Heterotic superstrings

1 Heterotic superstrings in bosonic formulation

1.1 Heteroticity

We have discussed that in closed string theories the left and right moving sectors have independent hamiltonian evolution. The only relation between both is in the construction of physical states, the level matching conditions.

We have also discussed two consistent (say, left moving) sectors. That of the bosonic string, given (in the light-cone gauge) by 24 2d bosons $X_L^i(\sigma+t)$, $i = 2, \ldots, 25$ and that of the superstring, given by 8 bosons $X_L^i(\sigma + t)$ and 8 fermions $\psi_L^i(\sigma + t)$, $i = 2, \ldots, 9$.

The basic idea in the construction of the heterotic string theories is to consider using the bosonic 2d content for the left moving sector and the superstring 2d content for the right moving sector [1]. Let us denote our right movers by $X_R^i(\sigma - t)$ $\psi_R^i(\sigma - t)$, and our left movers by $X_L^i(\sigma + t)$, $X_L^I(\sigma + t)$, with $i = 2, \ldots, 9$, $I = 1, \ldots, 16$.

The theory is rather peculiar at first sight. The left moving bosons $X_L^i(\sigma + t)$ can combine with the right moving ones $X_R^i(\sigma - t)$ to make out the coordinates of physical spacetime (which therefore has ten dimensions). On the other hand, it is not clear what meaning the remaining left moving

[1] That this can be done is already very non-trivial. In a Polyakov description we are coupling a 2d *chiral* field theory (since it is not invariant under 2d parity, i.e. exchange of left and right) to a 2d metric. In order for the path integral over 2d metrics to be well defined the 2d field theory must be free of 2d gravitational anomalies; this is true precisely for the matter content of left and right moving degrees of freedom that we have proposed.

bosons $X_L^I(\sigma + t)$ have. We will see that, in a precise sense to be explained below, they do not correspond to physical spacetime dimensions, but rather should be though of as parametrizing a 16d compact torus, with very small and fixed radius $R = \sqrt{\alpha'}$. Since this distance is of order the string scale, it is not very meaningful to assing a geometric interpretation to the corresponding dimensions.

1.2 Hamiltonian quantization

The worldsheet action is the expected one, namely the Polyakov action for left and right movers independently, with the right moving sector coupling also to a 2d gravitino. Since we will be interested in the light cone quantization, we simply say that it proceeds as usual, and that the only physical fields left over are those mentioned above. We now review the main features

Right movers

In the right moving sector, bosons parametrize non-compact directions, so they must be periodic in σ

$$X_R^i(\sigma - t + \ell) \;=\; X_R^i(\sigma - t) \tag{1}$$

They have the usual integer mode expansion

$$X_R^i(\sigma - t) \;=\; \frac{x^i}{2} + \frac{p_i}{2p^+}\,(t-\sigma) \;+\; i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbf{Z}-\{\mathbf{0}\}} \frac{\tilde{\alpha}_n^i}{n}\, e^{2\pi i\, n\,(\sigma - t)/\ell} \tag{2}$$

Right moving fermions can be either periodic (R) or antiperiodic (NS)

$$\begin{array}{ll} \text{NS} & \psi_L^i(\sigma + t + \ell) = -\psi_L^i(\sigma + t) \\ \text{R} & \psi_L^i(\sigma + t + \ell) = \psi_L^i(\sigma + t) \end{array} \tag{3}$$

so we have the mode expansion

$$\psi_R^i(\sigma - t) \;=\; i\sqrt{\frac{\alpha'}{2}} \sum_{r\in\mathbf{Z}} \tilde{\psi}_{r+\nu}^i\, e^{2\pi i(r+\nu)(\sigma-t)/\ell} \tag{4}$$

with $\nu = 0, 1/2$ for R, NS fermions.

The complete right moving hamiltonian is

$$\begin{aligned} H_R &= \frac{\sum_i p_i p_i}{4p^+} + \frac{1}{\alpha' p^+} \, (\tilde{N}_B + \tilde{N}_F + \tilde{E}_0) \\ \tilde{N}_B &= \sum_{n>0} \tilde{\alpha}^i_{-n} \tilde{\alpha}^i_n \quad ; \quad \tilde{N}_F = \sum_{r=0}^{\infty} (r+\nu) \, \psi^i_{-r-\nu} \, \psi^i_{r+\nu} \quad ; \quad \tilde{E}_0 = -2\nu(1-\nu) \end{aligned} \qquad (5)$$

Left movers

For the left sector, the bosons $X^i_L(\sigma+t)$ are paired with the right moving bosons, so they are pediodic

$$X^i_L(\sigma + t + \ell) = X^i_L(\sigma + t) \qquad (6)$$

and have a mode expansion

$$X^i_L(\sigma + t) = \frac{x^i}{2} + \frac{p_i}{2p^+} \, (t+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in \mathbf{Z}-\{\mathbf{0}\}} \frac{\alpha^i_n}{n} \, e^{-2\pi i \, n \, (\sigma+t)/\ell} \qquad (7)$$

We now need to propose mode expansions for the remaining left moving bosons $X^I(\sigma+t)$. To put it in a heuristic way, we propose a mode expansion that corresponds to the left moving sector of a bosonic theory compactified on a 16d torus, consistently with making the corresponding right moving degrees of freedom identically vanish.

