expression we simply apply the definition (1.1) and integrate by parts

$$\begin{aligned} \mathfrak{F}\left[\frac{\partial f}{\partial x}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\partial f}{\partial x}e^{iwx}dx \\ &= \frac{1}{\sqrt{2\pi}}f(x)e^{iwx}\Big|_{-\infty}^{+\infty} - \frac{iw}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)e^{iwx}dx\ . \end{aligned}$$

Since the Fourier integral must converge, $\lim_{x\to\pm\infty} f(x) = 0$, and expression (1.7) is recovered. Notice that if we had chosen a different convention for the sign of the complex exponential factor in the definition of the Fourier transform in equation (1.1) we would have obtained a different sign in (1.7). As a corollary, this can be easily generalised to the n^{th} derivative

$$\mathfrak{F}\left[\frac{\partial^n f}{\partial x^n}\right] = (-iw)^n\mathfrak{F}\left[f\right]\ . \tag{1.8}$$

The proof of this expression can be done by induction, assuming the expression holds for an arbitrary n, we demonstrate that it also holds for $n+1$ as follows

$$\begin{aligned} \mathfrak{F}\left[\frac{\partial^{n+1} f}{\partial x^{n+1}}\right] &= \mathfrak{F}\left[\frac{\partial}{\partial x}\frac{\partial^n f}{\partial x^n}\right] \\ &= -iw\mathfrak{F}\left[\frac{\partial^n f}{\partial x^n}\right] \\ &= (-iw)^{n+1}\mathfrak{F}\left[f\right]\ . \end{aligned}$$

1.2.2 Fourier transform of $x^n f(x)$

This is a combination which often appears in differential equations. We start by considering the case $n = 1$,

$$\begin{aligned} \mathfrak{F}\left[xf(x)\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}xf(x)e^{iwx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{f(x)}{i}\frac{\partial}{\partial w}e^{iwx}dx \\ &= -i\frac{\partial}{\partial w}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(x)e^{iwx}dx \\ &= -i\frac{\partial}{\partial w}\mathfrak{F}\left[f(x)\right]\ . \end{aligned}$$

This result can be readily generalised as

$$\mathfrak{F}\left[x^n f(x)\right] = (-i)^n \frac{\partial^n \mathfrak{F}\left[f\right]}{\partial w^n} \tag{1.9}$$

From equations (1.7) and (1.9) we can observe that the Fourier transform exchanges the multiplication and differentiation operators due to the properties of the exponential function.

1.2.3 Fourier transform and scaling

Scaling factors are also easy to incorporate in the Fourier transform, leading to a change of variables in the argument of the transform

$$\mathfrak{F}\left[f(ax)\right](w) = \frac{1}{a}\mathfrak{F}\left[f\right]\left(\frac{w}{a}\right) \tag{1.10}$$

Proof:

$$\begin{aligned}\mathfrak{F}\left[f(ax)\right](w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(ax)e^{iwx}dx \\ &= \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x')e^{iwx'/a}dx' \\ &= \frac{1}{a}\mathfrak{F}\left[f\right]\left(\frac{w}{a}\right)\end{aligned}$$

1.2.4 Fourier transform of a translated function

$$\mathfrak{F}\left[f(x+a)\right](w) = e^{-iwa}\mathfrak{F}\left[f\right](w) \tag{1.11}$$

This expression can easily be proved using the definition of the Fourier transform (see Problem 1.2).

1.2.5 Fourier transform and exponential multiplication

$$\mathfrak{F}\left[e^{iax}f(x)\right](w) = \mathfrak{F}\left[f\right](w+a) \tag{1.12}$$

This expression can easily be proved using the definition of the Fourier transform (see Problem 1.2).

1.2.6 Parity of the Fourier transform

Although in most of the physical applications $f(x)$ corresponds to a real function, the Fourier transform can also be defined for a complex function. In general, the transformed function will be a complex function, too. There are simple relations which are easy to prove as an exercise regarding the parity of the Fourier transform.

• if $f(x)$ is real, then the Fourier transform satisfies $\hat{f}(w)^* = \hat{f}(-w)$ (proof in exercise (1.4).

EXAMPLE 1.2

Consider a one-dimensional string, whose dynamics is described by the following *wave equation*

$$\frac{\partial^2 Y}{\partial x^2}(x,t) = \frac{1}{v^2}\frac{\partial^2 Y}{\partial t^2}(x,t) \ .$$

This partial differential equation can be easily solved by transforming it to *Fourier space* since the transformation allows us to redefine derivatives in terms of products. Let

$$\mathfrak{F}\left[Y(x,t)\right] = \hat{Y}(k,t) \ .$$

Then, applying the Fourier transform to the whole equation, and using relation (1.7), we obtain the simpler equation

$$\begin{aligned} \frac{\partial^2 Y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 Y}{\partial t^2} \quad &\xrightarrow{\mathfrak{F}} \quad (-ik)^2\hat{Y} = \frac{1}{v^2}\frac{\partial^2 \hat{Y}}{\partial t^2} \\ &\rightarrow \quad \frac{\partial^2 \hat{Y}}{\partial t^2} = -w^2\hat{Y} \end{aligned}$$

where we have defined $w \equiv kv$. The previous PDE is reduced to the equation of the harmonic oscillator, for which we know the general solution,

$$\hat{Y}(k,t) = A(k)e^{ikvt} + B(k)e^{-ikvt}$$

Applying the inverse Fourier transform to this solution we obtain the general solution in the original set of variables,

$$\begin{aligned} Y(x,t) &= \mathfrak{F}^{-1}\left[\hat{Y}(k,t)\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(A(k)e^{ikvt} + B(k)e^{-ikvt}\right)e^{-ikx}dk \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(A(k)e^{-ik(x-vt)} + B(k)e^{-ik(x+vt)}\right)dk \end{aligned}$$

As we can observe, the general solution is a superposition of plane waves, moving along the x axis in both positive and negative directions. The coefficients $A(k)$ and $B(k)$ can be determined from the initial conditions.

PROBLEMS

1.1 Calculate the Fourier transform of

$$f(x) = \frac{1}{\sqrt{\tau}}\, e^{-x^2/(2\tau)}\,.$$

What are the Fourier transforms of $x\,f(x)$ and $f^{(n)}(x)$?

In Example 1.1 we have shown that the Fourier transform of a Gaussian is a Gaussian. Therefore

$$\mathfrak{F}\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{w^2}{2}}$$

We can now define the function $g(x) \equiv e^{-\frac{x^2}{2}}$ such that

$$\mathfrak{F}\left[f(x)\right] = \frac{1}{\sqrt{\tau}}\mathfrak{F}\left[g\left(\frac{x}{\sqrt{\tau}}\right)\right]$$

This is a rescaling, and therefore, applying expression (1.10) the Fourier transform can easily be expressed as

$$\begin{aligned}
\mathfrak{F}\left[f(x)\right] &= \frac{1}{\sqrt{\tau}}\left(\sqrt{\tau}\mathfrak{F}\left[g(x)\right](\sqrt{\tau}w)\right) \\
&= e^{-\tau w^2/2}
\end{aligned}$$

On the other hand, making use of equation (1.9) we can write

$$\begin{aligned}
\mathfrak{F}\left[xf(x)\right] &= -i\frac{\partial}{\partial w}\mathfrak{F}\left[f(x)\right]\,, \\
&= -i\frac{\partial e^{-\tau w^2/2}}{\partial w} \\
&= i\tau w e^{-\tau w^2/2}
\end{aligned}$$

Finally, using equation (1.7) we can express

$$\begin{aligned}
\mathfrak{F}\left[\frac{\partial^n f}{\partial x^n}\right] &= (-iw)^n\mathfrak{F}\left[f\right]\,, \\
&= (-iw)^n e^{-\tau w^2/2}
\end{aligned}$$

1.2 Prove relation (1.11) for the Fourier transform of a translated function and (1.19) for the multiplication with an exponential factor.

In both cases, this is just solved by applying the definition of the Fourier transform and either performing a change of variables inside the integral symbol or in the transformed variable.

$$\begin{aligned}
\mathfrak{F}\left[f(x+a)\right](w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x+a)e^{iwx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')e^{iw(x'-a)}dx' \\
&= e^{-iwa}\mathfrak{F}\left[f(x')\right] \\
&= e^{-iwa}\mathfrak{F}\left[f(x)\right]
\end{aligned}$$

$$\begin{aligned}\mathfrak{F}\left[e^{iax}f(x)\right](w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{iax}f(x)e^{iwx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(w+a)x}dx \\ &= \mathfrak{F}\left[f(x)\right](w+a)\end{aligned}$$

1.3 Calculate the Fourier transform of the following odd function

$$f(x) = \begin{cases} e^{-x}\,, & x \geq 0\,, \\ -e^{x}\,, & x < 0\,. \end{cases}$$

$$\begin{aligned}\mathfrak{F}\left[f(x)\right] &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{x}e^{iwx}dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty} e^{-x}e^{iwx}dx \\ &= -\frac{1}{\sqrt{2\pi}}\left.\frac{e^{(iw+1)x}}{(iw+1)}\right|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}}\left.\frac{e^{(iw-1)x}}{(iw-1)}\right|_{0}^{+\infty} \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{-1}{iw+1} - \frac{1}{iw-1}\right) \\ &= \frac{2i}{\sqrt{2\pi}}\frac{w}{w^2+1}\end{aligned}$$

Notice that the resulting function $\hat{f}(w)$ is odd and pure imaginary as a consequence of $f(x)$ being real and odd.

1.4 Prove that $\hat{f}(w)^* = \hat{f}(-w)$ on the Fourier transform of $f(x)$ is a necessary and sufficient condition for $f(x)$ to be real.

If $f(x)$ is real, then we have

$$\begin{aligned}\hat{f}(w)^* &= \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{iwx}dx\right]^* \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)^*e^{-iwx}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)^*e^{i(-w)x}dx \\ &= \hat{f}(-w)\end{aligned}$$

If we now start by assuming that $\hat{f}(w)^* = \hat{f}(-w)$ we can prove that $f(x)$ is real

$$
\begin{aligned}
f(x)^* &= \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{f}(w)e^{-iwx}dw\right]^* \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{f}(w)^* e^{iwx}dw \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{f}(-w)e^{-i(-w)x}dw \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{f}(w)e^{-iwx}dw \\
&= f(x)
\end{aligned}
$$

1.3 The Dirac δ function and its Fourier Transform

1.3.1 The Dirac δ function

The Dirac delta function is defined as the "function" $\delta(x)$ such that

$$\begin{aligned} &\delta(x) = 0 \qquad \text{if} \qquad x \neq 0 \\ &\int_{-\infty}^{+\infty} \delta(x)dx = 1 \\ &\int_{-\infty}^{+\infty} \delta(x)f(x)dx = f(0) \end{aligned} \tag{1.13}$$

where we assume that $f(x)$ is continuous at $x = 0$.

In order to understand this object it is useful to consider it as the limit of well-behaved functions. For example, consider the following definition

$$f_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \tag{1.14}$$

This function satisfies the properties of the Dirac delta function in the limit $n \to \infty$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if} \qquad x \neq 0 \\ \infty & \text{if} \qquad x = 0 \end{cases}$$

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{+\infty} f_n(x)dx &= \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} dx \\ &= \lim_{n\to\infty} \frac{n}{\sqrt{\pi}} \frac{\sqrt{\pi}}{n} \\ &= 1 \end{aligned}$$

However, this is not the only way in which the delta function can be defined. For example, consider the function

$$g_n(x) = \frac{n}{\pi} \frac{1}{1 + n^2x^2} \ . \tag{1.15}$$

We can easily see that this function also satisfies the requirements of the Dirac Delta function. Namely,

$$\lim_{n\to\infty} g_n(x) = \begin{cases} 0 & \text{if} \qquad x \neq 0 \\ \infty & \text{if} \qquad x = 0 \end{cases}$$

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{+\infty} g_n(x)dx &= \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{n}{\pi} \frac{1}{1 + n^2x^2} \, dx \\ &= \frac{2\pi i}{\pi} \text{Res} \left(\frac{1}{n^2(z + i/n)(z - i/n)} \right)_{z=i/n} \\ &= 1 \end{aligned}$$

We now try to prove the last condition for an arbitrary function $f(x)$ which behaves nicely,

$$\begin{aligned}\lim_{n\to\infty}\int_{-\infty}^{+\infty} f(x)g_n(x)dx &= \lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{n}{\pi}\frac{f(x)}{1+n^2x^2}\,dx \\ &= \frac{2\pi i}{\pi}\mathrm{Res}\left(\frac{f(x)}{n^2(z+i/n)(z-i/n)}\right)_{z=i/n} \\ &= f\left(\frac{i}{n}\right)\xrightarrow{n\to\infty} f(0)\end{aligned}$$

Other possible definitions of the Dirac Delta functions include $\sin(nx)/(\pi x)$ and a square pulse whose hight scales up as the width decreases in such a way that the integration is kept constant and equal to one.

The Diract Delta function is not a real function and in fact there cannot be such a function! It is what we call a distribution function and it is only defined through this limiting process. We should rather regard it as an operator which returns the value of a function evaluated at a given point.

