Towards a quantitative understanding of the $\Delta I = 1/2$ rule

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Fermions and Extended Objects on the Lattice
Benasque, (25/02-02/03)/07
Outline

The (many difficulties of studying the) $\Delta I=1/2$ rule:
- Operator product expansion and long distance QCD effects.
- Problems with pions and chirality.
- Our strategy.

Computational setup:
- Low-energy description: $p$- vs $\epsilon$-regime.
- Chiral lattice fermions.

Computational details:
- Bare results and the approach to the chiral regime.
- Matching to physics: renormalisation and chiral fits.

Results and conclusions.
Outline

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  - Operator product expansion and long distance QCD effects.
  - Problems with pions and chirality.
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  - Low-energy description: $p$- vs $\epsilon$-regime.
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- Computational details.
  - Bare results and the approach to the chiral regime.
  - Matching to physics: renormalisation and chiral fits.

- Results and conclusions.
$K \rightarrow \pi \pi$ decays in a nutshell

- Hamiltonian for the dynamics of $K^0 - \bar{K}^0$ system determined by hermiticity+CPT:

$$H = M - \frac{i}{2} \Gamma = \begin{pmatrix} A & p^2 \\ q^2 & A \end{pmatrix}$$

- If CP is conserved the eigenstates of the Hamiltonian are $|K_{1,2}\rangle = \frac{1}{\sqrt{2}} (|K^0\rangle \pm |\bar{K}^0\rangle)$. 

  CP violation in the SM leads to mixing:

$$|K_S\rangle = \frac{1}{\sqrt{1 + |\bar{\epsilon}|^2}} (|K_1\rangle + \bar{\epsilon}|K_2\rangle) \quad |K_L\rangle = \frac{1}{\sqrt{1 + |\bar{\epsilon}|^2}} (|K_2\rangle + \bar{\epsilon}|K_1\rangle) \quad \bar{\epsilon} = \frac{p - q}{p + q}$$

- CP violation parameters accessible via decay amplitudes into two pions:

$$-iT[K^0 \rightarrow (\pi \pi)_I] = A_I e^{i\delta_I} \quad T[(\pi \pi)_I \rightarrow (\pi \pi)_I]_{l=0} = 2e^{i\delta_I} \sin \delta_I$$
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$$\epsilon = \frac{T[K_L \to (\pi\pi)_0]}{T[K_S \to (\pi\pi)_0]} \simeq \bar{\epsilon} + i \frac{\text{Im} A_0}{\text{Re} A_0}$$

$$\epsilon' = \frac{\epsilon}{\sqrt{2}} \left( \frac{T[K_L \to (\pi\pi)_2]}{T[K_S \to (\pi\pi)_0]} - \frac{T[K_S \to (\pi\pi)_2]}{T[K_L \to (\pi\pi)_0]} \right) \simeq \frac{1}{\sqrt{2}} e^{i(\delta_2 - \delta_0 + \pi/2)} \frac{\text{Re} A_2}{\text{Re} A_0} \left( \frac{\text{Im} A_2}{\text{Re} A_2} - \frac{\text{Im} A_0}{\text{Re} A_0} \right)$$
$K \to \pi\pi$ decays in a nutshell

Experiment:

\[
\left| \frac{A_0}{A_2} \right| \simeq 22.1
\]

\[
|\epsilon| = (2.282 \pm 0.017) \times 10^{-3}
\]

\[
\text{Re} \left( \frac{\epsilon'}{\epsilon} \right) = (16.7 \pm 2.3) \times 10^{-4}
\]
The $\Delta I = 1/2$ rule for kaon decays

$$T(K \to (\pi\pi)_x) = iA_\alpha e^{i\delta_\alpha}, \quad \alpha = 0, 2$$

$|A_0/A_2| = 22.1$

- Bulk of enhancement in the SM must come from long-distance strong interaction effects ... 
  
  Altarelli & Maiani, PLB 52 (1974) 351

- ... that have to be addressed non-perturbatively.
  
  Cabibbo, Martinelli & Petronzio, NPB 244 (1984) 381
  Brower, Maturana, Gavela & Gupta, PRL 53 (1984) 1318

- Lattice QCD studies hampered by no-go theorems on chiral fermions and multiparticle decays, almost no activity in the ‘90s.

- Theoretical breakthroughs in late ‘90s (mainly chiral lattice fermions) have led to a renewed interest and some “rough” lattice results.