Namely, recall the mode expansion for left and right moving bosons in a circle compactification of the bosonic theory (see lesson on toroidal compactification), in the sector of momentum k and winding w (k,

$$\begin{aligned} X_L(\sigma + t) &= \tfrac{x}{2} + \tfrac{p_L}{2p^+} \, (t+\sigma) + i\sqrt{\tfrac{\alpha'}{2}} \, \textstyle\sum_{n\in \mathbf{Z}-\{\mathbf{0}\}} \tfrac{\alpha^i_n}{n} \, e^{-2\pi i \, n \, (\sigma+t)/\ell} \\ X_R(\sigma - t) &= \tfrac{x}{2} + \tfrac{p_R}{2p^+} \, (t-\sigma) + i\sqrt{\tfrac{\alpha'}{2}} \, \textstyle\sum_{n\in \mathbf{Z}-\{\mathbf{0}\}} \tfrac{\tilde{\alpha}^i_n}{n} \, e^{2\pi i \, n \, (\sigma-t)/\ell} \end{aligned} \qquad (8)$$

with

$$p_L = \frac{k}{R} + \frac{wR}{\alpha'} \quad ; \quad p_R = \frac{k}{R} - \frac{wR}{\alpha'} \qquad (9)$$

In order to be compatible with making all right handed dynamics trivial, namely $X_R \equiv 0$, we need

$$x = 0 \quad ; \quad \tilde{\alpha}_n = 0 \quad ; \quad k = w \quad ; \quad R = \sqrt{\alpha'} \tag{10}$$

So the center of mass position degree of freedom is removed, momentum is related to winding, and the internal torus is frozen at fixed radius $\sqrt{\alpha'}$.

Generalizing to 16 dimensions, we propose the following expansion for the left moving fields $X^I(\sigma + t)$

$$X_L^I(\sigma + t) \;=\; \tfrac{P^I}{2p^+}\,(t+\sigma) \,+\, i\sqrt{\tfrac{\alpha'}{2}}\,\textstyle\sum_{n\in\mathbf{Z}-\{\mathbf{0}\}} \tfrac{\alpha_n^i}{n}\, e^{-2\pi i\, n\,(\sigma+t)/\ell} \tag{11}$$

where P^I is a 16d vector in a lattice Λ of internal quantized momenta. The whole right moving sector can be though of as consistenty truncated from the theory (to check complete consistency would require to verify that right handed dynamical modes are not excited in interactions, either; we skip this more involved issue).

The total left moving hamiltionian is

$$\begin{aligned} H_L &= \frac{\sum_i p_i^2}{4p^+} + \frac{\sum_I P^I P^I}{4p^+} + \frac{1}{\alpha' p^+}\,(N-1) \\ N &= \sum_i \sum_N \alpha_{-n}^i \alpha_n^i \sum_I \sum_N \alpha_{-n}^I \alpha_n^I \end{aligned} \tag{12}$$

We have the spacetime mass formulae

$$\begin{aligned} \alpha' m_R^2/2 &= \tilde{N}_B + \tilde{N}_F - 2\nu(1-\nu) \\ \alpha' m_L^2/2 &= N_B + \frac{P^2}{2} - 1 \end{aligned} \tag{13}$$

and the level matching conditions are given by

$$m_L^2 = m_R^2 \tag{14}$$

1.3 Modular invariance and lattices

Let us describe a modular invariant partition function and then discuss what kind of physical spectrum it is describing. We can assume the simple ansatz that the complete partition function factorizes as a product of a left and a right moving piece, namely

$$Z(\tau) \;=\; (4\pi\alpha'\tau_2)^{-4}\,|\eta(\tau)|^{-16}\,\overline{Z}_\psi(\tau)\,Z_{\alpha^I}(\tau) \tag{15}$$

The first factor corresponds to tracing over the 10d spacetime momentum degrees of freedom, the second to the trace over the oscillators of the X^i_R, X^i_L. The factor $\overline{Z}_\psi(\tau)$ is the trace over the right moving fermionic oscillators. From our experience with type II superstrings, an almost modular invariant partition function for this sector is

$$\overline{Z}_\psi \;=\; (\eta^{-4})^* \left(\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 - \vartheta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^4 - \vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4 \pm \vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}^4 \right)^* \tag{16}$$

The two choices for the sign eventually lead to the same theory (up to a 10d parity transformation), so for concreteness we pick the $-$ sign.