Section 8.7 of [1]

1.3.2 Properties of the δ function

1. $\delta(-x) = \delta(x)$
2. $\delta(x-a)f(x) = \delta(x-a)f(a)$, provided that $f(x)$ is continuous in $x = a$.
3. If a function $g(x)$ has at most simple zeroes then

$$\delta(g(x)) = \sum_{xi \text{ zeroes of } g(x)} \frac{\delta(x-x_i)}{|g'(x_i)|}$$

4. $\delta(ax) = \frac{\delta(x)}{|a|}$
5. The δ function is differentiable and satisfies

$$\int_{-\infty}^{+\infty}\delta'(x)f(x)dx = f(x)\delta(x)\big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\delta(x)f'(x)dx = -f'(0)$$

were we have used that $f(x)$ behaves nicely and vanishes in the limit $\pm\infty$. Higher derivatives can be derived in the same way and in general we have

$$\int_{-\infty}^{+\infty}\delta^{(n)}(x)f(x)dx = (-1)^n f^{(n)}(0)$$

6. The Heaviside step function is the primitive of the δ function

$$\Theta'(x) = \delta(x)$$

with

$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

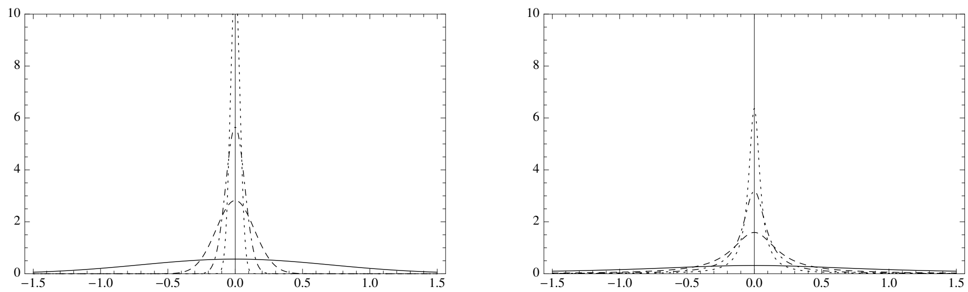


Figure 1.1 Representation of $f_n(x)$ (left) and $g_n(x)$ (right), for $n = 1,\ 5,\ 10$, and 20, as defined in equations (1.14) and (1.15), respectively.

This can be proved as follows

$$\begin{aligned}
\int_{-\infty}^{+\infty} \Theta'(x) f(x) dx &= -\int_{-\infty}^{+\infty} \Theta(x) f'(x) dx \\
&= -\int_{0}^{+\infty} \Theta(x) f'(x) dx \\
&= -f(\infty) + f(0) \\
&= \int_{-\infty}^{+\infty} \delta(x) f(x) dx
\end{aligned}$$

EXAMPLE 1.3

$$\delta(x^2 - a^2) \quad = \quad \frac{\delta(x-a)}{2a} + \frac{\delta(x+a)}{2a} \quad \text{with} \quad a \neq 0$$

1.3.3 Fourier representation of the δ function

The Fourier transform of the δ function yields

$$\mathfrak{F}\left[\delta(x)\right] \equiv \hat{\delta}(w) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(x) e^{iwx} dx = \tfrac{1}{\sqrt{2\pi}} \tag{1.16}$$

Applying the inverse transformation we also obtain

$$\mathfrak{F}^{-1}\left[\hat{\delta}(w)\right] = \delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-iwx} dw = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iwx} dw \tag{1.17}$$

Notice that at the same time we can prove that

$$\mathfrak{F}\left[1\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} 1\ e^{iwx} dx = \sqrt{2\pi}\ \delta(w) \tag{1.18}$$

Also

$$\mathfrak{F}\left[e^{iax}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iax}\ e^{iwx} dx = \sqrt{2\pi}\ \delta(w+a) \tag{1.19}$$

1.3.4 Convolutions

Let us consider two functions $f(x)$ and $g(x)$, whose Fourier transforms are $\hat{f}(w)$ and $\hat{g}(w)$, respectively. We define the operation

$$f(x) * g(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) f(x-y) dy \tag{1.20}$$

as the convolution of the two functions over the $(-\infty, +\infty)$ interval.

If we now introduce the Fourier transform

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) f(x-y) dy &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(w) e^{-iw(x-y)} dw \right) dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(w) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) e^{iwy} dy \right) e^{-iwx} dw \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(w) \hat{g}(w) e^{-iwx} dw
\end{aligned}$$

Notice that if we now apply the Fourier transform to both sides of the equality we are left with

$$\mathfrak{F}\left[f(x) * g(x)\right] = \hat{f}(w)\hat{g}(w) \tag{1.21}$$

or alternatively

$$f(x) * g(x) = \mathfrak{F}^{-1}\left[\hat{f}(w)\hat{g}(w)\right] \tag{1.22}$$

EXAMPLE 1.4

Consider the following differential equation

$$f''(x) - f(x) = e^{-x^2}$$

The general solution contains the two solutions of the homogeneous equation $f''(x) - f(x) = 0$, which can be written as

$$f_0(x) = Ae^{x} + Be^{-x}$$

The third particular solution can be found applying the Fourier transform to both sides of the equation and using the convolution theorem as follows.

$$\begin{aligned}
\mathfrak{F}\left[f''(x) - f(x)\right] &= \mathfrak{F}\left[e^{-x^2}\right] \\
(-iw)^2 \hat{f}(w) - \hat{f}(w) &= \mathfrak{F}\left[e^{-x^2}\right] \\
\hat{f}(w) &= \mathfrak{F}\left[e^{-x^2}\right] \frac{-1}{1+w^2}
\end{aligned}$$

Notice that this equation can be rewritten in a way which clearly simplifies the computation by making use of the convolution theorem (1.21)

$$\begin{aligned}
\mathfrak{F}\left[f(x)\right] &= -\mathfrak{F}\left[e^{-x^2} * \mathfrak{F}^{-1}\left[\frac{1}{1+w^2}\right]\right] \\
f(x) &= -e^{-x^2} * \mathfrak{F}^{-1}\left[\frac{1}{1+w^2}\right]
\end{aligned}$$

The inverse transform is easily calculated as

$$\mathfrak{F}^{-1}\left[\frac{1}{1+w^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{1+w^2}e^{-iwx}dw$$

This integral can be solved by going to the complex plane. For $x \geq 0$ the integration contour has to be closed on the lower half of the complex plane (as for $\pi < \theta < 2\pi$ we would have $Rx\sin\theta < 0$ and $e^{Rx\sin\theta}$ goes to zero as $R \to \infty$). On the other hand, when $x < 0$ the integral has to be closed on the upper half of the complex plane. This leads to

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}2\pi i\left(-\text{Res}\left(\frac{e^{-iwx}}{1+w^2}\right)_{x=-i}\Bigg|_{x\geq 0} + \text{Res}\left(\frac{e^{-iwx}}{1+w^2}\right)_{x=i}\Bigg|_{x<0}\right) \\
&= \sqrt{\frac{\pi}{2}}\left(e^{-x}\big|_{x\geq 0} + e^{x}\big|_{x<0}\right) \\
&= \sqrt{\frac{\pi}{2}}e^{-|x|}
\end{aligned}$$

This then leads to

$$\begin{aligned}
f(x) &= -e^{-x^2} * \sqrt{\frac{\pi}{2}}e^{-|x|} \\
&= -\frac{1}{2}\int_{-\infty}^{+\infty}e^{-(x-y)^2}e^{-|y|}dy
\end{aligned}$$

PROBLEMS

1.1 Find the Fourier transform of $f(x) = \cos(2\pi a x)$ and $g(x) = \sin(2\pi a x)$, where a is a constant.

We begin by considering the following simple Fourier transforms

$$\begin{aligned}
\mathfrak{F}\left[e^{2\pi a x i}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{2\pi a x i} e^{iwx} dx &= \mathfrak{F}[1](w+2\pi a) &= \sqrt{2\pi}\,\delta(w+2\pi a) \\
\mathfrak{F}\left[e^{-2\pi a x i}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-2\pi a x i} e^{iwx} dx &= \mathfrak{F}[1](w-2\pi a) &= \sqrt{2\pi}\,\delta(w-2\pi a)
\end{aligned}$$

Then we can simply express the sine and cosine functions in terms of the exponentials as follows

$$\mathfrak{F}[\cos(2\pi a x)] = \mathfrak{F}\left[\frac{e^{2\pi a x i}+e^{-2\pi a x i}}{2}\right] = \frac{\mathfrak{F}\left[e^{2\pi a x i}\right]+\mathfrak{F}\left[e^{-2\pi a x i}\right]}{2} = \sqrt{\frac{\pi}{2}}\left(\delta(w+2\pi a)+\delta(w-2\pi a)\right)$$

$$\mathfrak{F}[\sin(2\pi a x)] = \mathfrak{F}\left[\frac{e^{2\pi a x i}-e^{-2\pi a x i}}{2i}\right] = \frac{\mathfrak{F}\left[e^{2\pi a x i}\right]-\mathfrak{F}\left[e^{-2\pi a x i}\right]}{2i} = -i\sqrt{\frac{\pi}{2}}\left(\delta(w+2\pi a)-\delta(w-2\pi a)\right)$$

1.2 Prove the following relation shown in the lecture

$$\delta(ax) = \frac{1}{a}\,\delta(x)$$

We solve it by applying the delta function to a function $f(x)$ which behaves nicely

$$\int_{-\infty}^{+\infty} f(x)\delta(ax)dx = \frac{1}{a}\int_{-\infty}^{+\infty} f(x'/a)\delta(x')dx' = \frac{f(0)}{a} = \int_{-\infty}^{+\infty} f(x)\frac{\delta(x)}{a}dx$$

1.3 Show that the convolution product

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dy\, f(x-y)\, g(y)$$

is associative, commutative and admits $\delta(x)$ as a neutral element.

1. Associativity:

$$\begin{aligned}
(f*(g*h))(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dy\, f(x-y)\,(g*h)(y) \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dydz\, f(x-y)\, g(y-z)\, h(z)
\end{aligned}$$

Let $v = y - z$, so that we get

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dvdz\, f(x-v-z)\, g(v)\, h(z) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dz\,(f*g)(x-z)\, h(z) \\
&= ((f*g)*h)(x)
\end{aligned}$$

2. Commutativity:

$$\begin{aligned}(f*g)(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dy\, f(x-y)\, g(y) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} du\, f(u)\, g(x-u) \\ &= (g*f)(x)\end{aligned}$$

where we have taken $u = x - y$

3. Neutral element:

$$\begin{aligned}(f*\sqrt{2\pi}\delta)(x) &= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dy\, f(x-y)\, \delta(y) \\ &= f(x)\end{aligned}$$

1.4 Find the convolution of the function

$$f(x) = \delta(x+a) + \delta(x-a)$$

with the function

$$g(x) = \begin{cases} 1, & \text{if } -b \le x < b\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

where $a < b$.

Then find the Fourier transform of the function $h = f * g$ using the convolution theorem, that is considering the Fourier transform of the function f and g. Verify your result calculating $\hat{h}$ directly.

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-b}^{b} dy\, \left(\delta(x-y+a) + \delta(x-y-a)\right)$$

$$\begin{aligned}\int_{-b}^{b} dy\, \delta(x-y+a) &= \begin{cases} 1\,, & \text{if } -b-a < x < b-a\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \\ \int_{-b}^{b} dy\, \delta(x-y-a) &= \begin{cases} 1\,, & \text{if } a-b < x < a+b\,, \\ 0\,, & \text{otherwise}\,. \end{cases}\end{aligned}$$

Then

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}}\begin{cases} 0\,, & \text{if } x \le -a-b\,, \\ 1\,, & \text{if } -a-b < x \le a-b\,, \\ 2\,, & \text{if } a-b < x \le b-a\,, \\ 1\,, & \text{if } b-a < x \le a+b\,, \\ 0\,, & \text{if } x > a+b\,. \end{cases}$$

On the other hand

$$\begin{aligned}\mathfrak{F}[f(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(\delta(x+a)+\delta(x-a))e^{iwx}dx\\ &= \frac{1}{\sqrt{2\pi}}\left(e^{-iwa}+e^{iwa}\right)\\ &= \frac{2}{\sqrt{2\pi}}\cos(wa)\end{aligned}$$

$$\begin{aligned}\mathfrak{F}[g(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-b}^{b}e^{iwx}dx\\ &= \frac{1}{iw\sqrt{2\pi}}\left(e^{iwb}-e^{-iwb}\right)\\ &= \frac{2}{w\sqrt{2\pi}}\sin(wb)\end{aligned}$$

Hence,

$$\begin{aligned}\mathfrak{F}[f(x)*g(x)] &= \mathfrak{F}[f]\,\mathfrak{F}[g]\\ &= \frac{2}{w\pi}\cos(wa)\sin(wb)\end{aligned}$$

We can check this by direct computation using $h(x)$

$$\begin{aligned}\hat{h}(x) &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\left(\int_{-a-b}^{a-b}dx\,e^{iwx}+2\int_{a-b}^{b-a}dx\,e^{iwx}+\int_{b-a}^{a+b}dx\,e^{iwx}\right)\\ &= \frac{1}{2\pi}\left(\int_{-a-b}^{a+b}dx\,e^{iwx}+\int_{a-b}^{b-a}dx\,e^{iwx}\right)\\ &= \frac{1}{2\pi}\left(\frac{1}{iw}(e^{iw(a+b)}-e^{-iw(a+b)})+\frac{1}{iw}\left(e^{-iw(a-b)}-e^{iw(a-b)}\right)\right)\\ &= \frac{1}{w\pi}\left(\sin(w(a+b))-\sin(w(a-b))\right)\\ &= \frac{1}{w\pi}\left(\sin wa\,\cos wb+\cos wa\sin wb-\sin wa\,\cos wb+\cos wa\sin wb\right)\\ &= \frac{2}{w\pi}\cos(wa)\,\sin(wb)\end{aligned}$$

1.5 Prove the Wiener-Kinchin theorem, that states that $\hat{C}(k)=[\hat{f}(k)]^*\hat{g}(k)$, where $\hat{C}(k)$ is the cross-correlation function

$$\hat{C}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dx\,(f\otimes g)(x)\,e^{ikx}\ .$$

$$\hat{C}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dx\,(f\otimes g)(x)\,e^{ikx}=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx\,dy\,f(y)^*\,g(x+y)\,e^{ikx}$$

Letting $z = x + y$, we get

$$\begin{aligned}\hat{C}(k) &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dy\,dz\,f(y)^*\,g(z)\,e^{-ik(y-z)} \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} dy\,f(y)\,e^{-iky}\right]^* \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dz\,g(z)\,e^{ikz} = \hat{f}(k)^*\,\hat{g}(k)\end{aligned}$$

1.4 Parseval's theorem

$$\int_{-\infty}^{+\infty} f(x)^* g(x) dx = \int_{-\infty}^{+\infty} \hat{f}(w)^* \hat{g}(w) dw \tag{1.23}$$

This identity can be tested using the properties of the Dirac δ function as follows.