  CP-PACS & RBC Collaborations

- Still far from having an understanding of the mechanism(s) behind the enhancement.
Effective Weak Hamiltonian

\[ \mathcal{A}(i \rightarrow f) \approx \langle f | H_{\text{eff}}^W | i \rangle \]

\[ H_{\text{eff}}^W = \frac{G_F}{\sqrt{2}} \sum_k f_k (V_{\text{CKM}}) C_k (\mu / M_W) \bar{O}_k (\mu) \]
Effective Weak Hamiltonian

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CKM parameters

Wilson coefficients — high energy, NLO computation

Composite operators — low energy (hadronic) scales
Effective Weak Hamiltonian

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\[ H_{\text{eff}}^W = \frac{G_F}{\sqrt{2}} \sum_k f_k (V_{\text{CKM}}) C_k (\mu / M_W) \tilde{O}_k (\mu) \]

With an active charm quark (CP-violating effects neglected):

\[ H_w = \frac{g_w^2}{2M_w^2} (V_{us})^* (V_{ud}) \sum_{\sigma = \pm} \{ k_1^\sigma Q_1^\sigma + k_2^\sigma Q_2 \} \]

\[ Q_1^\pm = (\bar{s} \gamma_\mu P_u) (\bar{u} \gamma_\mu P_d) \pm (\bar{s} \gamma_\mu P_d) (\bar{u} \gamma_\mu P_u) - [u \rightarrow c] \]

\[ Q_2^\pm = (m_u^2 - m_c^2) \{ m_d (\bar{s} P_d) + m_s (\bar{s} P_d) \} \]

\( Q_1^\pm \) transform according to irreps of d=84 (+) and d=20 (-) of SU(4). \( Q_2^\pm \) do not contribute to the physical \( K \rightarrow \pi \pi \) transition.
Effective Weak Hamiltonian

\[ H_w = \frac{g_w^2}{2M_W^2} (V_{us})^* (V_{ud}) \sum_{\sigma=\pm} \{ k_1^{\sigma} Q_1^{\sigma} + k_2^{\sigma} Q_2^{\sigma} \} \]

\[ Q_1^{\pm} = (\bar{s}\gamma_\mu P_u)(\bar{u}\gamma_\mu P_d)^{\pm}(\bar{s}\gamma_\mu P_d)(\bar{u}\gamma_\mu P_u) - [u \rightarrow c] \]

\[ Q_2^{\pm} = (m_u^2 - m_c^2) \{ m_d(\bar{s}P_d) + m_s(\bar{s}P_d) \} \]

\[ \left| \frac{A_0}{A_2} \right| = \frac{k_1^- (M_W)}{k_1^+ (M_W)} \frac{\langle (\pi\pi)_{I=0} | \hat{Q}_1^- | K \rangle}{\langle (\pi\pi)_{I=2} | \hat{Q}_1^+ | K \rangle} \]

\[ \frac{k_1^- (M_W)}{k_1^+ (M_W)} = 2.8 \sim O(1) \]

Enhancement dominated by matrix elements of effective interaction vertices (long-distance regime of the strong interaction).
A flagrant failure of large $N_c$

$$H_w^{\Delta S=1} \sim G_F J_w^\mu J_w^\mu$$

$$O(N_C^2) \quad O(N_C) \quad O(1)$$

$$T(K^0 \to \pi^0 \pi^0) \sim 0 \Rightarrow \frac{A_0}{A_2} \bigg|_{N \to \infty} \sim \sqrt{2}$$

Fukugita et al. 1977; Chivukula, Flynn, Georgi 1986
Effective Weak Hamiltonian

\[ H_w = \frac{g^2}{2M_W^2}(V_{us})^*(V_{ud}) \sum_{\sigma=\pm} \{k_1^{\sigma} Q_1^{\sigma} + k_2^{\sigma} Q_2^{\sigma}\} \]

\[ Q_1^{\pm} = (\bar{s}\gamma_\mu P_u)(\bar{d}\gamma_\mu P_d) \pm (\bar{s}\gamma_\mu P_d)(\bar{u}\gamma_\mu P_u) - [u \rightarrow c] \]

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\[ \left| \frac{A_0}{A_2} \right| = \frac{k_1^{-}(M_W)}{k_1^{+}(M_W)} \frac{\langle (\pi\pi)_{I=0} | \hat{Q}_1^{-} | K \rangle}{\langle (\pi\pi)_{I=2} | \hat{Q}_1^{+} | K \rangle} \quad \frac{k_1^{-}(M_W)}{k_1^{+}(M_W)} = 2.8 \sim O(1) \]

Enhancement dominated by matrix elements of effective interaction vertices (long-distance regime of the strong interaction).