For $Z_{\alpha^I}(\tau)$ we have the trace over the oscillators and the 16d momentum degrees of freedom

$$Z_{\alpha^I}(\tau) \;=\; \eta(\tau)^{-16} \sum_{P\in\Lambda} q^{P^2/2} \tag{17}$$

Now we need to require modular invariance, and this will impose some restrictions on the possible choices of Λ.

i) As $\tau \to \tau+1$, the momentum and bosonic oscillator part is invariant, while we have

$$\vartheta\left[\begin{array}{c}0\\0\end{array}\right](\tau+1)=\vartheta\left[\begin{array}{c}0\\1/2\end{array}\right](\tau) \quad ; \quad \vartheta\left[\begin{array}{c}0\\1/2\end{array}\right](\tau+1)=\vartheta\left[\begin{array}{c}0\\0\end{array}\right](\tau)$$

$$\vartheta\left[\begin{array}{c}1/2\\0\end{array}\right](\tau+1)=e^{-\pi i/4}\,\vartheta\left[\begin{array}{c}1/2\\0\end{array}\right](\tau) \quad ; \quad \vartheta\left[\begin{array}{c}1/2\\1/2\end{array}\right](\tau+1)=e^{-\pi i/4}\,\vartheta\left[\begin{array}{c}1/2\\1/2\end{array}\right](\tau)$$

$$\eta(\tau+1)=e^{\pi i/12}\,\eta(\tau)$$

and hence

$$\overline{Z}_\psi(\tau+1) \;=\; e^{4\pi i/3}\,\overline{Z}_\psi \tag{18}$$

Hence we need

$$Z_{\alpha^I}(\tau+1) \;=\; e^{2\pi i/3}\,Z_{\alpha^I} \tag{19}$$

This is so, provided

$$\sum_{P\in\Lambda} e^{2\pi i(\tau+1)\,P^2/2} \;=\; \sum_{P\in\Lambda} e^{2\pi i\tau\,P^2/2} \tag{20}$$

Namely, we need $P^2 \in 2\mathbf{Z}$ for any $P \in \Lambda$. Lattices with this property are called *even*.

For future use (see next footnote), let us point out that even lattice are always *integer* lattices. An integer lattice is such that for any $v, w \in \Lambda$, we have $v \cdot w \in \Lambda$. To show this, notice that in an even lattice, for any v, w we have $(v+w)^2$ is even, but $(v+w)^2 = v^2 + w^2 + 2v\cdot w$. Since v^2, w^2 are enen, it follows that $v \cdot w \in \mathbf{Z}$ and Λ is integer.

ii) As $\tau \to -1/\tau$, the spacetime momentum times spactime bosonic oscillator piece is invariant. For the fermionic piece we have

$$\vartheta\left[\begin{array}{c}0\\0\end{array}\right](-1/\tau)=(-i\tau)^{1/2}\,\vartheta\left[\begin{array}{c}0\\0\end{array}\right](\tau) \qquad \vartheta\left[\begin{array}{c}0\\1/2\end{array}\right](-1/\tau)=(-i\tau)^{1/2}\,\vartheta\left[\begin{array}{c}1/2\\0\end{array}\right](\tau)$$

$$\vartheta\left[\begin{array}{c}1/2\\0\end{array}\right](-1/\tau)=(-i\tau)^{1/2}\,\vartheta\left[\begin{array}{c}0\\1/2\end{array}\right](\tau) \qquad \vartheta\left[\begin{array}{c}1/2\\1/2\end{array}\right](-1/\tau)=i(-i\tau)^{1/2}\,\vartheta\left[\begin{array}{c}1/2\\1/2\end{array}\right](\tau)$$

$$\eta(-1/\tau)=(-i\tau)^{1/2}\,\eta(\tau)$$

and hence

$$\overline{Z}_\psi(-1/\tau) \;=\; \overline{Z}_\psi(\tau) \tag{21}$$

So we need

$$Z_{\alpha^I}(-1/\tau) \;=\; Z_{\alpha^I}(\tau) \tag{22}$$

The left hand side reads

$$Z_{\alpha^I}(-1/\tau) \;=\; (-i\tau)^{-8}\,\eta(\tau)^{-16}\sum_{P\in\Lambda} e^{2\pi i(-1/\tau)P^2/2} \tag{23}$$

Using the Poisson resummation formula [2]

$$\begin{aligned} &\textstyle\sum_{v\in\Lambda}\,\exp\left[\,-\pi(v+\theta)\cdot A\cdot(v+\theta)+2\pi i\,(v+\theta)\cdot\phi\,\right] = \\ &= \tfrac{1}{|\Lambda^*/\Lambda|\sqrt{\det A}}\textstyle\sum_{k\in\Lambda^*}\,\exp\left[\,-\pi(k+\phi)\cdot A^{-1}\cdot(k+\phi)-2\pi i k\theta\,\right] \end{aligned} \tag{24}$$

we have

$$Z_{\alpha^I}(-1/\tau) \;=\; (-i\tau)^{-8}\,\eta(\tau)^{16}\,\frac{1}{|\Lambda^*/\Lambda|}(-i\tau)^{8}\sum_{K\in\Lambda^*} e^{-2\pi i\tau\,K^2/2} \tag{25}$$

So we have invariance if $\Lambda* = \Lambda$. Such lattices are called *self-dual.*

The compactification lattice Λ must be even and self-dual to obtain a consistent modular invariant theory. Even self-dual lattices (with euclidean signature scalar product) have been proved by mathematicians to be extremely constrained. They only exist in dimensions multiple of eight; happily we need 16d lattices, so the dimension is in the allowed set of values.