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x)^* g(x) dx &= \int_{-\infty}^{+\infty} \left(\mathfrak{F}^{-1}\left[\hat{f}(w)\right]\right)^* \mathfrak{F}^{-1}\left[\hat{g}(w')\right] dx \\
&= \int_{-\infty}^{+\infty} dx \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(w) e^{-iwx} dw\right)^* \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{g}(w') e^{-iw'x} dw'\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int \int \int_{-\infty}^{+\infty} dw\, dw'\, dx \hat{f}(w)^* \hat{g}(w') e^{-ix(w-w')} \\
&= \int \int_{-\infty}^{+\infty} dw\, dw' \hat{f}(w)^* \hat{g}(w') \delta(w - w') \\
&= \int_{-\infty}^{+\infty} \hat{f}(w)^* \hat{g}(w) dw
\end{aligned}$$

Notice that if we take the case where $g(x) = f(x)$ then the Parseval relation returns a normalization integral. Relation (1.23) then guarantees that if a function $f(x)$ is normalised to unity, then its Fourier transform is also normalised to unity. The Fourier transform is a unitary operation in the Hilbert space of square-integrable functions L^2 and the Parseval identity is a manifestation of this property.

The Parseval relations can also be derived from the corresponding Sturm Liouville problem for the operator of the wave function, in which case they can be understood as a consequence of the orthogonality relations among the different eigenfunctions.

The Parseval relation is sometimes useful in order to compute integrals

EXAMPLE 1.5

Compute the following integral

$$I = \int_{0}^{+\infty} \frac{dw}{(a^2 + w^2)^2}$$

Defining $k = w/a$ we can write this integral as

$$I = \frac{1}{2a^3} \int_{-\infty}^{+\infty} \frac{dk}{(1 + k^2)^2}$$

We know already that

$$\mathfrak{F}^{-1}\left[\frac{1}{(1 + k^2)}\right] = \sqrt{\frac{\pi}{2}} e^{-|x|}$$

Hence, by Parseval's relation we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dk}{(1+k^2)^2} &= \int_{-\infty}^{+\infty} \left(\sqrt{\frac{\pi}{2}} e^{-|x|}\right)^2 dx \\ &= \pi \int_{0}^{+\infty} e^{-2x} dx \\ &= \frac{\pi}{2}\end{aligned}$$

from where we get

$$I = \frac{\pi}{4a^3}$$

1.4.1 Fourier transform and integral equations

The Fourier transform can also be used to solve integral equations that involve convolution of functions.

$$h(x) = e^{i3x} + \int_{-\infty}^{+\infty} e^{-|y|} h(x-y)\, dy$$

Noticing that the integral on the right hand side corresponds to the convolution of the two functions $\sqrt{2\pi}(h(x) * e^{-|x|})$ we can apply the Fourier transform to the whole equation, and use relations (1.21) and (1.19) to obtain

$$\hat{h}(w) = \sqrt{2\pi}\, \delta(w+3) + \frac{2}{1+w^2} \hat{h}(w)$$

This is an equation that can be rewritten as follows

$$\hat{h}(w) = \sqrt{2\pi}\, \delta(w+3) \left(1 - \frac{2}{1+w^2}\right)^{-1}$$

we now only need to invert this relation

$$\begin{aligned}h(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{2\pi}\, \delta(w+3) \left(1 - \frac{2}{1+w^2}\right)^{-1} e^{-iwx} dw \\ &= \frac{5}{4} e^{i3x}\end{aligned}$$

1.5 Discrete Fourier Transforms

The Fourier transform allows us to study the frequency content of a signal $h(t)$. In practise, however, the signal may not be know in the continuous, but only in certain points. This is the case, for example, if $h(t)$ was experimentally measured at given times which we can assume are equally spaced.

Let us assume that we know that a function $h(t)$ has a period T and we perform a set of measurements h_j within that interval. We will denote by $t_j = j \frac{T}{2N}$ the set of times at which the function is measured, with $0 \leq j \leq 2N-1$ (we consider an even number of points).

The Discrete Fourier Transform (DFT) is defined from this set of data as

$$\hat{h}(w_p) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} h_j e^{iw_p t_j} \tag{1.24}$$

where $w_p = \frac{2\pi p}{T}$ with $0 \leq p \leq 2N - 1$. Notice that the decomposition includes as many frequencies as time measurements were performed. This relation can be inverted, but now considering that the time is a continuous variable

$$h^{DFT}(t) = \frac{1}{\sqrt{2N}} \sum_{p=0}^{2N-1} \hat{h}_p e^{-iw_p t} \tag{1.25}$$

Notice that the resulting function is defined *for all* values of t, retains the periodic properties of the function, and satisfies the measurements that were performed, this is $h^{DFT}(t_j) = h_j$. This does not mean that $h^{DFT}(t) = h(t)$, but it converges to it when the number of measurements increases and goes to infinity.

1.5.1 The Fourier Matrix:

The above procedure can be expressed in terms of the *Fourier Matrix* as follows

$$\begin{pmatrix} \hat{h}_1 \\ \vdots \\ \hat{h}_{2N-1} \end{pmatrix} = \left(\frac{e^{iw_p t_j}}{\sqrt{2N}} \right) \begin{pmatrix} h_1 \\ \vdots \\ h_{2N-1} \end{pmatrix}$$

where

$$F(2N)_{pj} = \left(\frac{e^{iw_p t_j}}{\sqrt{2N}} \right) = \left(\frac{e^{i\pi pj/N}}{\sqrt{2N}} \right)$$

is the Fourier Matrix which provides the basis of plane waves in which the signal is expanded.

EXAMPLE 1.6

Consider the periodic signal $h(t) = \cos(t)$, for which we know the period $T = 2\pi$. Assume that we only have the measurements of t his signal at four given times,

$$\begin{array}{lll} h(t_0) = 1 & ; & t_0 = 0 \\ h(t_1) = 0 & ; & t_1 = \frac{\pi}{2} \\ h(t_2) = -1 & ; & t_2 = \pi \\ h(t_3) = 0 & ; & t_3 = \frac{3\pi}{2} \end{array}$$

In our notation this corresponds to $N = 2$ and the frequencies in which the DFT is expanded correspond to

$$\begin{aligned} w_0 &= \frac{2\pi 0}{T} = 0 \\ w_1 &= \frac{2\pi}{T} = 1 \\ w_2 &= \frac{4\pi}{T} = 2 \\ w_3 &= \frac{6\pi}{T} = 3 \end{aligned}$$

We can compute the Fourier Matrix for this basis and arrive to

$$\begin{pmatrix} \hat{h}_0 \\ \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

For where $\hat{h}(w_p) = (0, 1, 0, 1)$. We can now recover a continuous function applying the inverse DFT to obtain

$$\begin{aligned} h(t) &= \frac{1}{2} \sum_{p=0}^{2N-1} \hat{h}_p e^{-iw_p t} \\ &= \frac{1}{2} \left(0 \cdot e^{-i0t} + 1 \cdot e^{-it} + 0 \cdot e^{-i2t} + 1 \cdot e^{-i3t} \right) \\ &= \frac{1}{2} \left(e^{-it} + e^{-i3t} \right) \end{aligned}$$

Given that the function is real, we concentrate only on the real part of this expansion.

$$h(t) = \frac{1}{2} (\cos t + \cos 3t)$$

Notice that it is *not* the original signal, but has the right period and satisfies the measurements. A more accurate response can be obtained if we increase the number of measurements.

1.5.2 Fast Fourier Transform

Notice that the Discrete Fourier Transform requires a large number of operations. Thus, although more precision can be obtained when the number of measurements increases, in practise the number of computations also increases as N^2. This is a clear drawback in signal processing. The *Fast Fourier Transform* is a way of factoring and rearranging the terms in the sums of the DFT that greatly reduces the number of calculations needed and speeds up the processing of signals.

PROBLEMS

1.1 Use the Fourier transform to solve

$$y''(x) + 2y'(x) + y(x) = g(x)$$

expressing the solution as a convolution.
(Hint: $\mathfrak{F}\left[-xe^x\Theta(-x)\right] = \frac{1}{\sqrt{2\pi}(1-iw)^2}$ Notice that we could also relate it by symmetry arguments to $xe^x\Theta(x)$ but this function is not L^2.)

The general solution is the sum of the homogeneous solution and a particular solution. The homogeneous solution is trivial to obtain and reads $y_0(x) = (Ax + B)e^{-x}$.

We apply the Fourier transform to the whole equation

$$\left((-iw)^2 + 2(-iw) + 1\right)\hat{y}(w) = \mathfrak{F}\left[g(x)\right]$$

from where

$$\begin{aligned}
\hat{y}(w) &= \mathfrak{F}\left[g(x)\right]\frac{1}{-w^2 - 2iw + 1} \\
&= \mathfrak{F}\left[g(x)\right]\frac{1}{(1-iw)^2} \\
&= \sqrt{2\pi}\mathfrak{F}\left[g(x)\right]\mathfrak{F}\left[-xe^x\Theta(-x)\right] \\
&= \sqrt{2\pi}\mathfrak{F}\left[g(x) * (-xe^x\Theta(-x))\right]
\end{aligned}$$

This equation can be easily inverted now by applying $\mathfrak{F}^{-1}\left[\right]$

$$\begin{aligned}
y(x) &= \sqrt{2\pi}g(x) * (-xe^x\Theta(-x)) \\
&= -\int_{-\infty}^{+\infty} g(x-y)ye^y\Theta(-y)dy \\
&= \int_0^{+\infty} g(x+y)ye^{-y}dy
\end{aligned}$$

Thus, the general solution reads

$$y(x) = (Ax + B)e^{-x} + \int_0^{+\infty} g(x+y)ye^{-y}dy$$

1.2 Use the convolution theorem to find the solution $f(x)$ of the equation

$$\int_{-\infty}^{+\infty} dy\, f(y)\, f(x-y) = \frac{1}{1+x^2}$$

(Hint: Remember that $\mathfrak{F}\left[1/(1+x^2)\right] = \sqrt{\pi/2}\, e^{-|w|}$).

Applying the Fourier transform to the left-hand side of the equation we get

$$\begin{aligned}
\mathfrak{F}\left[\int_{-\infty}^{+\infty} dy\, f(y)\, f(x-y)\right] &= \sqrt{2\pi}\,\mathfrak{F}\left[f(x) * f(x)\right] \\
&= \sqrt{2\pi}\,\mathfrak{F}\left[f(x)\right]\mathfrak{F}\left[f(x)\right]
\end{aligned}$$

On the other hand, from the right-hand side of the equation we have

$$\mathfrak{F}\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{\pi}{2}}e^{-|w|}$$

Equating both sides implies that

$$\mathfrak{F}\left[f(x)\right] = \frac{1}{\sqrt{2}}e^{-|w|/2}$$

An now we only need to invert this equation. Defining $w' = w/2$ and $x' = 2x$ we have

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2}}e^{-|w|/2}e^{-iwx}dw \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-|w'|}e^{iw'(2x)}dw' \\ &= \frac{2}{\sqrt{\pi}}\mathfrak{F}^{-1}\left[e^{-|w^2|}\right](x') \\ &= \frac{2}{\sqrt{\pi}\,(1+4x^2)} \end{aligned}$$

1.3 Find the solution of the equation

$$h(x) = f(x) + \lambda\int_{-\infty}^{+\infty}dy\,\frac{\sin y}{y}\,h(x-y)$$

by taking the inverse Fourier transform of

$$g_a(w) = \begin{cases} 1\,, & |w| \le a\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

and comparing the form of this Fourier transform to the integral above.

We apply the inverse Fourier Transform to $g(w)$ to obtain

$$\begin{aligned} \mathfrak{F}^{-1}\left[g(w)\right] &= \frac{1}{\sqrt{2\pi}}\int_{-a}^{+a}e^{-iwx}dw \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{ixa}-e^{-ixa}}{ix}\right) \\ &= a\sqrt{\frac{2}{\pi}}\frac{\sin(xa)}{xa} \end{aligned}$$

Which implies that

$$\mathfrak{F}\left[\frac{\sin y}{y}\right] = \sqrt{\frac{\pi}{2}}\,g_1(w)$$

Thus, we can apply the Fourier transform to the whole expression and use formula (1.21) on the Fourier transform of the integral to arrive to

$$\begin{aligned}
\mathfrak{F}[h(x)] &= \mathfrak{F}[f(x)] + \lambda\sqrt{2\pi}\,\mathfrak{F}\left[\frac{\sin x}{x} * h(x)\right] \\
\mathfrak{F}[h(x)] &= \mathfrak{F}[f(x)] + \lambda\sqrt{2\pi}\,\mathfrak{F}\left[\frac{\sin x}{x}\right]\mathfrak{F}[h(x)] \\
\mathfrak{F}[h(x)] &= \mathfrak{F}[f(x)] + \lambda\pi\, g_1(w)\,\mathfrak{F}[h(x)] \\
\mathfrak{F}[h(x)] &= \frac{\mathfrak{F}[f(x)]}{1-\lambda\pi\, g_1(w)}
\end{aligned}$$

We only need to invert the Fourier transform now

$$\begin{aligned}
h(x) &= \mathfrak{F}^{-1}\left[\frac{\mathfrak{F}[f(x)]}{1-\lambda\pi\, g_1(w)}\right] \\
&= f(x) * \mathfrak{F}^{-1}\left[\frac{1}{1-\lambda\pi\, g_1(w)}\right]
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathfrak{F}^{-1}\left[\frac{1}{1-\lambda\pi\, g_1(w)}\right] &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{-1} e^{-iwx}dw + \int_{-1}^{1}\frac{1}{1-\lambda\pi}e^{-iwx}dw + \int_{1}^{+\infty} e^{-iwx}dw\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{-iwx}dw + \int_{-1}^{1}\left(\frac{1}{1-\lambda\pi}-1\right)e^{-iwx}dw\right) \\
&= \sqrt{2\pi}\delta(x) + \sqrt{\frac{2}{\pi}}\frac{\sin x}{x}\left(\frac{\lambda\pi}{1-\lambda\pi}\right)
\end{aligned}$$

From where, finally, we get to

$$\begin{aligned}
h(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x-y)\mathfrak{F}^{-1}\left[\frac{1}{1-\lambda\pi\, g_1(w)}\right](y)\,dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x-y)\left(\sqrt{2\pi}\delta(y) + \sqrt{\frac{2}{\pi}}\frac{\sin y}{y}\left(\frac{\lambda\pi}{1-\lambda\pi}\right)\right)\,dy \\
&= f(x) + \left(\frac{\lambda}{1-\lambda\pi}\right)\int_{-\infty}^{+\infty} f(x-y)\frac{\sin y}{y}dy
\end{aligned}$$

1.4 Write out the Fourier matrix for a sample with $N = 4$.

For $N = 4$ the Fourier matrix is 8×8 and reads

$$F(2N)_{pj} = \left(\frac{e^{i\pi pj/N}}{\sqrt{2N}}\right)$$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{i\pi}{4}} & i & e^{\frac{3i\pi}{4}} & -1 & e^{-\frac{3i\pi}{4}} & -i & e^{-\frac{i\pi}{4}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & e^{\frac{3i\pi}{4}} & -i & e^{\frac{i\pi}{4}} & -1 & e^{-\frac{i\pi}{4}} & i & e^{-\frac{3i\pi}{4}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & e^{-\frac{3i\pi}{4}} & i & e^{-\frac{i\pi}{4}} & -1 & e^{\frac{i\pi}{4}} & -i & e^{\frac{3i\pi}{4}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & e^{-\frac{i\pi}{4}} & -i & e^{-\frac{3i\pi}{4}} & -1 & e^{\frac{3i\pi}{4}} & i & e^{\frac{i\pi}{4}} \end{pmatrix}$$

1.5 Find the DFT of the signal given by

$$f(t_j) = (0, 1, 4, 9)$$

Then take the inverse transform to recover the original signal.