Well, let’s compute the matrix elements ...
The realm of no-go theorems

Maiani-Testa theorem: physical $A(i \rightarrow f_1 \ldots f_n)$ cannot be extracted from Euclidean lattice amplitudes in the infinite volume limit.

Maiani & Testa, PLB 245 (1990) 585
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Maiani & Testa, PLB 245 (1990) 585

Physical matrix elements can still be extracted by matching QCD at infinite and finite volume — but the volumes required are prohibitively large.

Lellouch & Lüscher, CMP 219 (2001) 31
Lin, Martinelli, Sachrajda & Testa, NPB 619 (2001) 467
The realm of no-go theorems

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Use chiral (low-energy) expansion to relate the physical $K \rightarrow \pi\pi$ amplitudes to computable quantities.

Bernard et al., PRD 32 (1985) 2343
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Nielsen-Ninomiya theorem: no ultralocal lattice regularisation of QCD preserves chiral symmetry.

Nielsen & Ninomiya, NPB 185 (1981) 20
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**Nielsen-Ninomiya theorem:** no ultralocal lattice regularisation of QCD preserves chiral symmetry.

Nielsen & Ninomiya, NPB 185 (1981) 20

Absence of chiral symmetry induces operator mixing with (severely) power-divergent coefficients $\to$ it is very difficult to construct the renormalised $H_w$.

Bochicchio et al., NPB 262 (1985) 331
Maiani, Martinelli, Rossi & Testa, NPB 289 (1987) 505
The realm of no-go theorems

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Nielsen-Ninomiya theorem: no ultralocal lattice regularisation of QCD preserves chiral symmetry.

Nielsen & Ninomiya, NPB 185 (1981) 20

Use regularisations with exact chiral symmetry, or with better chiral properties.

Capitani & Giusti, PRD 64 (2001) 014506
CP, Sint & Vladikas, JHEP 09 (2005) 069
Frezzotti & Rossi, JHEP 10 (2005) 070
A tale of various scales

\[ M_W \quad \mathcal{H}_{SM} \to \mathcal{H}_{\Delta S=1}^{N_f=4} = \sqrt{2} G_F V_{us}^* V_{ud} (k_{+Q_+} + k_{-Q_-}) \]

\[ Q_{\pm} \equiv [\bar{s}u]_{V-A} [\bar{u}d]_{V-A} \pm [\bar{s}d]_{V-A} [\bar{u}u]_{V-A} - (u \leftrightarrow c) \]

\[ SU(4)_L \times SU(4)_R: \quad Q_+ \to (84, 1) \quad Q_- \to (20, 1) \]

\[ m_c \quad \mathcal{H}_{\Delta S=1}^{N_f=4} \to \mathcal{H}_{\Delta S=1}^{N_f=3} = \sqrt{2} G_F V_{us}^* V_{ud} \sum_{\sigma=1,10} C_{\sigma} Q_{\sigma} \]

\[ Q_{\sigma} : \ldots, [\bar{s}d]_{V-A} [\bar{q}q]_{V+A}, \ldots \]

\[ SU(3)_L \times SU(3)_R: \quad (27, 1) \to A_2, A_0, (8, 1) \to A_0 \]

\[ \Lambda_\chi \quad \mathcal{H}_{\Delta S=1}^{N_f=3} \to \mathcal{H}_{\chi PT}^{N_f=3} \]
A tale of various scales

The standard [?] lore:

- Resummation of $O(1/N) \log(\mu/M_W)$ up to $\mu > m_c$ gives a moderate enhancement.

- Charm threshold: $\mu < m_c \rightarrow$ penguins.

- Penguin matrix elements can be large compared to that of left-left operators.

Still to be verified/discarded via an explicit computation ...
Existing results for $A_0, A_2$?

- Use of $\chi$PT for weak decays already developed in the ‘80s.
  
  Georgi 84; Bernard et al. 85; Kambor et al. 91

- Exploratory lattice computations have obtained statistical signals for the relevant matrix elements in the quenched approximation, but suffer from uncontrolled systematic uncertainties.
  
  Kilcup, Pekurovsky 98; Blum et al. 01; Ali Khan et al. 01

- Approximate chiral symmetry.
- Charm integrated out: severe ultraviolet problems (effective Hamiltonian contains 10 operators, ultraviolet-divergent mixing even with exact chiral symmetry).
- Large quark masses.

- Many works rely in models for low-energy strong interactions.
  
  See reviews in Bertolini et al. 00, Pallante et al. 01
Existing results for $A_0, A_2$?