Moreover there are only *two* inequivalent 16d even and self-dual lattices. These are the following

[2]Here Λ^* is the lattice dual to Λ, which is formed by the vectors k such that $k\cdot v\in\mathbf{Z}$ for any $v\in\Lambda$. For integer lattices, Λ is a sublattice of Λ^*, and the quotient Λ^*/Λ is a finite set. Its cardinal $|\Lambda^*/\Lambda|$ is called the index of Λ in Λ^*.

i) The $E_8 \times E_8$ lattice

It is spanned by vectors of the form

$$(n_1,\ldots,n_8;n'_1,\ldots,n'_8) \quad ; \qquad (n_1+\tfrac{1}{2},\ldots,n_8+\tfrac{1}{2};n'_1,\ldots,n'_8) \quad (26)$$
$$(n_1,\ldots,n_8;n'_1+\frac{1}{2},\ldots,n'_8+\frac{1}{2}) \quad ; \qquad (n_1+\tfrac{1}{2},\ldots,n_8+\tfrac{1}{2};n'_1+\tfrac{1}{2},\ldots,n'_8+\tfrac{1}{2})$$

with $n_I, n'_I \in \mathbf{Z}$, and $\sum_I n_I = \text{even}$, $\sum_{I'} n'_I = \text{even}$

ii) The Spin(32)/$\mathbf{Z_2}$ lattice

Spanned by vectors of the form

$$\begin{gathered}(n_1,\ldots,n_{16})\\ (n_1+\tfrac{1}{2},\ldots,n_{16}+\tfrac{1}{2})\end{gathered} \qquad (27)$$

So these define two consistent heterotic superstring theories.

1.4 Spectrum

The spectrum of these theories is found by constructing left and right moving states in the usual way (constructing ladder operators and Hilbert spaces, and applying the GSO projections dictated by the partition function), and glueing them together satisfying level-matching.

We will simply discuss massless states, although the rules to build the whole tower of string states should be clear.

The right moving sector is exactly the same as one of the sides of the type II superstrings. The two choices of Z_ψ give two final theories which differ by a 10d parity operation, so are equivalent; hence we choose one of them. The massless states suriviving the GSO projection are

Sector	State	$SO(8)$
NS	$\tilde{\psi}^i_{-1/2}\|0\rangle$	8_V
R	$\tilde{A}^+_a\|0\rangle$	8_C
	$\tilde{A}^+_{a_1}\tilde{A}^+_{a_2}\tilde{A}^+_{a_3}\|0\rangle$	

We will denote the states in the R sector by $\frac{1}{2}(\pm,\pm,\pm,\pm)$ (with odd number of $-$'s), i.e. by the $SO(8)$ weights.

For the left movers, the mass formula is given by

$$\alpha' m_L^2/2 \;=\; N_B \,+\, \frac{P^2}{2} \,-\, 1 \tag{28}$$

Lightest states are

	State	$\alpha' m_L^2/2$	$SO(8)$
$N_B=0, P=0$	$\|0\rangle$	-1	1
$N_B=1, P=0$	$\alpha^i_{-1}\|0\rangle$	0	8_V
$N_B=1, P=0$	$\alpha^I_{-1}\|0\rangle$	0	1
$N_B=0, P^2=2$	$\|P\rangle$	0	1

Notice that there is a tachyon, but it will not lead to any physical state in spacetime since it has no tachyonic right-moving state to be level-matched with.

The latter states with $P^2=2$ are different for the two choices of lattice. For the $E_8\times E_8$ lattice, these states have internal momentum P of the form

$$\begin{array}{ll} (\underline{\pm,\pm,0,0,0,0,0,0};0,0,0,0,0,0,0,0) & \\ \frac{1}{2}(\pm,\pm,\pm,\pm,\pm,\pm,\pm,\pm;0,0,0,0,0,0,0,0) & \#- = \text{even} \\ (0,0,0,0,0,0,0,0;\underline{\pm,\pm,0,0,0,0,0,0}) & \\ \frac{1}{2}(0,0,0,0,0,0,0,0;\pm,\pm,\pm,\pm,\pm,\pm,\pm,\pm) & \#- = \text{even} \end{array} \tag{29}$$

We note that these are the non-zero root vectors of $E_8\times E_8$ (hence the name of the lattice).