We start by determining the Fourier Matrix for $N = 2$. This matrix is given in Example (1.6). The values of the four measurements can be implemented as a four vector in order to extract the DFT as follows

$$\begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \end{pmatrix}$$

This leads to the transformed sample

$$\begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2-4i \\ -3 \\ -2+4i \end{pmatrix}$$

Inverting this transformation, this leads to the following signal

$$f(t) = \frac{7}{2} - (1+2i)e^{-it} - \frac{3}{2}e^{-2it} - (1-2i)e^{-3it}$$

and taking only the real part we are left with

$$f(t) = \frac{7}{2} - \cos t - 2\sin t - \frac{3}{2}\cos 2t - \cos 3t + 2\sin 3t$$

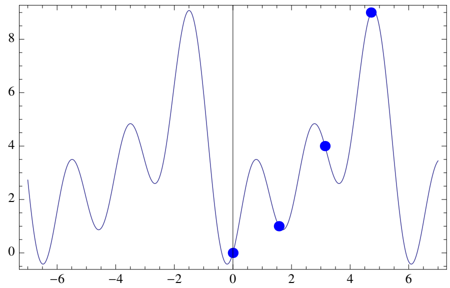


Figure 1.2 Resulting $f(t)$ in exercise 1.5. Blue dots denote the actual measurements and the blue curve is the reconstructed signal.

1.6 The Fourier transform in quantum mechanics

PROBLEMS

NOTE: For this page we will define the Fourier operator $\hat{F} \equiv \mathfrak{F}^{-1}$ (which simply implies a change of sign in the complex argument of the exponential in our definition of the Fourier transform)

1.1 Use Parseval's theorem to compute, for $a, b > 0$, $\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$.

We already know that

$$\mathfrak{F}\left[\frac{1}{1+x^2}\right](k) = \pi\, e^{-|k|}\,.$$

Hence

$$\mathfrak{F}\left[\frac{1}{a^2+x^2}\right](k) = \frac{1}{a^2}\,\mathfrak{F}\left[\frac{1}{1+(x/a)^2}\right](k) = \frac{1}{a}\,\mathfrak{F}\left[\frac{1}{1+x^2}\right](ak) = \frac{\pi}{a}\,e^{-a|k|}\,.$$

Using Parseval's theorem, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)} &= \int_{-\infty}^{+\infty} \tfrac{dk}{2\pi}\,\mathfrak{F}\left[\tfrac{1}{a^2+x^2}\right](k)\,\mathfrak{F}\left[\tfrac{1}{b^2+x^2}\right](k)\\ &= \tfrac{\pi}{2ab}\int_{-\infty}^{+\infty} dk\, e^{-(a+b)|k|}\\ &= \tfrac{\pi}{ab}\int_{0}^{+\infty} dk\, e^{-(a+b)k}\\ &= \tfrac{\pi}{ab(a+b)}\,.\end{aligned}$$

1.2 Assume you are given a wave function $\psi(x) = \frac{N}{a^2+x^2}$, with $a > 0$. Determine N such that $\psi(x)$ is correctly normalised ($||\psi|| = 1$) and find the momentum space wave function $\overline{\psi}(p)$. Then compute the mean and the variance of the momentum operator $\hat{p}$,

$$\langle \hat{p} \rangle = \int_{-\infty}^{+\infty} dx\, \psi(x)^*\, \hat{p}\psi(x) \;\text{ and }\; (\Delta\hat{p})^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2\,.$$

In order to compute the normalisation constant, we compute

$$1 = ||\psi||^2 = |N|^2 \int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)^2} = \frac{\pi\,|N|^2}{2a^3}\,,$$

where we use the previous result. Hence

$$N = e^{i\alpha}\sqrt{\frac{2a^3}{\pi}}\,, \qquad \text{for some real number } \alpha\,,$$

In the following we choose $\alpha = 0$. The momentum space wave function is

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2a^3}{\pi}}\,\mathfrak{F}\left[\frac{1}{a^2+x^2}\right](p) = \sqrt{a}\,e^{-a|p|}\,.$$

Using Parseval's theorem, we get

$$\langle\hat{p}\rangle = \int_{-\infty}^{+\infty} dp\,a\,p\,e^{-2a|p|} = 0\,,$$

and

$$(\Delta\hat{p})^2 = \langle\hat{p}^2\rangle = \int_{-\infty}^{+\infty} dp\,a\,p^2\,e^{-2a|p|} = 2a\int_0^{\infty} dp\,p^2 e^{-2ap} = \frac{1}{2a^2}\,.$$

1.3 Let $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2$ be the Hamiltonian of the quantum harmonic oscillator. The eigenfunctions of $\hat{H}$ are

$$\varphi_n(x) = \frac{1}{\sqrt{2^n\,n!\,\sqrt{\pi}}}\,H_n(x)\,e^{-x^2/2}\,, \qquad n \geq 0\,,$$

where $H_n(x)$ are the Hermite polynomials $H_n(x) = (-1)^n\,e^{x^2}\,\frac{d^n}{dx^n}\,e^{-x^2}$. The aim of this problem is to prove that the functions $\varphi_n(x)$ are also eigenfunctions of the Fourier operator $\hat{F}$ with eigenvalue $(-i)^n$.

1. Let $\mathcal{H}(t,x) = e^{-t^2+2xt} = e^{x^2}\,e^{-(t-x)^2}$. Our first goal is to compute the Taylor expansion of $\mathcal{H}(t,x)$ with respect to t and to show that the Taylor coefficients are precisely the Hermite polynomials

   $$\mathcal{H}(t,x) = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}\,.$$

   To prove this, follow the following steps:

   (a) Show that $\frac{\partial}{\partial t}\mathcal{H}(t,x) = -e^{x^2}\frac{\partial}{\partial x}e^{-(t-x)^2}$ and deduce that $\frac{\partial^n}{\partial t^n}\mathcal{H}(t,x) = (-1)^n\,e^{x^2}\,\frac{\partial^n}{\partial x^n}e^{-(t-x)^2}$.

   (b) Deduce that that the Taylor coefficients are precisely the Hermite polynomials, by comparing to the definition of the Hermite polynomials given above.

2. Compute the Fourier transform of $e^{-x^2/2}\,\mathcal{H}(t,x)$ with respect to x and show that it is given by $e^{-p^2/2}\mathcal{H}(-it,p)$. Write down its Taylor expansion with respect to t using the result of step 1.

3. By comparing the Taylor coefficients before and after Fourier transforming, what do you conclude?

1. (a) By direct computation, we get

$$\frac{\partial}{\partial t}\mathcal{H}(t,x) = -2(t-x)\, e^{x^2}\, e^{-(t-x)^2} = -e^{x^2}\,\frac{\partial}{\partial x}e^{-(t-x)^2}\,.$$

For $n > 1$, we just iterate this procedure, e.g.,

$$\frac{\partial^2}{\partial t^2}\mathcal{H}(t,x) = -e^{x^2}\,\frac{\partial^2}{\partial x\,\partial t}e^{-(t-x)^2} = e^{x^2}\,\frac{\partial^2}{\partial x^2}\,e^{-(t-x)^2}\,,$$

and so on.

(b) We know that the Taylor series is

$$\mathcal{H}(t,x) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,a_n\,,$$

where

$$a_n = \left(\frac{\partial^n}{\partial t^n}\mathcal{H}(t,x)\right)_{|t=0} = (-1)^n\, e^{x^2}\left(\frac{\partial^n}{\partial x^n}e^{-(t-x)^2}\right)_{|t=0} = (-1)^n\, e^{x^2}\,\frac{d^n}{dx^n}e^{-x^2} = H_n(x)\,.$$

2.

$$\begin{aligned}\hat{F}\left[e^{-x^2/2}\,\mathcal{H}(t,x)\right] &= \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2-2xt-t^2-ipx} = e^{-t^2}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2+2xt-ipx}\\ &= e^{-t^2}\,e^{(2t-ip)^2/2}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-2t+ip)^2}\end{aligned}$$

Using the argument as in Lecture I, we get

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}(x-2t+ip)^2} = \sqrt{2\pi}\,.$$

Hence

$$\hat{F}\left[e^{-x^2/2}\,\mathcal{H}(t,x)\right] = e^{t^2-p^2/2-2ipt} = e^{-p^2/2}\,\mathcal{H}(-it,p)\,.$$

Using the previous result, we see that

$$\hat{F}\left[e^{-x^2/2}\,\mathcal{H}(t,x)\right] = e^{-p^2/2}\sum_{n=0}^{\infty}(-i)^n\,H_n(p)\,\frac{t^n}{n!}\,.$$

3. We can compute the Fourier transform before and after Taylor expansion:

$$\begin{aligned}\hat{F}\left[e^{-x^2/2}\,\mathcal{H}(t,x)\right] &= \textstyle\sum_{n=0}^{\infty}\frac{t^n}{n!}\,\hat{F}\left[e^{-x^2/2}\,H_n(x)\right]\,,\\ \hat{F}\left[e^{-x^2/2}\,\mathcal{H}(t,x)\right] &= \textstyle\sum_{n=0}^{\infty}\frac{t^n}{n!}\,(-i)^n\,e^{-p^2/2}\,H_n(p)\;.\end{aligned}$$

By comparing the coefficients of the Taylor series, we see that

$$\hat{F}\left[e^{-x^2/2}\,H_n(x)\right] = (-i)^n\,e^{-p^2/2}\,H_n(p)\,.$$

and so

$$\hat{F}\varphi_n = (-i)^n\,\varphi_n\,.$$

CHAPTER 2

LAPLACE TRANSFORMS

2.1 Introduction to the Laplace Transform

In this section we introduce a useful integral transform. The Laplace transform of a function $f(t)$, where the paramter t is usually real (and normally refers to the time "time" variable) reads

$$\mathfrak{L}\left[f(t)\right](s) \equiv \bar{f}(s) = \int_0^\infty f(t) e^{-ts} ds \tag{2.1}$$

The transform is not defined for all values of s. The range of specific values corresponding to those for which eq.(2.1) converges, and is usually $s > 0$ (since the sign of the exponential factor and the fact that the "time" variable, t, is usually positive).

EXAMPLE 2.1

$$\begin{aligned} \mathfrak{L}\left[1\right] &= \int_0^\infty e^{-ts} dt \\ &= \frac{1}{s} \end{aligned} \tag{2.2}$$

where the equation above only converges for $s > 0$.

EXAMPLE 2.2

The Laplace transform of $f(t) = e^{at}$ with $a > 0$ reads

$$\begin{aligned}\mathfrak{L}\left[e^{at}\right] &= \int_0^\infty e^{at}e^{-st}dt \\ &= \int_0^\infty e^{-(s-a)t}dt \\ &= \frac{1}{s-a}\end{aligned}$$

where the integral only converges if $s - a > 0$ or equivalently $s > a$.

The Laplace transform is related to the Fourier transform. If we assume that $s = x + iy$ we can write

$$\begin{aligned}\mathfrak{L}\left[f(t)\right] &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^\infty f(t)e^{-(x+iy)t}dt \\ &= \int_0^\infty f(t)e^{-xt}e^{-iyt}dt \\ &= \int_{-\infty}^{+\infty} f(t)e^{-xt}\Theta(t)e^{-iyt}dt \\ &= \sqrt{2\pi}\,\mathfrak{F}^{-1}\left[f(t)e^{-xt}\Theta(t)\right] \qquad (2.3)\end{aligned}$$

EXAMPLE 2.3

Consider the function $f(t) = \cosh(kt) = (e^{kt} + e^{-kt})/2$.

$$\begin{aligned}\mathfrak{L}\left[\cosh(t)\right] &= \frac{1}{2}\left(\mathfrak{L}\left[e^{kt}\right] + \mathfrak{L}\left[e^{-kt}\right]\right) \\ &= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) \\ &= \frac{s}{s^2-k^2}\end{aligned}$$

where the result only converges for $s - k > 0$ and $s + k > 0$, i.e., $s > |k|$.

2.1.1 Laplace Transform and derivatives

The Laplace transform is commonly used when solving differential equations, since it simplifies the derivative operator, usually converting the differential equation in a polynomial one.

$$\begin{aligned}\mathfrak{L}\left[\frac{\partial f}{\partial t}\right] &= \int_0^\infty \frac{\partial f}{\partial t}e^{-st}dt \\ &= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)\frac{\partial e^{-st}}{\partial t}dt \\ &= -f(0) + s\int_0^\infty f(t)e^{-st}dt\end{aligned}$$

where we have used that $f(t)e^{-st}$ vanishes at $t = \infty$ due to the convergence condition on the Laplace transform. From here we obtain

$$\mathfrak{L}\left[\frac{\partial f}{\partial t}\right] = s\mathfrak{L}\left[f(t)\right] - f(0) \tag{2.4}$$

This expression can be easily generalised to the n-th derivative

$$\mathfrak{L}\left[\frac{\partial^n f}{\partial t^n}\right] = s^n\mathfrak{L}\left[f(t)\right] - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0) \tag{2.5}$$

EXAMPLE 2.4

Let us consider the function $f(t) = \sinh kt$. Since we can write $f(t) = \sinh kt = \frac{1}{k}\frac{\partial \cosh kt}{\partial t}$ then its Laplace transform is

$$\begin{aligned}
\mathfrak{L}\left[\sinh kt\right] &= \frac{1}{k}\mathfrak{L}\left[\partial \cosh kt\right]\partial t \\
&= \frac{1}{k}\left(s\mathfrak{L}\left[\cosh kt\right] - \cosh 0\right) \\
&= \frac{1}{k}\left(s\frac{s}{s^2 - k^2} - 1\right) \\
&= \frac{k}{s^2 - k^2}
\end{aligned}$$

which converges for $s > |k|$.