Lightest pion mass around 495 MeV.

CP-PACS Collaboration (Ali Khan et al.) 01
Our strategy to reveal the role of the charm

Disentangle several possible origins/contributions:

- Physics at the charm scale (via penguins).
- Physics at intrinsic QCD scale \( \sim 200\text{-}300 \text{ MeV} \).
- Final state interactions.
- All of the above (no dominating “mechanism”).

Separate “intrinsic QCD” effects from physics at the charm scale:

Consider effective weak Hamiltonian with an active charm and study \( A_0, A_2 \) as a function of \( m_c \).

\[
\begin{align*}
m_u & = m_d = m_s = m_c \\
m_u & = m_d = m_s \ll m_c
\end{align*}
\]

Our strategy to reveal the role of the charm

Disentangle several possible origins/contributions:

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- Physics at intrinsic QCD scale ~200-300 MeV.
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Separate “intrinsic QCD” effects from physics at the charm scale:

Consider effective weak Hamiltonian with an active charm and study $A_0, A_2$ as a function of $m_c$.

$m_u = m_d = m_s = m_c$

$mu = md = ms \ll mc$

Outline

- The (many difficulties of studying the) $\Delta I=1/2$ rule.
  - Operator product expansion and long distance QCD effects.
  - Problems with pions and chirality.
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- Results and conclusions.
Effective low-energy description

Dynamics of Goldstone bosons @ LO:

\[
\mathcal{L}_E = \frac{1}{4} F^2 \text{Tr} \left[ \partial_\mu U \partial_\mu U^\dagger \right] - \frac{1}{2} \Sigma \text{Tr} \left[ U M^\dagger e^{i\theta/N_f} + \text{h.c.} \right]
\]

\[
U \in \text{SU}(4), \quad M = \text{mass matrix}
\]

Low-energy counterpart of the weak effective Hamiltonian @ LO:

\[
\mathcal{H}_{\chi PT}^W = \frac{g_w^2}{2 M_W^2} (V_{us})^* (V_{ud}) \sum_{\sigma = \pm} g_1^\sigma \left\{ [\hat{O}_1^\sigma]_{suud} - [\hat{O}_1^\sigma]_{sccd} \right\}
\]

\[
[\hat{O}_1]_{\alpha \beta \gamma \delta} = \frac{1}{4} F^4 (U \partial_\mu U^\dagger)_{\gamma \alpha} (U \partial_\mu U^\dagger)_{\delta \beta}
\]

Relation of LEC’s to $K \to \pi \pi$ transition amplitudes @ LO in $\chi$PT:

\[
\frac{A_0}{A_2} = \frac{1}{\sqrt{2}} \left( \frac{1}{2} + \frac{3 g_1^-}{2 g_1^+} \right)
\]

⇒ Determine LEC’s using lattice QCD
Effective low-energy description

Dynamics of Goldstone bosons @ LO:

\[
L_E = \frac{1}{4} F^2 \text{Tr} \left[ \partial_\mu U \partial_\mu U^\dagger \right] - \frac{1}{2} \Sigma \text{Tr} \left[ UM^\dagger e^{i\theta/N_f} + \text{h.c.} \right]
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\( U \in \text{SU}(4), \ M = \text{mass matrix} \)

Low-energy counterpart of the weak effective Hamiltonian @ LO:

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\mathcal{H}_w^{\chi PT} = \frac{g_w^2}{2M_w^2} (V_{us})^* (V_{ud}) \sum_{\sigma = \pm} g_1^\sigma \left\{ [\hat{O}_1^\sigma]_{sudd} - [\hat{O}_1^\sigma]_{sccd} \right\}
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\[
[\hat{O}_1^\sigma]_{\alpha \beta \gamma \delta} = \frac{1}{4} F^4 (U\partial_\mu U^\dagger)_{\gamma \alpha} (U\partial_\mu U^\dagger)_{\delta \beta}
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Relation of LEC’s to \( K \rightarrow \pi \pi \) transition amplitudes @ LO in \( \chi PT \):

\[
\frac{A_0}{A_2} = \frac{1}{\sqrt{2}} \left( \frac{1}{2} + \frac{3}{2} g_1^- \right)
\]