States with $P^2=2$ in the $Spin(32)/\mathbf{Z_2}$ lattice have P of the form

$$(\underline{\pm,\pm,0,0,0,0,0,0,0,0,0,0,0,0,0,0}) \tag{30}$$

We note that these are the non-zero root vectors of $SO(32)$ (hence the name of the lattice). Notice that momenta of the form $P = 1/2(\pm, \ldots, \pm)$ have $P^2 = 4$ and give rise to massive states.

We should now glue together left and right states. The schematic structure of massless states is

$$(\tilde{8}_V + \tilde{8}_C) \times (8_V + \alpha^I + |P\rangle) \tag{31}$$

Namely, we have the states

$$\begin{array}{cc} \tilde{\psi}^i_{-1/2}|0\rangle \times \alpha^j_{-1}|0\rangle & 8_V \times 8_V = 1 + 28_V + 35_V \\ \frac{1}{2}(\pm,\pm,\pm,\pm) \times \alpha^j_{-1}|0\rangle & 8_C \times 8_V = 8_S + 56_S \end{array}$$

The massless fields are a scalar dilaton ϕ, a graviton $G_{\mu\nu}$, a 2-form B_2, and fermion superpartners, including a 10d chiral gravitino (56_S). This is the $N = 1$ 10d supergravity multiplet, so the theory turns out to have $N = 1$ spacetime susy. Notice that this is half the susy of type II theories, since we have GSO projection only on one of the sides, and this produces half as many gravitinos.

We also obtain the states

$$\begin{array}{cc} \psi^i_{-1/2}|0\rangle \times \alpha^I_{-1}|0\rangle & 8_v \\ \frac{1}{2}(\pm,\pm,\pm,\pm) \times \alpha^I_{-1}|0\rangle & 8_C \end{array}$$

they correspond to 16 gauge bosons and superpartner gauginos. The gauge group is $U(1)^{16}$.

Finally we have the states

$$\begin{array}{cc} \tilde{\psi}^i_{-1/2}|0\rangle \times |P\rangle & 8_v \\ \frac{1}{2}(\pm,\pm,\pm,\pm) \times |P\rangle & 8_C \end{array}$$

These are also gauge bosons and gauginos. It is possible to see that they are charged under the $U(1)^{16}$ gauge symmetries (this is analogous to how winding and momentum states are charged with respect to the gauge symmetries obtained in toroidal compactifications), so the gauge group will be enhanced to a non-abelian symmetry. We would like to identify what is the final gauge group, for each of the two choices of internal lattice. The $U(1)^{16}$ gives the Cartan subalgebra of the group, which hence has rank 16. The charge of a state $|P\rangle$ under the I^{th} $U(1)$ factor is given by P^I, hence the vectors P must correspond to the non-zero roots of the gauge group. As we have mentioned before, the $P^2 = 2$ states of the compactification lattices precisely correspond to the non-zero roots of the groups $E_8 \times E_8$ and $SO(32)$, respectively for each of the lattices. Hence states from the α^I oscillators and from momentum P give altogether 10d $N = 1$ vector multiplets of $E_8 \times E_8$ or $SO(32)$.

The complete massless spectrum for the two consistent (spacetime supersymmetric) heterotic theories is 10d $N = 1$ supergravity coupled to $E_8 \times E_8$ or $SO(32)$ vector multiplets. These theories are chiral, so there is a very stringent consistency issue arising from 10d anomalies. This will be reviewed later on in this lecture.

Notice that the spectrum of these theories is very exciting. It contains non-abelian gauge symmetries and charged chiral fermions. In later lectures we will see that this structure allows to obtain interesting theories with charged chiral 4d fermions upon compactification. In particular this is possible due to the existence of fundamental vector multiplets in the higher dimensional theory, therefore avoiding diverse no-go theorems about getting charged chiral fermions in Kaluza-Klein theories with pure (super)gravity in the higher dimensional theory.

2 Heterotic strings in the fermionic formulation

In this section we discuss a different construction of the same heterotic string theories as before. Readers comfortable with the above bosonic formulation may therefore skip this section.

We refer the reader to the last section in the lesson about type II superstring to the discussion of bosonization/fermionization. There we discussed that a theory of k left-moving boson paramerizing compactified directions is equivalent to a theory $2k$ fermions with a sum over boundary conditions determined by the compactification lattice.

This motivates introducing a different description of the heterotic strings we have constructed. Indeed, we construct a string theory whose worldsheet degrees of freedom (already in the light-cone gauge) are right moving fields $X_R^i(\sigma - t)$, $\psi_R^i(\sigma - t)$, $i = 2, \ldots, 9$ and left-moving fields $X_L^i(\sigma + t)$, $\lambda_L^A(\sigma + t)$, with $i = 2, \ldots, 9$ and $A = 1, \ldots, 32$.

The quantization of these is standard: Bosons $X_{L,R}^i$ are periodic in σ and give rise to integer-modded oscillators, fermions ψ^i can be NS or R and have consequently half-integer or integer modded oscillators. Finally fermions λ^A can also be NS or R, but in contrast with the preivous λ^A's with different boundary condition can coexist in the same sector (recall the ψ's must be all NS or all R in order not to violate spacetime Lorentz invariance).