This formula can be verified by direct computation.

2.1.2 Laplace transforms and integrals

$$\begin{aligned}
\mathfrak{L}\left[\int_a^t f(t')dt'\right] &= \int_0^t \left(\int_a^t f(t')dt'\right) e^{-st} dt \\
&= -\frac{1}{s}\int_a^t f(t')dt' e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt
\end{aligned}$$

Using the convergence argument on the first term, we arrive at

$$\mathfrak{L}\left[\int_a^t f(t')dt'\right] = \frac{1}{s}\mathfrak{L}\left[f(t)\right] - \frac{1}{s}\int_0^a f(t')dt' \tag{2.6}$$

Laplace transforms can be useful in solving integrals.

EXAMPLE 2.5

Consider the following integral

$$f(t) = \int_0^\infty \frac{\sin tx}{x} dx$$

Let us solve it by transforming the whole equation

$$\begin{aligned}\mathfrak{L}[f(t)] &= \mathfrak{L}\left[\int_0^\infty \frac{\sin tx}{x}dx\right] \\ &= \int_0^\infty \left(\int_0^\infty \frac{\sin tx}{x}dx\right) e^{-st}dt \\ &= \int_0^\infty \mathfrak{L}[\sin tx]\,\frac{dx}{x}\end{aligned}$$

Using that $\mathfrak{L}[\sin xt] = \frac{x}{s^2+x^2}$ we have

$$\mathfrak{L}[f(t)] = \int_0^\infty \frac{1}{s^2+x^2}dx = \frac{\pi}{2s}$$

where the last integral can be easily solved using the residue theorem.

We are left with a solution in the "transformed space". The question now is how do we transform it back. We will see this in the following sections. For now, we can readily identify $\mathfrak{L}[1] = \frac{1}{s}$ add thus

$$f(t) = \frac{\pi}{2}$$

with $t > 0$ so that the Laplace transform converges.

For $t < 0$ we can use $\sin tx = -\sin(-tx)$ in the equations above and arrive to

$$f(t) = -\frac{\pi}{2}$$

with $t < 0$.

Thus, the complete equation reads

$$f(t) = \frac{\pi}{2}\text{sign}(t)$$

PROBLEMS

2.1 Fill in the following table of Laplace transforms by direct evaluation of the integral

$$\bar{f}(s) = \mathfrak{L}[f](s) = \int_0^\infty dt\, e^{-st}\, f(t)\,,$$

where s is taken to be real. In each case state for which values of s the integral exists. In the following a is a positive constant and n a positive integer.

$f(t)$	$\mathfrak{L}[f](s)$	valid for	$f(t)$	$\mathfrak{L}[f](s)$	valid for
a			$\sqrt{t}$		
$\sin at$			$1/\sqrt{t}$		
e^{-at}			$\delta(t-a)$		
$\cos at$			$\theta(t-a)$		

$f(t)$	$\mathfrak{L}[f](s)$	valid for	$f(t)$	$\mathfrak{L}[f](s)$	valid for
a	a/s	$s>0$	$\sqrt{t}$	$\frac{1}{2}\sqrt{\pi/s^3}$	$s>0$
$\sin at$	$a/(s^2+a^2)$	$s>0$	$1/\sqrt{t}$	$\sqrt{\pi/s}$	$s>0$
e^{-at}	$1/(s+a)$	$s>-a$	$\delta(t-a)$	e^{-as}	$s>0$
$\cos at$	$s/(s^2+a^2)$	$s>0$	$\theta(t-a)$	e^{-as}/s	$s>0$

2.2 Prove that

$$\mathfrak{L}\left[\int_a^t \int_a^{t'} dt'\, dx\, f(x)\right](s) = \frac{1}{s^2}\mathfrak{L}[f](s) - \frac{1}{s^2}\int_0^a dx\, f(x) - \frac{1}{s}\int_0^a \int_a^{t'} dt'\, dx\, f(x)\,.$$

$$\begin{aligned}\mathfrak{L}\left[\int_a^t \int_a^{t'} dt'\, dx\, f(x)\right](s) &= \tfrac{1}{s}\,\mathfrak{L}\left[\textstyle\int_a^t dx\, f(x)\right](s) - \tfrac{1}{s}\textstyle\int_0^a \int_a^{t'} dt'\, dx\, f(x)\\ &= \tfrac{1}{s}\left[\tfrac{1}{s}\mathfrak{L}[f](s) - \tfrac{1}{s}\textstyle\int_0^a dx\, f(x)\right] - \tfrac{1}{s}\textstyle\int_0^a \int_a^{t'} dt'\, dx\, f(x)\\ &= \tfrac{1}{s^2}\mathfrak{L}[f](s) - \tfrac{1}{s^2}\textstyle\int_0^a dx\, f(x) - \tfrac{1}{s}\textstyle\int_0^a \int_a^{t'} dt'\, dx\, f(x)\,.\end{aligned}$$

2.3 Use the differentiation formula to compute the Laplace transform of t^n.
Note that $\frac{d^n}{dt^n}t^n = n!$. Taking the Laplace transform on both sides, we get

$$\mathfrak{L}\left[\frac{d^n}{dt^n}t^n\right](s) = n!\,\mathfrak{L}[1](s) = n!/s\,.$$

Using the formula for the derivative, we get

$$\mathfrak{L}\left[\frac{d^n}{dt^n}t^n\right](s) = s^n\,\mathfrak{L}[t^n](s) - \sum_{k=0}^{n-1} s^{n-1-k}\,\frac{d^k}{dt^k}t^n{}_{|t=0}\,.$$

The last term obviously vanishes, and so we get

$$\mathfrak{L}[t^n](s) = \frac{n!}{s^{n+1}}\,.$$

2.4 Prove that $\lim_{s\to\infty} s\,\bar{f}(s) = f(0)$ [Hint: Write f as a Taylor series around $t = 0$].
Close to $t = 0$, we can write

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(0)\,\frac{t^n}{n!}\,. \tag{2.7}$$

Using the result from Problem 3, we get

$$\lim_{s\to\infty} s\,\mathfrak{L}[f](s) = \lim_{s\to\infty} s \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,\mathfrak{L}[t^n](s) = \lim_{s\to\infty} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{s^n} = f(0)\,. \tag{2.8}$$

2.5 Compute the Laplace transform of the following function

$$f(t) = \begin{cases} 1\,, & \text{if } 2n < t < 2n+1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

where $n = 0, 1, 2, 3\ldots$.
We can write

$$f(t) = \sum_{n=0}^{\infty} g_n(t)\,, \tag{2.9}$$

with

$$g_n(t) = \theta(t-2n) - \theta(t-2n-1)\,. \tag{2.10}$$

Thus, using the result from Problem 1,

$$\mathfrak{L}[g_n](s) = \mathfrak{L}[\theta(t-2n)](s) - \mathfrak{L}[\theta(t-2n-1)](s) = \frac{e^{-2ns}}{s} - \frac{e^{-(2n+1)s}}{s} = \frac{e^{-2ns}}{s}(1-e^{-s}) \tag{2.11}$$

Hence

$$\mathfrak{L}[f](s) = \sum_{n=0}^{\infty} \mathfrak{L}[g_n](s) = \frac{1-e^{-s}}{s} \sum_{n=0}^{\infty} e^{-2ns} = \frac{1-e^{-s}}{s\,(1-e^{-2s})} = \frac{1}{s\,(1+e^{-s})}\,. \tag{2.12}$$

2.2 Properties of the Laplace Transform

The following transforms can be solved by direct calculation

$$\mathfrak{L}[1] = \frac{1}{s} \tag{2.13}$$

$$\mathfrak{L}[t] = \frac{1}{s^2} \tag{2.14}$$

There are some properties of the Laplace Transform which are easy to prove by doing a variable change inside the integral symbol

$$\mathfrak{L}\left[e^{-at}f(t)\right](s) = \mathfrak{L}[f(t)](s+a), \tag{2.15}$$

$$\mathfrak{L}\left[f\left(\frac{t}{a}\right)\right](s) = a\mathfrak{L}[f(t)](as), \tag{2.16}$$

$$\mathfrak{L}[f(t-a)\Theta(t-a)](s) = e^{-as}\mathfrak{L}[f(t)](s) \tag{2.17}$$

The last property is very useful in order to determine the Laplace transform of a function which is defined piecewise.

EXAMPLE 2.6

Let us consider the following function

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 2-t, & \text{if } 1 \leq t \leq 2 \\ 0, & \text{if } 2 \leq t \end{cases}$$

We can write the function as

$$f(t) = \Theta(t) + (1-t)\Theta(t-1) - (2-t)\Theta(t-2)$$

Then, the Laplace Transform can be written as

$$\begin{aligned} \mathfrak{L}[f(t)] &= \mathfrak{L}[\Theta(t)] - \mathfrak{L}[(t-1)\Theta(t-1)] - \mathfrak{L}[(2-t)\Theta(t-2)] \\ &= \mathfrak{L}[1] - e^{-s}\mathfrak{L}[t] - e^{-2s}\mathfrak{L}[t] \\ &= \frac{1}{s} - \frac{e^{-s}}{s^2}\left(1-e^{-s}\right) \end{aligned}$$

Another interesting property, useful in specific differential equations, is the following

$$\mathfrak{L}[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n}\mathfrak{L}[f(t)](s) \tag{2.18}$$

which can be proved by induction. We start by proving it for $n = 0$ (which holds trivially), and $n = 1$

$$\begin{aligned} \mathfrak{L}\left[t^1 f(t)\right](s) &= \int_0^\infty t f(t) e^{-st} dt \\ &= -\int_0^\infty f(t) \frac{d}{ds} e^{-st} dt \\ &= -\frac{d}{ds}\mathfrak{L}[f(t)](s) \end{aligned}$$

and, assuming it is valid for an arbitrary n, we prove it for $n+1$

$$\begin{aligned}\mathfrak{L}\left[t^{n+1}f(t)\right](s) &= \int_0^\infty t^{n+1}f(t)e^{-st}dt \\ &= -\frac{d}{ds}\mathfrak{L}\left[t^n f(t)\right](s) \\ &= (-1)^{n+1}\frac{d^{n+1}}{ds^{n+1}}\mathfrak{L}\left[f(t)\right](s)\end{aligned}$$

EXAMPLE 2.7

As an example of the above relation we can consider the following set of Laplace transforms,

$$\mathfrak{L}\left[t\right](s) = -\frac{d}{ds}\mathfrak{L}\left[1\right](s) = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$\mathfrak{L}\left[t^2\right](s) = (-1)^2\frac{d^2}{ds^2}\mathfrak{L}\left[1\right](s) = \frac{d^2}{ds^2}\left(\frac{1}{s}\right) = \frac{2}{s^3}$$

$$\mathfrak{L}\left[t\sin t\right] = -\frac{d}{ds}\mathfrak{L}\left[\sin t\right](s) = -\frac{d}{ds}\frac{1}{1+s^2} = \frac{2s}{(1+s^2)^2}$$

Theorem 2.1 *If a function $f(t)$ is periodic with period p, then its Laplace transform satisfies the following property*

$$\mathfrak{L}\left[f(t)\right](s) = \frac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}dt \tag{2.19}$$

Proof:

$$\begin{aligned}\mathfrak{L}\left[f(t)\right](s) &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^p f(t)e^{-st}dt + \int_p^{2p} f(t)e^{-st}dt + \ldots + \int_{np}^{(n+1)p} f(t)e^{-st}dt + \ldots\end{aligned}$$

In the n-th integral we can consider a shift $t = x + np$

$$\begin{aligned}\int_{np}^{(n+1)p} f(t)e^{-st}dt &= \int_0^p f(x+np)e^{-s(x+np)}dx \\ &= e^{-nps}\int_0^p f(x)e^{-sx}dx\end{aligned}$$

Applying this to every integral we are left with

$$\begin{aligned}\mathfrak{L}\left[f(t)\right](s) &= \left(1+e^{-ps}+e^{-2ps}+\ldots+e^{-nps}+\ldots\right)\int_0^p f(x)e^{-sx}dx \\ &= \frac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}dt\end{aligned}$$

EXAMPLE 2.8

Compute the Laplace transform of $f(t)$ defined piecewise in terms of $n = 1,\ 2,\ \ldots$.

$$f(t) = \begin{cases} 1, & \text{if} \quad n+0 \leq t < n+1 \\ 0, & \text{if} \quad n+1 \leq t < n+2 \end{cases}$$

This is a periodic function with period $p = 2$ and therefore

$$\begin{aligned}
\mathfrak{L}\left[f(t)\right](s) &= \frac{1}{1-e^{-2s}} \int_0^2 f(t) e^{-st} dt \\
&= \frac{1}{1-e^{-2s}} \int_0^1 e^{-st} dt \\
&= \frac{1}{1-e^{-2s}} \left.\frac{e^{-st}}{-s}\right|_0^1 \\
&= \frac{1-e^{-s}}{s(1-e^{-2s})} \\
&= \frac{1}{s(1+e^{-s})}
\end{aligned}$$

2.2.1 Laplace Transform of the Dirac δ function

$$\mathfrak{L}\left[\delta(t-t')\right](s) = e^{-st'}, \qquad t', s \geq 0$$

The proof is trivial, but let us just note as a particular example that

$$\mathfrak{L}\left[\delta(t)\right](s) = 1$$

EXAMPLE 2.9

Newton's second law applied to an impulse at time $t = 0$ reads

$$m\ddot{x} = P\delta(t)$$

and we consider the set of initial conditions with a velocity $\dot{x}(0) = v_0$ and position $x(0) = x_0$. This equation can be easily solved by applying the Laplace transform on both sides of the equation and making use of its properties.

$$\begin{aligned}
\mathfrak{L}\left[m\ddot{x}\right] &= \mathfrak{L}\left[P\delta(t)\right] \\
m\left(s^2 \mathfrak{L}\left[x\right] - \dot{x}(0) - sx(0)\right) &= P \\
\mathfrak{L}\left[x\right] &= \left(\frac{P}{m} + v_0\right)\frac{1}{s^2} + \frac{x_0}{s}
\end{aligned}$$

Although we have not yet defined the inverse Laplace transform, we easily identify the transforms on the right-hand side of the equation above, and thus can come to the solution

$$x(t) = \left(\frac{P}{m} + v_0\right) t + x_0$$

Notice that the interpretation of this result is that the impulse is transmitted instantaneously and leads to a change of the velocity $v_0 \to \frac{P}{m} + v_0$.