\( \Rightarrow \) Determine LEC’s using lattice QCD

n.b.: \( g_1^\pm|_{N\rightarrow\infty} = 1 \)
Matching QCD to the chiral expansion

\[ R^{\pm}(x_0, y_0) = \frac{C^{\pm}(x_0, y_0)}{C(x_0)C(y_0)} \]

\[ C^{\pm}(x_0, y_0) = \sum_{x,y} \langle [J_0(x)]_{du} [Q_1^{\pm}(0)] [J_0(y)]_{us} \rangle \]

\[ C(x_0) = \sum_{x} \langle [J_0(x)]_{ds} [J_0(0)]_{sd} \rangle \]

**QCD**

\[ k_{\text{RGI}}^{\pm} \left[ \frac{Z^{\pm}}{Z_A^2} \right]_{\text{RGI}} R^{\pm} = g^{\pm} R^{\pm}(m, V, \text{LECs}) \]

\[ \hat{C}^{\pm}(x_0, y_0) = \int d^3x d^3y \langle J_0(x) O_1^{\pm}(0) J_0(y) \rangle \]

\[ C(x_0) = \int d^3x \langle J_0(x) J_0(0) \rangle \]

\[ \hat{R}^{\pm}(x_0, y_0) = \frac{\hat{C}^{\pm}(x_0, y_0)}{C(x_0)C(y_0)} \]

**p-regime:** new LECs appear at NLO

**ε-regime:** no additional $\Delta S=1$ interaction terms at $O(\epsilon^2)$ ⇒ enables matching at NLO!
**ε- vs p-regime of Chiral Perturbation Theory**


**p-regime:** \( m \Sigma V \gg 1 \)

standard \( \chi \text{PT} \) in finite \( V \):

\[
m \sim p^2 \quad L^{-1}, T^{-1} \sim p
\]

**ε-regime:** \( m \Sigma V \lesssim 1 \)

reordering of the \( \chi \) expansion:

\[
m \sim p^4 \sim \epsilon^4 \quad L^{-1}, T^{-1} \sim \epsilon
\]

- Constant field configurations (zero modes) are factored out and treated as collective variables.
- Gauge field topology and the low-lying spectrum of the Dirac operator play a crucial rôle in this regime.
- No additional interactions in the effective chiral theory at \( O(\epsilon^2) \).
$R^\pm$ in the $\epsilon$-regime


$$2 \mathcal{R}^\pm(x_0, y_0) = 1 \pm \frac{2}{(FL)^2} \left[ \rho^{-1/2} \beta_1 - \rho k_{00} \right] = 1 \pm K$$

with $\rho \equiv T/L$ and $\beta_1, k_{00}$ are shape coefficients of the box.

- same for all $\nu$

- independent of $x_0$ and $y_0$

- same in (partially-)quenched theory

- no higher order weak or strong LECS $K$
$R^\pm$ in the p-regime

For these observables the $\epsilon$-regime and $\infty$-volume results can be smoothly reached from the $p$-regime expressions

$N_f = 3$, $L = 2\text{fm}$, $T/L = 2$, $\Lambda_+ = 500 - 2000\text{MeV}$ :

$2R^\pm(-T/3, T/3)$

Deviations from the infinite volume expectation are significant for $ML \leq 5$
Lattice setup: chiral fermions

Breaking of chiral symmetry $\Rightarrow$ less protection against mixing of composite operators under renormalisation.

- Power divergences in mixing with lower dimension operators.
- Mixing with operators in different chiral multiplets.

Bochicchio, Maiani, Martinelli, Rossi, Testa 1985
Maiani, Martinelli, Rossi, Testa 1987

Obvious alternatives: “better” Wilson regularisations (tmQCD-based) ...

CP, Sint, Vladikas 2005
Frezzotti, Rossi 2005

... or exact chiral symmetry $\Rightarrow$ continuum-like renormalisation.

$$Q_1^\pm = Z_{11}^\pm Q_1^{\pm,\text{bare}} + Z_{12}^\pm Q_2^{\pm,\text{bare}}$$
$$Q_2^\pm = Z_{21}^\pm Q_1^{\pm,\text{bare}} + Z_{22}^\pm Q_2^{\pm,\text{bare}}$$

- exact chiral symmetry $\Rightarrow$ arbitrarily low masses $\Rightarrow \epsilon$-regime.
Lattice setup: chiral fermions

Ginsparg-Wilson fermions:

\[ \gamma_5 D + D \gamma_5 = \tilde{a} D \gamma_5 D, \quad \tilde{a} = \frac{a}{1 + s} \]

Ginsparg, Wilson 1982
Kaplan; Hasenfratz, Laliena, Niedermayer; Neuberger; ...