With these ingredients, we can construct two possible modular invariant partition functions, which have the familiar GSO projection on the right-moving piece. They define two consistent heterotic string theories, which will turn out to be the two heterotic strings constructed above, but described in 2d fermionic language.

The two partition functions have the structure

$$Z(\tau) \;=\; (4\pi\alpha'\tau_2)^{-4}\,|\eta(\tau)|^{-16}\,\overline{Z}_\psi(\tau)\,Z_\lambda(\tau) \tag{32}$$

with two possible opctions for Z_λ

$$\begin{aligned}
\textbf{i)}\quad Z_\lambda(\tau) \;&=\; \frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^{16}+\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^{16}+\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^{16}+\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{16}}{\eta(\tau)^{16}}\\
\textbf{ii)}\quad Z_\lambda(\tau) \;&=\; \left(\frac{\vartheta\begin{bmatrix}0\\0\end{bmatrix}^{8}+\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^{8}+\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^{8}+\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{8}}{\eta(\tau)^{8}}\right)^2
\end{aligned} \tag{33}$$

They differ in the way the 32 fermions λ^A are grouped. It is possible to use the expressions of the ϑ functions as infinite sums and write the above partition functions as sums over momenta in the $Spin(32)/\mathbf{Z_2}$ and $E_8\times E_8$ lattices, thus showing the equivalence with the bosonic formulations above. We have recovered exactly the same heterotic string theories starting from a different world-sheet formulation (related to the previous by bosonization/fermionization).

It is however interesting to construct the spectrum directly in the fermionic formulation. We review it now, with special emphasis on the massless sector.

The right-moving sector is very familiar, and works exactly as one of the sides of the type II superstring. At the massless level, we obtain NS states $\tilde{\psi}^i_{-1/2}|0\rangle$ in the 8_V of $SO(8)$ and R states in the 8_C.

For the left-moving sector, we treat the two possible cases separately.

The $SO(32)$ heterotic in fermionic language

We start with **i)**, the partition function Z_λ has the structure

$$\mathrm{tr}_{\mathcal{H}_{NS}}(1+(-)^F)+\mathrm{tr}_{\mathcal{H}_R}(1+(-)^F) \tag{34}$$

Hence the 32 fermions are all with NS or all with R boundary conditions. In each sector there is an overall GSO projection.

NS sector

The mass formula is given by

$$\alpha' m_L^2/2 \;=\; N_B + N_F - 1 \tag{35}$$

There are no fermion zero modes, so the vacuum is non-degenerate; the Hilbert space is obtained by applying negative modding oscillators on it. The GSO projection requires the number of fermion oscillators to be even for physical states. The lightest states are

State	$\alpha' m_L^2/2$
$\|0\rangle$	-1
$\alpha_{-1}^i\|0\rangle$	0
$\lambda_{-1/2}^A\lambda_{-1/2}^B\|0\rangle$	0

The latter states correspond to antisymmetric combinations of the indices A and B. Therefore and for future convenience we associate them to the generators of an $SO(32)$ Lie algebra (whose generators in the vector representation are given by antisymmetric matrices).

As before, the left-moving tachyon cannot be level-matched with any right-moving state and does not lead to spacetime tachyon states.

R sector The mass formula is given by

$$\alpha' m_L^2/2 \;=\; N_B + N_F + 1 \tag{36}$$

There are 32 fermion zero modes, so the vacuum is 2^{16}-fold degenerate, split in two chiral spinor irreps of the underlying $SO(32)$ symmetry (acting on the Λ^A). The GSO projection selects states with even number of fermion oscillators on one of them, and states with odd number of fermion oscillators on the other. All states in the R sector are however massive, hence we will not be too interested in them.

The total spectrum is found by glueing left and right moving states in a level-matched way. The states

$$8_V \times \alpha^i_{-1}|0\rangle \quad ; \quad 8_C \times \alpha^i_{-1}|0\rangle \quad ; \tag{37}$$

reproduce the 10d $N=1$ supergravity multiplet $1+28_V+35_V+8_S+56_S$. The states

$$8_V \times \lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle \quad ; \quad 8_C \times \lambda^A_{-1/2}\lambda^B_{-1/2}|0\rangle \quad ; \tag{38}$$

reproduce 10d $N=1$ vector multiplets with gauge group $SO(32)$ (as can be guessed by noticing that we have $32\times 32/2$ states associated with antisymmetric combinations of indices in the vector of $SO(32)$.

Hence we have reproduced the (massless) spectrum of the $SO(32)$ heterotic superstring.