2.2.2 Convolution Theorem

We define a new type of convolution

$$(f \times g) \quad = \quad \int_0^t f(x)g(t-x)dx$$

It can be proved that this definition is associative, commutative and has $\delta(t)$ as neutral element.

Theorem 2.2 *Convolution theorem for the Laplace transform*

$$\mathfrak{L}\left[f \times g\right](s) \quad = \quad \mathfrak{L}\left[f\right](s)\mathfrak{L}\left[g\right](s)$$

Proof:

$$\begin{aligned}
\mathfrak{L}\left[f \times g\right](s) &= \int_0^\infty \left(\int_0^t f(x)g(t-x)dx\right) e^{-ts}dt \\
&= \int_0^\infty dx \left(\int_x^\infty f(x)g(t-x)e^{-ts}dt\right)
\end{aligned}$$

We can now identify $y = t - x$ such that $dy = dx$ and

$$\begin{aligned}
&= \int_0^\infty dx \left(\int_0^\infty f(x)g(y)e^{-(y+x)s}dy\right) \\
&= \left(\int_0^\infty f(x)e^{-xs}dx\right)\left(\int_0^\infty g(y)e^{-ys}dy\right) \\
&= \mathfrak{L}\left[f\right](s)\mathfrak{L}\left[g\right](s)
\end{aligned}$$

EXAMPLE 2.10

Compute the Laplace transform of the following integral

$$I(x) \quad = \quad \int_0^x \cos(b(x-u))e^{au}du$$

We can do the following definitions, $g(x-u) = \cos(b-(x-u))$ and $f(u) = e^{au}$ and then apply the convolution theorem

$$\begin{aligned}
\mathfrak{L}\left[I(x)\right] &= \mathfrak{L}\left[\int_0^x \cos(b(x-u))e^{au}du\right] \\
&= \mathfrak{L}\left[\int_0^x g(x-u)f(u)du\right] \\
&= \mathfrak{L}\left[g\right](s)\mathfrak{L}\left[f\right](s) \\
&= \frac{s}{s^2+b^2}\,\frac{1}{s-a}
\end{aligned}$$

PROBLEMS

2.1 Prove the following relation

$$\mathfrak{L}[\Theta(t-a)\, f(t-a)](s) = e^{-as}\, \mathfrak{L}[f(t)](s)\,,$$

where f is some arbitrary function whose Laplace transform exists. Then apply this rule to evaluate the Laplace transform of

$$f(t) = \Theta(t-a)\, \sin t\,.$$

Letting $u = t - a$, we get

$$\begin{aligned}\mathfrak{L}[\Theta(t-a)\, f(t-a)](s) &= \int_0^\infty dt\, \Theta(t-a)\, f(t-a)\, e^{-st} = \int_a^\infty dt\, f(t-a)\, e^{-st} \\ &= \int_0^\infty du\, f(u)\, e^{-s(u+a)} = e^{-as}\, \mathfrak{L}[f(t)](s)\,.\end{aligned}$$

Let $g(t) = \sin(t+a)$. Then $f(t) = \Theta(t-a)\, g(t-a)$. Hence

$$\mathfrak{L}[f(t)](s) = e^{-as}\, \mathfrak{L}[\sin(t+a)](s) = e^{-as}\, \cos a\, \mathfrak{L}[\sin t](s) + e^{-as}\, \sin a\, \mathfrak{L}[\cos t](s)\,.$$

The Laplace transforms of $\sin t$ and $\cos t$ have been computed in the previous Problem sheet. We have

$$\mathfrak{L}[\sin t](s) = \frac{1}{1+s^2} \quad \text{and} \quad \mathfrak{L}[\cos t](s) = \frac{s}{1+s^2}\,,$$

and so

$$\mathfrak{L}[f(t)](s) = \frac{e^{-as}}{1+s^2}\, (s\, \sin a + \cos a)\,.$$

2.2 Show that $\mathfrak{L}[f(t)](s) = \frac{\log(1+s)}{s}$ where $s > 0$ and $f(t) = \int_1^\infty dx\, \frac{e^{-xt}}{x}$.

$$\mathfrak{L}[f(t)](s) = \int_1^\infty \frac{dx}{x}\, \mathfrak{L}[e^{-xt}](s) = \int_1^\infty \frac{dx}{x\,(x+s)} = \frac{1}{s}\left[\log \frac{x}{x+s}\right]_1^\infty = \frac{\log(1+s)}{s}\,.$$

2.3 Evaluate the function $f(t) = e^{at} * e^{bt}$ where $a \neq b$ by applying the Laplace transform to f. Then check your result calculating the convolution explicitly.

$$\begin{aligned}\mathfrak{L}[f(t)](s) &= \mathfrak{L}[e^{as}](s)\, \mathfrak{L}[e^{bs}](s) = \frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] \\ &= \mathfrak{L}\left[\frac{1}{a-b}\left(e^{at} - e^{bt}\right)\right](s)\,.\end{aligned}$$

Hence

$$f(t) = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)\,.$$

By direct computation

$$\begin{aligned} f(t) &= \int_0^t du\, e^{au}\, e^{b(t-u)} = e^{bt} \int_0^t du\, e^{(a-b)u} = \frac{e^{bt}}{a-b} \left[e^{(a-b)u}\right]_0^t \\ &= \frac{1}{a-b}\left(e^{at} - e^{bt}\right) . \end{aligned}$$

2.4 Assuming that the order of integration can be reversed in the computation, prove that

$$\mathfrak{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty du\, \mathfrak{L}[f](u) .$$

Then use this formula to calculate the Laplace transform of $f(t) = \frac{\sin at}{at}$.

$$\mathfrak{L}\left[\frac{f(t)}{t}\right](s) = \int_0^\infty dt\, \frac{f(t)}{t}\, e^{-st} = \int_0^\infty dt\, f(t) \int_s^\infty du\; e^{-ut} = \int_s^\infty du\, \mathfrak{L}[f](u) .$$

$$\begin{aligned} \mathfrak{L}[f(t)](s) &= \frac{1}{a}\int_s^\infty du\, \mathfrak{L}[\sin at](u) = \frac{1}{a}\int_s^\infty du\, \frac{a}{a^2+u^2} \\ &= \frac{1}{a}\left[\arctan\frac{u}{a}\right]_s^\infty = \frac{\pi}{2a} - \frac{\arctan\frac{s}{a}}{a} . \end{aligned}$$

2.5 Find the Laplace transform of $f(t) = \sin^2 t$ using the formula of the Laplace transform for a periodic function.

$f(t)$ is periodic with period π.

$$\begin{aligned} \mathfrak{L}[f(t)](s) &= \tfrac{1}{1-e^{-\pi s}} \int_0^\pi dt\, e^{-st}\, \sin^2 t = \tfrac{1}{2(1-e^{-\pi s})} \int_0^\pi dt\, e^{-st}\,(1-\cos 2t) \\ &= \tfrac{1}{2s} - \tfrac{1}{2(1-e^{-\pi s})} \int_0^\pi dt\, e^{-st}\, \cos 2t . \end{aligned}$$

The integral gives

$$\begin{aligned} \int_0^\pi dt\, e^{-st}\, \cos 2t &= \frac{1}{2}\left[\int_0^\pi dt\, e^{(2i-s)t} + \int_0^\pi dt\, e^{-(2i+s)t}\right] \\ &= \frac{1}{2(2i-s)}\left(e^{-\pi s}-1\right) - \frac{1}{2(2i+s)}\left(e^{-\pi s}-1\right) \\ &= \frac{1}{2}\left(1-e^{-\pi s}\right)\frac{2s}{s^2+4} \\ &= \frac{s\left(1-e^{-\pi s}\right)}{s^2+4} . \end{aligned}$$

Hence

$$\mathfrak{L}[f(t)](s) = \frac{2}{s\,(s^2+4)} .$$

2.6 The error function is defined as

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} du\, e^{-u^2} .$$

Compute the Laplace transform of $\mathrm{erf}(a/(2\sqrt{t}))$. Be careful in swapping the order of integration.

$$\begin{aligned} \mathfrak{L}\left[\mathrm{erf}\left(\frac{a}{2\sqrt{t}}\right)\right](s) &= \tfrac{2}{\sqrt{\pi}} \int_0^\infty dt \int_{a/(2\sqrt{t})}^\infty du\, e^{-u^2-st} \\ &= \tfrac{2}{\sqrt{\pi}} \int_0^\infty du \int_{a^2/(4u^2)}^\infty dt\, e^{-u^2-st} \\ &= \tfrac{2}{s\sqrt{\pi}} \int_0^\infty du\, e^{-\left(u^2+\frac{a^2 s}{4u^2}\right)} \\ &= \tfrac{1}{s\sqrt{\pi}} \int_{-\infty}^\infty du\, e^{-\left(u^2+\frac{a^2 s}{4u^2}\right)} . \end{aligned}$$

We have

$$u^2 + \frac{a^2 s}{4u^2} = \left(u - \frac{a\sqrt{s}}{2u}\right)^2 + a\sqrt{s} .$$

Letting $v = u - \frac{a\sqrt{s}}{2u}$, we get

$$du = \frac{1}{2} dv + \frac{v\, dv}{2\sqrt{v^2 + 2a\sqrt{s}}} ,$$

and so

$$\mathfrak{L}\left[\mathrm{erf}\left(\frac{a}{2\sqrt{t}}\right)\right](s) = \frac{e^{-a\sqrt{s}}}{2s\sqrt{\pi}} \left[\int_{-\infty}^\infty dv\, e^{-v^2} + \int_{-\infty}^\infty dv\, \frac{v\, e^{-v^2}}{\sqrt{v^2+2a\sqrt{s}}}\right] .$$

The second integral vanishes, because the integrand is odd. So,

$$\mathfrak{L}\left[\mathrm{erf}\left(\frac{a}{2\sqrt{t}}\right)\right](s) = \frac{e^{-a\sqrt{s}}}{2s} .$$

2.3 Solving differential equations with Laplace transform

Laplace transform are very useful to solve differential equations. As we already saw in expression () the Laplace transform of the differential of a function can be easily related to the Laplace transform of the function itself. We reproduce here this expression for convenience

$$\mathfrak{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^n\mathfrak{L}\left[f(t)\right](s) - \sum_{k=0}^{n-1} s^{n-1-k}f^{(k)}(0) \qquad (2.20)$$

This property allows us to convert a differential equation into a polynomial one, which is generally easier to solve.

EXAMPLE 2.11

Consider the following differential equation with constant coefficients

$$\begin{aligned} y'' + 3y' + 2y &= 2 - 2\Theta(t-1) \\ y(0) = 0 & \\ y'(0) = 2 & \end{aligned}$$

Applying the Laplace transform to each of the elements in the above equation

$$\begin{aligned} \mathfrak{L}\left[y''\right] &= s^2\mathfrak{L}\left[y\right] - sy(0) - Y'(0) \\ &= s^2\mathfrak{L}\left[y\right] - 2 \\ \mathfrak{L}\left[y'\right] &= s\mathfrak{L}\left[y\right] - y(0) \\ &= s\mathfrak{L}\left[y\right] \\ \mathfrak{L}\left[2\right] &= \frac{2}{s} \\ \mathfrak{L}\left[\Theta(t-1)\right] &= e^{-s}\mathfrak{L}\left[1\right] \\ &= \frac{e^{-s}}{s} \end{aligned}$$

it can be converted into a polynomial equation

$$\begin{aligned} s^2\mathfrak{L}\left[y\right] - 2 + 3s\mathfrak{L}\left[y\right] + 2\mathfrak{L}\left[y\right] &= \frac{2}{s} - \frac{2e^{-s}}{s} \\ \mathfrak{L}\left[y\right]\left(s^2+3s+2\right) &= 2\left(1 + \frac{1}{s} - \frac{e^{-s}}{s}\right) \end{aligned}$$

Then

$$\begin{aligned} \mathfrak{L}\left[y\right] &= \frac{2}{(s+1)(s+2)}\left(\frac{1+s}{s} - \frac{e^{-s}}{s}\right) \\ &= \frac{2}{s(s+2)} - \frac{2e^{-s}}{s(s+1)(s+2)} \\ &= \frac{1}{s} - \frac{1}{(s+2)} - e^{-s}\left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right) \end{aligned}$$

Notice that all these transforms can be readily identified from previous examples in these notes, so that the inverse of the Laplace transform can be done by mere inspection, yielding

$$y(t) = 1 - e^{-2t} - \Theta(x-1)\left(1 - 2e^{-(t-1)} + e^{-2(t-1)}\right)$$

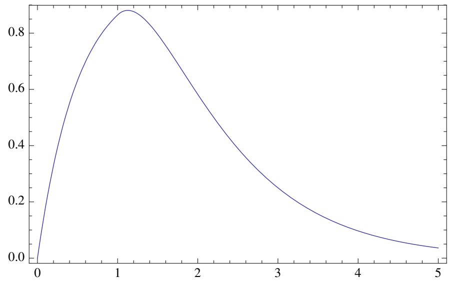


Figure 2.1 Resulting $y(t)$ in Example 2.11

In the same way, systems of coupled ordinary differential equations can also be transformed into a linear system

EXAMPLE 2.12

Consider the following set of coupled ODEs describing the charges in an electrical circuit

$$\begin{cases} L\ddot{q}_1 + M\ddot{q}_2 + \frac{1}{C}q_1 = 0 \\ M\ddot{q}_1 + L\ddot{q}_2 + \frac{1}{C}q_2 = 0 \end{cases}$$
$$q_1(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0$$
$$q_2(0) = V_0 C$$

This can be transformed using

$$\begin{aligned} \mathfrak{L}[\ddot{q}_1] &= s^2\mathfrak{L}[q_1] - \dot{q}_1(0) - sq_1(0) = s^2\mathfrak{L}[q_1] \\ \mathfrak{L}[\ddot{q}_2] &= s^2\mathfrak{L}[q_2] - \dot{q}_2(0) - sq_2(0) = s^2\mathfrak{L}[q_2] - sV_0C \end{aligned}$$

into

$$\begin{cases} \left(Ls^2 + \frac{1}{C}\right)\mathfrak{L}[q_1] + Ms^2\mathfrak{L}[q_2] = sMV_0C \\ \left(Ms^2\mathfrak{L}[q_1] + Ls^2 + \frac{1}{C}\right)\mathfrak{L}[q_2] = sLV_0C \end{cases}$$

From where we can simplify for $\mathfrak{L}[q_1]$ and $\mathfrak{L}[q_2]$ as follows

$$\begin{aligned} \mathfrak{L}[q_1] &= \frac{MV_0 s}{\left[(L+M)s^2 + \frac{1}{C^2}\right]\left[(L-M)s^2 + \frac{1}{C^2}\right]} \\ &= \frac{V_0 C}{2}\left[\frac{(L+M)s}{(L+M)s^2 + \frac{1}{C^2}} - \frac{(L-M)s}{(L-M)s^2 + \frac{1}{C^2}}\right] \end{aligned}$$

Given that $\mathfrak{L}[\cos bt] = s/(s^2+b^2)$ we can readily identify the inverse transform and write

$$q_1(t) = \frac{V_0 C}{2}\left(\cos\frac{t}{\sqrt{C(L+M)}} - \cos\frac{t}{\sqrt{C(L-M)}}\right)$$

and a similar solution for $q_2(t)$.