Lattice QCD action enjoys an exact chiral symmetry:

\[ \delta \psi = i \epsilon \hat{\gamma}_5 \psi, \quad \hat{\gamma}_5 = \gamma_5 (1 - \tilde{a}D) \]

\[ \delta \bar{\psi} = i \epsilon \bar{\psi} \gamma_5 \]

Lüscher 1998

Renormalisation and mixing patterns as in the formal continuum theory, provided:

\[ \psi \rightarrow \tilde{\psi} = (1 - \frac{1}{2} \tilde{a}D) \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \]

In particular, there is no dangerous mixing with lower dim. operators.
**Lattice setup: chiral fermions**

Ginsparg-Wilson fermions:

\[ \gamma_5 D + D \gamma_5 = \bar{a} D \gamma_5 D, \quad \bar{a} = \frac{a}{1 + s} \]

Ginsparg, Wilson 1982
Kaplan; Neuberger; Hasenfratz, Laliena, Niedermayer; ...

Our choice: Neuberger-Dirac operator.

\[ D_N = \frac{1}{\bar{a}} \left\{ 1 - \frac{A}{(A^+A)^{1/2}} \right\}, \quad A = 1 - aD_w \]

Neuberger 1997

Numerical treatment challenging and expensive.

Giusti, Hoelbling, Lüscher, Wittig 2002
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Numerical simulations

\[
R_{\pm}(x_0, y_0) = \frac{\sum_{x\bar{y}} \langle J_0(x) \hat{Q}_1^{\pm}(0) J_0(y) \rangle}{\sum_{x} \langle J_0(x) J_0(0) \rangle \cdot \sum_{y} \langle J_0(0) J_0(y) \rangle} \propto g_1^{\pm}
\]

Simulation parameters:

\[\beta = 5.8485 \quad \frac{V}{a^4} = 16^3 \cdot 32 \quad a \approx 0.125 \text{ fm} \quad V \approx 2^3 \cdot 4 \text{ fm}^4\]

Quark masses: p-regime \( m \sim m_s / 2 - m_s / 6 \) O(200) cfgs
\( \epsilon \)-regime \( m \sim m_s / 40, m_s / 60 \) O(800) cfgs

Quenched approximation.
Numerical simulations

- **p-regime**: ratios of correlation functions display plateaux at large enough separations ⇒ ratios of physical matrix elements.

- **ε-regime**: expansion performed at fixed topological charge, ratios of correlators fitted to constants and averaged over topological charges (no -dependence at (N)LO in χPT).

Simulation parameters:

\[ \beta = 5.8485 \quad \frac{V}{a^4} = 16^3 \cdot 32 \quad a \approx 0.125 \text{ fm} \quad V \approx 2^3 \cdot 4 \text{ fm}^4 \]

Quark masses: p-regime \( m \sim m_s/2 - m_s/6 \) \( \text{O(200) cdfs} \)

ε-regime \( m \sim m_s/40, m_s/60 \) \( \text{O(800) cdfs} \)

Quenched approximation.
Numerical simulations

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Quenched approximation.
The origin of statistical fluctuations

\[ S(x, y) = \frac{1}{V} \sum_k \frac{\eta_k(x) \otimes \eta_k(y)^\dagger}{\bar{\lambda}_k + m}, \quad \bar{\lambda}_k = (1 - \frac{1}{2} \bar{a} m) \lambda_k \]

\( m \gg \frac{1}{\Sigma V} \) ➤ Low-lying spectrum of \( D_m \) dense near \( m \).
➤ Contributions from low modes averaged with same weight.

\( m \ll \frac{1}{\Sigma V} \) ➤ Sizeable contribution of configurations with very small ev’s.
➤ Strong dependence on the observable considered.
➤ Fluctuations reduced by unquenching.

\( m \lesssim \frac{1}{\Sigma V} \) ➤ Low-lying spectrum of \( D_m \) discrete: \( m \approx \Delta \lambda = \frac{1}{\Sigma V} \)
➤ A few low modes give sizeable contributions.
➤ “Bumpy” wave functions can induce large fluctuations.
The origin of statistical fluctuations

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S(x, y) = \frac{1}{V} \sum_k \frac{\eta_k(x) \otimes \eta_k(y)^\dagger}{\bar{\lambda}_k + m}, \quad \bar{\lambda}_k = (1 - \frac{1}{2} \bar{a}m) \lambda_k
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\[
\langle \lambda_i \rangle_\nu = \frac{O(1)}{\Sigma V}
\]

\[
\Delta \lambda = \lambda_{i+1} - \lambda_i \sim \frac{O(1)}{\Sigma V} \geq m
\]

» Low-lying spectrum of \(D_m\) discrete.