The $E_8 \times E_8$ heterotic in fermionic language

We now study **ii)**, the partition function Z_λ has the structure

$$\left[\mathrm{tr}\,_{\mathcal{H}_{NS}} + \mathrm{tr}\,_{\mathcal{H}_R}(1+(-)^F)\right]^2 \tag{39}$$

Hence the 32 fermions are split in two sets of 16, which we denote λ^A, $\lambda^{A'}$. They have equal boundary conditions within each set, but with independent boundary conditions. For each set of 16 fermions: the NS boundary conditions imply the groundstate is unique, and GSO requires an even number of fermion oscillators to be applies; the R boundary conditions imply a 2^8-fold degenerate groundstate, split as two chiral spinor irreps of the underlying $SO(16)$, denoted 128 and 128', with GSO requiring even number of fermion oscillators acting on 128 and odd number on 128'.

With this information we can construct the complete left-moving spectrum. The lightest states which will finally level-match with right-moving ones are the massless ones, so we look only at these

$\mathbf{NS}_{16}\mathbf{NS}_{16}$

The mass formula is

$$\alpha' m_L^2/2 \;=\; N_B + N_F - 1 \tag{40}$$

The massless states are

State	Remark
$\alpha^i_{-1}\vert 0\rangle$	8_V of $SO(8)$
$\lambda^A_{-1/2}\lambda^B_{-1/2}\vert 0\rangle$	Adj. of $SO(16)$
$\lambda^{A'}_{-1/2}\lambda^{B'}_{-1/2}\vert 0\rangle$	Adj. of $SO(16)'$

$\mathbf{R}_{16}\mathbf{NS}_{16}$

The vacuum is 2^7-fold degenerate due to the 16 R fermion zero modes. The mass formula is

$$\alpha' m_L^2/2 \;=\; N_B + N_F \tag{41}$$

The massless states are the groundstates, which transform as 128 of the $SO(16)$

$\mathbf{NS}_{16}\mathbf{R}_{16}$

Similarly to the above, the massless states are the groundstates, which transform as 128 of the $SO(16)'$

$\mathbf{R}_{16}\mathbf{R}_{16}$

In this sector even the groundstate is massive.

The total massless spectrum is obtained by tensoring the right-moving $8_V + 8_C$ with the above left handed states. It is easyy to recover the 10d $N = 1$ supergravity multiplet by tensoring the right-moving $8_V + 8_C$ with the left-moving 8_V. On the other hand, by tensoring the right-moving $8_V + 8_C$ with the left-moving $SO(8)$ singlets we obtain 10d $N = 1$ vector multiplets with gauge group $E_8 \times E_8'$. The gauge group can be guessed by rememebering that the adjoint of E_8 decomposes as an adjoint plus a 128 of $SO(16)$. Hence we recover the complete massless spectrum of the $E_8 \times E_8$ heterotic.

3 Spacetime Non-susy heterotic string theories

There are other ways to construct modular invariant partition functions, beyond the factorized proposal used above. These are more easily constructed using the fermionic formulation of superstrings (a bosonized formulation is also possible, but more involved since it would require lattices mixing the internal bosons and spacetime fermionic degrees of freedom).

Without aiming at a general classification, let us simply give one example of such a modular invariant partition function

$$\frac{1}{\overline{\eta}^4\,\eta^{16}} \quad \left(\overline{\vartheta} \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^{16} - \overline{\vartheta} \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4 \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^{16} - \right.$$
$$\left. \overline{\vartheta} \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^4 \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^{16} - \overline{\vartheta} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}^4 \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}^{16} \right) \qquad (42)$$

The interpretation in terms of the GSO projection is that we correlate the $(-1)^F$ quantum number of the right moving fermions with the $(-1)^F$ quantum number of the internal left-moving fermions.

Schematically the spectrum at the massless level is

Sector	State	$\alpha' m^2$	$SO(8)$	internal
NS-NS	$\|0\rangle \otimes \psi^I_{-1/2}\|0\rangle$	-2	1	32
	$\tilde{\psi}^i_{-1/2}\|0\rangle \otimes \alpha^i_{-1}\|0\rangle$	0	$1 + 28_V + 35_V$	1
	$\tilde{\psi}^i_{-1/2}\|0\rangle \otimes \psi^I_{-1/2}\psi^J_{-1/2}\|0\rangle$	0	8_V	$SO(32)$

Notice that the left moving R states has only massive modes, so by level matching the NS-R, R-NS and R-R sector have only massive modes. The theory contains the graviton, 2-form and dilaton field, as well as $SO(32)$ gauge bosons. The theory is spacetime non-supersymmetric, and contains tachyons, transforming in the 32 of $SO(32)$. As in other cases of tachyons in closed string theories, the fate of this instability is not known. Finally, the theory contains fermions, but all of them are massive. Overall, the theory is not too interesting, and is given just as an example of non-supersymmetric heterotic strings.

This heterotic string can also be constructed in the bosonic formulation, by reading off the required lattice from the above partition function. Note as we said that the lattice would involve the internal bosons as well as the bosonization of the right moving fermions.

We conclude by pointing out that all 10d non-supersymmetric heterotic theories contains tachyons, except for the so-called $SO(16) \times SO(16)$ heterotic. Details on this can be found in [2] (although discussed in a language perhaps not too transparent).