If the coefficients in the differential equations are not constant then the resulting equation is not directly a polynomial one, but often is simplified. Also, if we manage to find a solution to the Laplace transform in terms of a series, then we can apply the following theorem to invert it.

Theorem 2.3 *Suppose $\mathfrak{L}[f](s)$ admits a series expansion of the form*

$$\mathfrak{L}[f](s) = \sum_{n=0}^{\infty} a_n s^{-n-1} \tag{2.21}$$

and the series in the right-hand side converges $|s| > \rho$. Then, for $t > 0$,

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \tag{2.22}$$

EXAMPLE 2.13

Consider Bessel's equation of order 0

$$\begin{aligned} ty'' + y' + ty &= 0 \\ y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

Let us denote $\bar{y} \equiv \mathfrak{L}[y]$. Using the properties of the Laplace transform we arrive to

$$\begin{aligned} \mathfrak{L}[ty] &= -\frac{d}{ds}\bar{y} \\ \mathfrak{L}[y'] &= s\bar{y} - y(0) = s\bar{y} - 1 \\ \mathfrak{L}[ty''] &= -\frac{d}{ds}\mathfrak{L}[y''] \\ &= -\frac{d}{ds}\left(s^2\bar{y} - sy(0) - y'(0)\right) \\ &= -\left(2s\bar{y} + s^2\frac{d}{ds}\bar{y} - 1\right) \end{aligned}$$

which implies that the differential equation can be transformed into

$$\begin{aligned} -\left(2s\bar{y} + s^2\frac{d}{ds}\bar{y} - 1\right) + s\bar{y} - 1 - \frac{d}{ds}\bar{y} &= 0 \\ \frac{d}{ds}\bar{y}(s^2+1) + s\bar{y} &= 0 \\ \frac{d}{ds}\bar{y} &= \frac{-s}{s^2+1}\bar{y} \\ \frac{d\bar{y}}{\bar{y}} &= \frac{-sds}{s^2+1} \end{aligned}$$

This can be easily solved

$$\begin{aligned}\log \bar{y} &= C - \int \frac{sds}{s^2+1} \\ \log \bar{y} &= C - \frac{1}{2}\log(s^2+1)\end{aligned}$$

From where

$$\bar{y} = \frac{C}{\sqrt{s^2+1}}$$

The question is now to invert this relation $y = \mathfrak{L}^{-1}[\bar{y}]$. One easy way to do it is to Taylor expand the expression above around $s = \infty$ and use the known expressions as follows

$$\begin{aligned}\bar{y} &= \frac{C}{s}\left(1+\frac{1}{s^2}\right)^{-1/2} \\ &= \frac{C}{s}\left(1-\frac{1}{2s^2}+\frac{3}{8s^4}+\dots\right) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{(n!)^2 2^{2n}}\frac{1}{s^{2n+1}}\end{aligned}$$

Since this is a series expansion for the Laplace transform, we can readily identify its inverse as

$$y(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{t}{2}\right)^{2n} = J_0(t)$$

Where $J_0(t)$ is the Bessel function of the first kind with $\alpha = 0$.

PROBLEMS

2.1 Find the electrical current in a circuit with a capacitor (capacitance C), inductor (inductance L) and resistor (resistance R), connected in series if a potential difference v is applied for times $t > 0$ given that the equation for the circuit current $i(t)$ can be obtained from

$$L\,\frac{di}{dt} + R\,i + \frac{1}{C}\left(q_0 + \int_0^t dt'\,i(t')\right) = v\,.$$

The initial condition is $i(0) = 0$, because the potential difference is only applied for $t > 0$, so there is no current initially. Laplace transforming, we get, with $\bar{\imath}(s) = \mathfrak{L}[i](s)$,

$$L\,s\,\bar{\imath} + R\,\bar{\imath} + \frac{1}{C}\left(\frac{q_0}{s} + \frac{\bar{\imath}}{s}\right) = \frac{v}{s} \quad \Rightarrow \quad \bar{\imath}(s) = \frac{v - q_0/C}{L((s+a)^2 + b^2)}\,,$$

with

$$a = \frac{R}{2L} \text{ and } b = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,.$$

Transforming back, we find

$$i(t) = \frac{v - q_0/C}{b\,L}\,e^{-at}\,\sin bt\,.$$

2.2 At time $t > 0$ an antenna produces an EM wave with electric field $E(x,t)$ in the positive direction. It is described by

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{v^2}\,\frac{\partial^2 E(x,t)}{\partial t^2} = 0\,.$$

Use Laplace transforms to find $E(x,t)$ for $t > 0$ and $x > 0$ given that $E(x,0) = 0$, $\frac{\partial E}{\partial t}_{|t=0} = 0$ and $E(0,t) = \phi(t)$. [Hint: Laplace transform with respect to the time variable.]

Laplace transforming with respect to t, we find

$$\frac{\partial^2 \overline{E}(x,s)}{\partial x^2} - \frac{1}{v^2}\left[s^2\,\overline{E}(x,s) - s\,E(x,0) - \frac{\partial E}{\partial t}_{|t=0}\right] = 0$$
$$\Downarrow$$
$$\frac{\partial^2 \overline{E}(x,s)}{\partial x^2} = \frac{s^2}{v^2}\,\overline{E}(x,s)\,.$$

In other words, we obtain an ODE (in fact, a harmonic oscillator) in Laplace space:

$$\overline{E}(x,s) = A(s)\,e^{sx/v} + B(s)\,e^{-sx/v}\,.$$

The EM wave propagates in the positive x direction, and so if we want to have a finite solution in this direction, we need to chose $A(s) = 0$. At $x = 0$, we have $B(s) = \overline{\phi}(s)$. So,

$$\overline{E}(x,s) = \overline{\phi}(s)\,e^{-sx/v}\,.$$

Using the convolution theorem, we can obtain the solution in (x,t)-space, for $t>0$,

$$\begin{aligned} E(x,t) = \phi(t) * \mathfrak{L}^{-1}[e^{-sx/v}](t) &= \phi(t) * \delta\left(t - \frac{x}{v}\right) \\ &= \int_0^t dt'\, \phi(t-t')\, \delta\left(t' - \frac{x}{v}\right) \\ &= \phi\left(t - \frac{x}{v}\right). \end{aligned}$$

2.3 Obtain the functions $x(t)$ and $y(t)$ using Laplace transforms when

$$\begin{cases} x' + x + 4y &= 10\,, \\ -y' + x - y &= 0\,, \end{cases}$$

with initial conditions $x(0) = 4$ and $y(0) = 3$.

Laplace transforming the system, we get

$$\begin{cases} s\,\bar{x} - x(0) + \bar{x} + 4\bar{y} &= \frac{10}{s}\,, \\ -s\,\bar{y} + y(0) + \bar{x} - \bar{y} &= 0\,, \end{cases} \quad \Rightarrow \quad \begin{cases} (1+s)\,\bar{x} + 4\bar{y} &= 4 + \frac{10}{s}\,, \\ \bar{x} - (1+s)\,\bar{y} &= -3\,. \end{cases}$$

We obtain an ordinary linear system in Laplace space. The solution is

$$\bar{x}(s) = \frac{2\left(2s^2 + s + 5\right)}{s\left(s^2 + 2s + 5\right)} \quad \text{and} \quad \bar{y}(s) = \frac{3s^2 + 7s + 10}{s\left(s^2 + 2s + 5\right)}\,.$$

Transforming back:

$$\begin{aligned} x(t) &= 2e^{-t}(\cos 2t - \sin 2t) + 2\,, \\ y(t) &= e^{-t}(\sin 2t + \cos 2t) + 2\,. \end{aligned}$$

2.4 Use Laplace transforms to find the solution of the differential equation

$$y''(x) + 3y'(x) + 2y(x) = g(x)\,,$$

where

$$g(x) = \begin{cases} 1\,, & \text{if } 0 \le x < 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

and the boundary conditions are $y(0) = 0$, $y'(0) = 1$.

Laplace transforming gives

$$\begin{gathered} s^2\,\bar{y} - s\,y(0) - y'(0) + 3(s\,\bar{y} - y(0)) + 2\bar{y} = (s+2)\,(s+1)\,\bar{y} - 1 = \bar{g}(s) \\ \Downarrow \\ \bar{y}(s) = \tfrac{1+\bar{g}(s)}{(s+1)\,(s+2)}\,. \end{gathered}$$

The Laplace transform of the inhomogeneous part is

$$\bar{g}(s) = \mathfrak{L}[\theta(x)](s) - \mathfrak{L}[\theta(x-1)](s) = \frac{1 - e^{-s}}{s}\,.$$

Hence,

$$\bar{y}(s) = \frac{1 + s - e^{-s}}{s\,(s+1)\,(s+2)} = \frac{1}{s\,(s+2)} - \frac{e^{-s}}{s\,(s+1)\,(s+2)}\,.$$

We have

$$\frac{1}{s\,(s+2)} = \frac{1}{2s} - \frac{1}{2(s+2)} = \mathfrak{L}\left[\frac{1-e^{-2x}}{2}\right](s)\,,$$

$$\frac{1}{s\,(s+1)\,(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)} = \mathfrak{L}\left[\frac{1-2e^{-x}+e^{-2x}}{2}\right](s)\,.$$

Recalling that $\mathfrak{L}[\theta(x-1)\,f(x-1)](s) = e^{-s}\,\mathfrak{L}[f(x)](s)$, we have

$$\begin{aligned}\frac{e^{-s}}{s\,(s+1)\,(s+2)} &= e^{-s}\,\mathfrak{L}\left[\frac{1-2e^{-x}+e^{-2x}}{2}\right](s)\\ &= \mathfrak{L}\left[\frac{1}{2}\,\theta(x-1)\,\left(1-2e^{-(x-1)}+e^{-2(x-1)}\right)\right](s)\,.\end{aligned}$$

Hence

$$y(x) = \frac{1}{2}\left[1 - e^{-2x} + \theta(x-1)\,\left(1-2e^{-(x-1)}+e^{-2(x-1)}\right)\right]\,.$$

2.4 Inverse Laplace transform

2.4.1 By inspection (or inspiration)

The Laplace transform often gives rise to rational functions whose inverse is easy to identify. It is useful to remember some of the basic transforms

$$\begin{aligned}
\mathfrak{L}[1] &= \frac{1}{s} \\
\mathfrak{L}[t^n] &= \frac{n!}{s^{n+1}} \\
\mathfrak{L}\left[e^{at}\right] &= \frac{1}{s-a} \\
\mathfrak{L}[\cos t] &= \frac{s}{s^2+1} \\
\mathfrak{L}[\sin t] &= \frac{1}{s^2+1}
\end{aligned}$$

EXAMPLE 2.14

$$\begin{aligned}
\mathfrak{L}[f](s) &= \frac{4}{s(s+2)} \\
&= \frac{2}{s} - \frac{2}{s+2} \\
&= 2\mathfrak{L}[1](s) - 2\mathfrak{L}\left[e^{-2t}\right](s)
\end{aligned}$$

which leads to

$$\begin{aligned}
f(t) &= \mathfrak{L}^{-1}\left[\frac{4}{s(s+2)}\right](t) \\
&= 2 - 2e^{-2t}
\end{aligned}$$

2.4.2 By the Convolution theorem

If we have the product of several functions whose inverse Laplace transform is known, we can do the inverse transform factor by factor and then use the convolution theorem.