» “Bumpy” wave functions of few low-lying modes ⇒ large fluctuations.
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\[ \Rightarrow \]

Low-lying spectrum of \( D_m \) discrete.

\[ \Rightarrow \]

“Bumpy” wave functions of few low-lying modes \( \Rightarrow \) large fluctuations.

Low-mode averaging: treat exactly a few low-modes — full translational invariance enforced in the most fluctuating contribution to the quark propagator.

Giusti, Hernández, Laine, Weisz, Wittig 2004
Giusti, Hernández, Laine, CP, Wennekers, Wittig 2005
The origin of statistical fluctuations

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\[ \gg \] Low-lying spectrum of \( D_m \) discrete.

\[ \gg \] “Bumpy” wave functions of few low-lying modes ⇒ large fluctuations.

Alternative: extract physics from topological zero-mode wave functions.

Giusti, Hernández, Laine, Weisz, Wittig 2004
Hernández, Laine, CP, Torró, Wennekers, Wittig in progress
The origin of statistical fluctuations

\[ P_- S(x, y) P_+ = P_- \left\{ \sum_{k=1}^{n} \frac{1}{\alpha_k} e_k(x) \otimes e_k(y)^\dagger + S^h(x, y) \right\} P_+ \]

\[ e_k = P_s u_k + P_- s D P_s u_k, \quad P_s (D_m^+ D_m) P_s u_k = \alpha_k u_k + r_k \]

\[ C_t(x_0) = C_t^{hh}(x_0) + C_t^{hl}(x_0) + C_t^{ll}(x_0) \]

\[ C_t^{hh}(x_0) = - \sum_x \langle \text{Tr} [\gamma_0 P_- S^h(x, 0)^\dagger \gamma_0 P_- S^h(x, 0)] \rangle \]

\[ C_t^{ll}(x_0) = - \frac{1}{V} \sum_{k,l=1}^{n} \sum_{y,z} \delta_{x_0,y_0-z_0} \frac{1}{\alpha_k \alpha_l} \langle [e_k(y)^\dagger \gamma_0 P_- e_l(y)][e_k(z)^\dagger \gamma_0 P_- e_l(z)] \rangle \]

\[ C_t^{hl}(x_0) = - \frac{1}{L^3} \sum_{k=1}^{n} \sum_{y,z} \delta_{x_0,y_0-z_0} \frac{1}{\alpha_k} \langle e_k(y)^\dagger \gamma_0 P_- S^h(y, z) \gamma_0 P_- e_k(z) \rangle + [y \leftrightarrow z] \]

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Numerical simulations: final (bare) results

Simulation parameters:

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Quenched approximation.
Numerical simulations: final (bare) results

Expected $\epsilon$-regime features — independence of $R^\pm$ on $(x_0,y_0)$, $m$ and $\nu$ — are all well reproduced by the data.

Simulation parameters:

$$\beta = 5.8485 \quad \frac{V}{a^4} = 16^3 \cdot 32 \quad a \approx 0.125 \text{ fm} \quad V \approx 2^3 \cdot 4 \text{ fm}^4$$

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Quenched approximation.
Connecting bare results with physical amplitudes

\[ g_1^\pm (1 + K_\pm) = k_1^\pm (M_W / \Lambda_{QCD}) \frac{\hat{Z}^\pm (g_0)}{Z_A^2 (g_0)} R_\pm \]

- **Chiral correction** → LO/NLO chiPT.
  
  Giusti, Hernández, Laine, Weisz, Wittig 2004
  Hernández and Laine 2006

- **Wilson coefficients** → NLO QCD PT.
  
  Ciuchini et al. 1998
  Buras, Misiak, Urban 2000

- **Renormalisation** → QCD PT / nonperturbative result.
  
  Dimopoulos et al. 2006
Renormalisation factor $\hat{Z}^\sigma$ relates bare and RGI operators:

$$\hat{Z}^\pm(g_0) = c^\pm_S(\mu/\Lambda_{QCD}) Z^\pm_S(g_0, a\mu)$$

$$c^\pm_S(\mu/\Lambda_{QCD}) = (2b_0\bar{g}^2(\mu))^{\gamma^\pm_0/(2b_0)} \exp\left\{-\int_0^{\bar{g}(\mu)} dg \left[ \frac{\gamma^\pm(g)}{\beta(g)} + \frac{\gamma^\pm_0}{b_0g} \right]\right\}$$

- NLO estimate in RI/MOM scheme available.  
  Capitani, Giusti 2000

- Nonperturbative renormalisation of chiral observables possible via a matching procedure to results obtained with Schrödinger Functional techniques.  
  Dimopoulos et al. 2006
Renormalisation