EXAMPLE 2.15

$$\mathfrak{L}[f](s) = \frac{4}{s^2(s+2)^2}$$

We know that

$$\begin{aligned} \frac{1}{s^2} &= \mathfrak{L}\left[t\right](s) \\ \frac{1}{(s+2)^2} &= -\frac{d}{ds}\frac{1}{s+2} \\ &= -\frac{d}{ds}\mathfrak{L}\left[e^{-2t}\right](s) \\ &= \mathfrak{L}\left[te^{-2t}\right](s) \end{aligned}$$

Using the convolution theorem, this leads to

$$\begin{aligned} \mathfrak{L}\left[f\right](s) &= 4\mathfrak{L}\left[t\right]\mathfrak{L}\left[te^{-2t}\right] \\ &= 4\mathfrak{L}\left[t \times (te^{-2t})\right] \end{aligned}$$

Then

$$\begin{aligned} f(x) &= 4t \times (te^{-2t}) \\ &= 4\int_0^t x(t-x)e^{-2(t-x)}dx \\ &= 4te^{-2t}\int_0^t xe^{2x}dx - 4e^{-2t}\int_0^t x^2e^{2x}dx \\ &= (1+t)e^{-2t} + t - 1 \end{aligned}$$

2.4.3 By the Inversion Theorem

Theorem 2.4 *Consider a function $f(t)$ that is piece smooth in $[0,\infty]$ and that its Laplace transform $\mathfrak{L}\left[f(t)\right](s)$ exists for $Re(s) > 0$. Then, for $t > 0$*

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathfrak{L}\left[f\right](s)e^{st}ds \qquad (2.23)$$

where the integral is independent of the choice of σ as long as $\sigma > 0$.

Proof: Consider $s = \sigma + ix$ such that

$$\mathfrak{L}\left[f\right](s) = \sqrt{2\pi}\,\mathfrak{F}^{-1}\left[f(t)\Theta(t)e^{-\sigma t}\right](x)$$

This equation can be inverted using the Fourier transform such that

$$\begin{aligned} f(t)\Theta(t)e^{-\sigma t} &= \frac{1}{\sqrt{2\pi}}\mathfrak{F}\left[\mathfrak{L}[f](s)\right](t) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathfrak{L}\left[f\right](\sigma+ix)\,e^{ixt}dx \end{aligned}$$

Thus

$$f(t)\Theta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathfrak{L}\left[f\right](\sigma+ix)\,e^{(\sigma+ix)t}dx$$

Changing variables to $s = \sigma + ix$ such that $ds = dx$ the integral limits change to $\sigma \pm i\infty$ and expression (2.23) is recovered.

This integral is also referred to as the Bromwich integral or the Fourier-Mellin integral and can be often computed using the residue theorem.

Theorem 2.5 *Let $\mathfrak{L}[f](s)$ be meromorphic*[1] *with a finite number of poles at $\{a_i\}$ with $i \in \{1, \ldots, n\}$, and suppose that there is a positive real number M such that*

$$|\mathfrak{L}[f](s)| \leq M|s|^{-k} \tag{2.24}$$

for large s and some integer k. Then, for $t > 0$ and $\sigma > \mathrm{Re}\, a_i$, $\forall i$ (i.e., all the poles lie to the left) we have

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathfrak{L}[f](s)e^{st}ds \\ &= \sum_{i=1}^{n} \mathrm{Res}\left(\mathfrak{L}[f](s)e^{ts}\right)_{s=a_i} \end{aligned} \tag{2.25}$$

Proof: *We choose the contour C, described by the straight line $\mathrm{Re}(s) = \sigma$ and the semi circle $C_R : s = \sigma + Re^{i\theta}$, with $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, as depicted in Fig. 2.2*

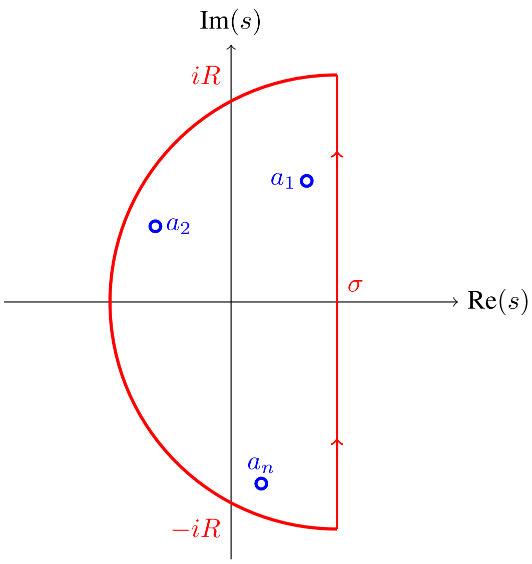


Figure 2.2 The roots $a_1, \ldots, a_n$ of g and the curve C_R.

[1]A meromorphic function can be expressed as the ratio of two holomorphic functions.

Given that $|s| = |\sigma + R\,e^{i\theta}| \geq \left||\sigma| - |R\,e^{i\theta}|\right| = |\sigma - R| = R - \sigma$*, (which holds for* $R > \sigma$ *as we are assuming the limit where* $R \to \infty$*), then along the semicircle we have*

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{C_R}\mathfrak{L}\left[f\right](s)e^{ts}ds\right| &= \left|\frac{1}{2\pi i}\int_{\pi/2}^{3\pi/2}\mathfrak{L}\left[f\right](\sigma + Re^{i\theta})e^{t(\sigma+re^{i\theta})}iRe^{i\theta}d\theta\right| \\ &\leq \frac{1}{2\pi i}\int_{\pi/2}^{3\pi/2}\left|\mathfrak{L}\left[f\right](\sigma + Re^{i\theta})\right|\left|e^{t(\sigma+re^{i\theta})}iRe^{i\theta}\right|d\theta\end{aligned}$$

Now, considering that $|s| \geq R - \sigma \to \frac{1}{|s|} \leq \frac{1}{R-\sigma}$*, we have*

$$\begin{aligned}\left|\mathfrak{L}\left[f\right](s + Re^{i\theta})\right| &\leq M\left|\sigma + Re^{i\theta}\right|^{-k} \\ &\leq M(R-\sigma)^{-k}\end{aligned}$$

Also,

$$\left|e^{t(\sigma+Re^{i\theta})}\right| = \left|e^{t(\sigma+R\cos\theta)}e^{itR\sin\theta}\right| = e^{t(\sigma+R\cos\theta)}$$

Then,

$$\left|\frac{1}{2\pi i}\int_{C_R}\mathfrak{L}\left[f\right](s)e^{ts}ds\right| \leq \frac{MR}{2\pi}\int_{\pi/2}^{3\pi/2}(R-\sigma)^{-k}e^{t(\sigma+R\cos\theta)}d\theta$$

Given that $\cos\theta < 0$ *for* $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$*, then the integrand vanishes in the limit* $R \to \infty$*, which implies that the integral along* C_R *vanishes in that limit, too. We are thus left with the line integral and the result follows from the application of the residue theorem to the whole integral on* C*.*

It is crucial that all the poles are on the left and that we close the contour on the left as well (so as the contribution of the semi circle C_R to the integral vanishes). If there is a pole on the right then the function $\mathfrak{L}\left[f\right](s)$ is not defined everywhere to the right and the inversion theorem does not apply.

EXAMPLE 2.16

$$\mathfrak{L}\left[f\right](s) = \frac{4}{s^2(s+2)^2}$$

We have poles in $s = 0$ and $s = -2$. The residues are easily computed

$$\begin{aligned}\text{Res}\left(\frac{4\,e^{ts}}{s^2(s+2)^2}\right)_{s=0} &= \frac{d}{ds}\frac{4\,e^{ts}}{(s+2)^2}\bigg|_{s=0} = t-1 \\ \text{Res}\left(\frac{4\,e^{ts}}{s^2(s+2)^2}\right)_{s=-2} &= \frac{d}{ds}\frac{4\,e^{ts}}{s^2}\bigg|_{s=-2} = e^{-2t}(1+t)\end{aligned}$$

Then,

$$f(t) = t - 1 + e^{-2t}(1+t), \qquad t > 0$$

PROBLEMS

2.1 Given that $f(t) = \mathfrak{L}^{-1}[\bar{f}(s)](t)$, show that

$$\mathfrak{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right](t) = \int_0^t dt'\, f(t')\,.$$

[Hint: Use the convolution theorem]

$$\mathfrak{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right](t) = f(t) * \mathfrak{L}^{-1}\left[\frac{1}{s}\right](t) = f(t) * 1 = \int_0^t dt'\, f(t')\,.$$

2.2 Use the inversion theorem to find the function whose Laplace transform is $\bar{f}(s) = \frac{s}{s^2-k^2}$ where k is a real constant.

For large $|s|$, $|\bar{f}(s)|\ 1/|s|$, and so the inversion theorem applies:

$$f(t) = \mathrm{Res}_{s=k}\frac{s\,e^{st}}{s^2-k^2} + \mathrm{Res}_{s=-k}\frac{s\,e^{st}}{s^2-k^2} = \frac{1}{2}e^{kt} + \frac{1}{2}e^{-kt} = \cosh kt\,.$$

2.3 Compute

$$\mathfrak{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right](t)$$

(a) by ‘inspection’, (b) using the convolution theorem, (c) using the inversion theorem.

(a) We see that

$$\frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{d}{ds}\frac{1}{s^2+a^2}\,.$$

Furthermore, from

$$\mathfrak{L}[t\,f(t)](s) = -\frac{d}{ds}\,\mathfrak{L}[f](s)\,,$$

we get

$$\mathfrak{L}^{-1}\left[\frac{d\bar{f}}{ds}\right](t) = t\,\mathfrak{L}^{-1}[\bar{f}(s)](t)\,.$$

So,

$$\mathfrak{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right](t) = \frac{1}{2a}\,t\,\mathfrak{L}^{-1}\left[\frac{a}{s^2+a^2}\right](t) = \frac{t\,\sin at}{2a}\,.$$

(b)

$$\begin{aligned}
\mathfrak{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right](t) &= \mathfrak{L}^{-1}\left[\frac{s}{s^2+a^2}\right](t) * \mathfrak{L}^{-1}\left[\frac{1}{s^2+a^2}\right](t) \\
&= \cos at * \frac{\sin at}{a} = \frac{1}{a}\int_0^t dt'\;\cos at'\;\sin(a(t-t')) \\
&= \frac{1}{a}\left[\sin at\int_0^t dt'\;\cos^2 at' - \cos at\int_0^t dt'\;\cos at'\;\sin at'\right] \\
&= \frac{1}{a}\left[\frac{t}{2}\;\sin at + \frac{1}{2}\sin at\int_0^t dt'\;\cos 2at' - \frac{1}{2}\cos at\int_0^t dt'\;\sin 2at'\right]
\end{aligned}$$

$$\begin{aligned} &= \frac{t}{2a}\sin at + \frac{1}{4a^2}\sin at\,\sin 2at + \frac{1}{4a^2}\cos at(\cos 2at - 1) \\ &= \frac{t}{2a}\sin at + \frac{1}{2a^2}\sin^2 at\,\cos at - \frac{1}{2a^2}\cos at\sin^2 at \\ &= \frac{t}{2a}\sin at\,. \end{aligned}$$

(c) $|\frac{s}{(s^2+a^2)^2}| \sim 1/|s|^3$ for large $|s|$, so the inversion theorem applies for $t > 0$:

$$\begin{aligned} \mathfrak{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right](t) &= \mathrm{Res}_{s=ia}\frac{s\,e^{st}}{(s^2+a^2)^2} + \mathrm{Res}_{s=-ia}\frac{s\,e^{st}}{(s^2+a^2)^2} \\ &= -\frac{it}{4a}\,e^{iat} + \frac{it}{4a}\,e^{-iat} = \frac{t\,\sin at}{2a}\,. \end{aligned}$$

2.4 Find the function whose Laplace transform is

$$\bar{f}(s) = \frac{2s^2+3s-4}{(s-2)(s^2+2s+2)}$$

(a) by inspection and (b) using the inversion theorem.

(a) We have $s^2 + 2s + 2 = (s+1-i)(s+1+i)$. Partial fractioning, and using $\mathfrak{L}^{-1}[1/(s+a)](t) = e^{-at}$, we get

$$\begin{aligned} f(s) &= \mathfrak{L}^{-1}\left[\tfrac{2s^2+3s-4}{(s-2)(s^2+2s+2)}\right](t) \\ &= \mathfrak{L}^{-1}\left[\tfrac{1}{s-2}\right](t) + \left(\tfrac{1}{2}-i\right)\,\mathfrak{L}^{-1}\left[\tfrac{1}{s+1-i}\right](t) + \left(\tfrac{1}{2}+i\right)\,\mathfrak{L}^{-1}\left[\tfrac{1}{s+1+i}\right](t) \\ &= e^{2t} + \left(\tfrac{1}{2}-i\right)\,e^{-(1-i)t} + \left(\tfrac{1}{2}+i\right)\,e^{-(1+i)t} \\ &= e^{2t} + 2e^{-t}\sin t + e^{-t}\cos t\,. \end{aligned}$$

(b) $|\bar{f}(s)\;1/|s|^2$ for large $|s|$, so the inversion theorem applies for $t > 0$:

$$\begin{aligned} \bar{f}(s) &= \mathrm{Res}_{s=2}\frac{2s^2+3s-4}{(s-2)(s^2+2s+2)}e^{st} \\ &\quad +\mathrm{Res}_{s=-1-i}\frac{2s^2+3s-4}{(s-2)(s^2+2s+2)}e^{st} \\ &\quad +\mathrm{Res}_{s=-1+i}\frac{2s^2+3s-4}{(s-2)(s^2+2s+2)}e^{st} \end{aligned}$$

$$\begin{aligned} \mathrm{Res}_{s=2}\frac{2s^2+3s-4}{(s-2)(s^2+2s+2)}e^{st} &= e^{2t}\,, \\ \mathrm{Res}_{s=-1+i}\frac{2s^2+3s-4}{(s-2)(s^2+2s+2)}e^{st} &= \left(\frac{1}{2}-i\right)\,e^{-(1-i)t}\,, \\ \mathrm{Res}_{s=-1-i}\frac{2s^2+3s-4}{(s-2)(s^2+2s+2)}e^{st} &= \left(\frac{1}{2}+i\right)\,e^{-(1+i)t}\,. \end{aligned}$$

These are precisely the three terms we found in (a).

2.5 Calculate the inverse Laplace transform of

$$\bar{f}(s) = \frac{e^{-ks}}{s^2}\,, \qquad k > 0,$$

using the inversion theorem.

In order for the inversion theorem to apply, we need to have that $e^{(t-k)s}$ vanishes for large $|s|$, i.e., we need $t > k$:

$$f(t) = \text{Res}_{s=0} \frac{e^{(t-k)s}}{s^2} = t - k\,.$$

Since this expression in only valid for $t > k$, we have

$$f(t) = \theta(t-k)\,(t-k)\,.$$

REFERENCES

[1] Mathematical Methods for Physicists, G. B. Arken and H. J. Webber