Renormalisation factor $\hat{Z}^\sigma$ relates bare and RGI operators:

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<table>
<thead>
<tr>
<th></th>
<th>bare P.T.</th>
<th>MFI P.T.</th>
<th>non-perturbative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{Z}_1^+ / Z_A^2$</td>
<td>1.242</td>
<td>1.193</td>
<td>1.15(12)</td>
</tr>
<tr>
<td>$\hat{Z}_1^- / Z_A^2$</td>
<td>0.657</td>
<td>0.705</td>
<td>0.561(61)</td>
</tr>
<tr>
<td>$\hat{Z}_1^- / \hat{Z}_1^+$</td>
<td>0.525</td>
<td>0.582</td>
<td>0.584(62)</td>
</tr>
</tbody>
</table>
Fits to extract LECs

- Choose quantities with smaller mass corrections and statistical errors: $R^+, R^+R^-$

- Fit to NLO $\chi$PT to extract $g^\pm$ and $\Lambda^\pm$ (exploit smooth $\epsilon/p$-regime transition).

Tension between $\epsilon$- and $p$-regime may indicate non-negligible higher order corrections $\rightarrow$ systematic error included to account for this.
Outline

- The (many difficulties of studying the) $\Delta l=1/2$ rule.
  - Operator product expansion and long distance QCD effects.
  - Problems with pions and chirality.
  - Our strategy.

- Computational setup.
  - Low-energy description: p- vs $\epsilon$-regime.
  - Chiral lattice fermions.

- Computational details.
  - Bare results and the approach to the chiral regime.
  - Matching to physics: renormalisation and chiral fits

- Results and conclusions.
Results: $K \to \pi\pi$ amplitudes in the chiral limit

Giusti, Hernández, Laine, CP, Wennekers, Wittig 2006

- $\Delta I=3/2$ comes in the right ballpark (N.B.: charm effects enter only via quark loops).

- $\Delta I=1/2$ channel and amplitude ratio are a factor $\sim 4$ too small.

- Enhancement of the $\Delta I=1/2$ channel already present with an unphysically light charm quark ($A_0/A_2 \sim 6$): “pure no-penguin” effect.
Decoupling the charm quark

Matching between the SU(3)- and SU(4)-symmetric low-energy descriptions possible for non-degenerate but light charm masses: additional enhancement predicted to come out in the right direction.

When the charm is heavy enough the usual SU(3) chiral expansion is recovered.

\[ \bar{g}_1^\pm \rightarrow g_1^\pm (m_c) \quad \bar{g}_2^\pm \rightarrow g_2^\pm (m_c) \]

Numerical simulation requires the computation of penguin contractions, much more difficult to deal with. Hope is that low-mode averaging will tame the dominant statistical fluctuations.
Decoupling the charm quark

Matching between the SU(3)- and SU(4)-symmetric low-energy descriptions possible for non-degenerate but light charm masses: additional enhancement predicted to come out in the right direction.

\[ g_8(m_c) = \frac{1}{2} \left[ \frac{1}{5} g^+ \left( 1 + 15 \frac{M_c^2}{(4\pi F)^2} \ln \frac{\Lambda_X}{M_c} \right) + g^- \left( 1 + 3 \frac{M_c^2}{(4\pi F)^2} \ln \frac{\Lambda_X}{M_c} \right) \right] \]

\[ g_{27}(m_c) = \frac{3}{5} g^+ \]

- Logarithmic enhancement of octet.
- Many unknown LECs.

Hernández, Laine 2004-2006

Graph showing the ratio of \( \hat{g}_w^- / \hat{g}_w^+ = 5 \) and \( \hat{g}_w^- / \hat{g}_w^+ = 1 \) as a function of mass in MeV.
Decoupling the charm quark

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Conclusions and outlook

- SU(4) strategy allows to disentangle contributions to the $\Delta I=1/2$ enhancement with qualitatively different origins.

- First step: computation of low-energy couplings with an unphysically light charm quark ($\to$ “pure QCD” contribution):
  - Chiral systematics under control via access to $\epsilon$-regime. First time such light masses have been reached in numerical simulations.
  - UV effects under control (GW fermions, non-perturbative renormalisation).

- Computation of the couplings completed. Moderate enhancement found. Factor $\sim 4$ still missing. Main suspect: charm dependence of amplitudes.

- Next step: go to heavier charm masses and monitor amplitudes.

- Future: unquenching.