4 A few words on anomalies

Anomaly cancellation in theories with chiral 10d spectrum is an astonishing example of self-consistency of string theory. Therefore it is an interesting topic to be covered. We leave its discussion for the evaluation project.

4.1 What is an anomaly?

Let us start giving a set of basic facts about anomalies, directed towards understanding in what situations they may appear. A complete but formal introduction may be found in [3].

When a classical theory has a symmetry which is not present in the quantum theory, we say that the symmetry has an anomaly or that the theory is anomalous. Namely, what happens is that quantum corrections generate terms in the effective action which are not invariant under the symmetry. Since the classical lagrangian was invariant, such terms cannot be removed with local counterterms, and the quantum theory is not invariant.

In the path integral formalism of quantum field theory, the lack of invariance of the quantum theory (the anomaly) arises from the non-invariance of the measure of the functional integration (this is Fujikawa's method of computing anomalies).

Notice that if there exists some regularization which preserves a classical symmetry of the classical theory, then the symmetry is not anomalous. Namely, the regularized theory is still invariant under the symmetry, so regularized quantum corrections preserve the symmetry, and when the cutoff is taken to infinity the symmmetry is still preserved. Hence the only symmetries which can be anomalous are those for which no symmetry-preserving regularization exists.

This has the important consequence that only chiral fields can contribute to anomalies. The contribution from non-chiral fields can always be regularized by using the Pauli-Villars regularization, which preserves all the symmetries of the system.

This implies that anomalies can arise only in even dimensions [3] $D = 2n$

[3]There exists different class of anomalies, called global anomalies (what I mean here is different from anomalies for global symmetries), which are different from the ones we

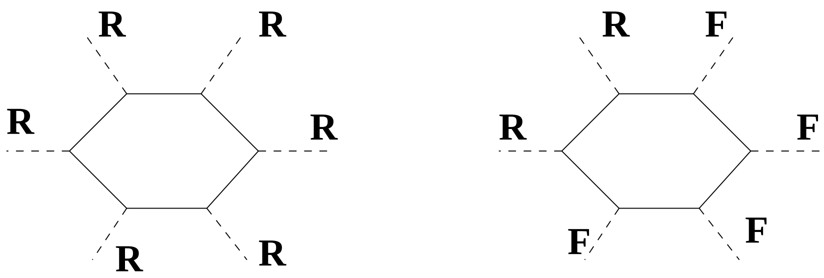

Figure 1: Different hexagon diagrams contributing to gravitaional, gauge and mixed anomalies.

because only then there exist chiral representations of the Lorentz group. Anomalies arise from very precise diagrams, they appear only from contributions at one loop (and not at higher order, this is Adler's theorem), in a diagram of one loop of chiral fields (usually fermions) with $n+1$ external legs of the fields associated to the symmetry (gauge bosons for gauge symmetries, gravitons for diffeomorphism invariance (gravitaional anomalies), and external currents for global symmetries). For instance, in 10d theories, anomalies arise from hexagon diagrams (see fig **??** with external legs corresponding to gravitons and/or gauge bosons, if they are present in the theory.

We will center on gauge anomalies, which are lethal for the theories. Namely, in preserving unitarity of the theory it is essential that unphysical polarization modes decouple, and this happens as a consequence of gauge invariance. If an anomaly spoils the gauge invariance in the quantum theory, the latter is inconsistent (non-unitary, etc). Namely, by scattering physical polarization modes we can create unphysical ones by processes mediated by the anomaly diagram. Hence, the latter must vanish in order to have a

study here and may also exist in odd dimensions; for instance the parity anomaly in odd dimensions.

consistent unitary theory.

Anomalous gauge variations of the effective action can be obtained from the so-called anomaly polynomial I which is a formal $(2n+2)$-form constructed as a polylnomial in the gravitational and gauge curvature 2-forms, R and F, resp. It is therefore closed and gauge invariant. For instance, in a 10d theory with gravitons and gauge bosons, the anomaly polynomial is a linear combination of things like $\mathrm{tr}\, R^6$, $\mathrm{tr}\, R^4 \mathrm{tr}\, R^2$, $\mathrm{tr}\, F^6$, $\mathrm{tr}\, F^4 \mathrm{tr}\, F^2$, $(\mathrm{tr}\, F^2)^3$, etc, with wedge products implied. Coefficients of the anomaly polynomial are determined by the spectrum of chiral fields of the theory. The anomalous variantion of the 1-loop effective action under a symmetry transformation with gauge parameter λ is of the form

$$\delta_\lambda S_{\mathrm{eff}} = \int I^{(1)} \tag{43}$$

where $I^{(1)}$ is an n-form, obtained by the so-called Wess-Zumino descent procedure, as follows. Since the anomaly polynomial I is closed, it is locally exact and can be written as $I = dI^{(0)}$, with $I^{(0)}$ a $(2n+1)$-form. It can be shown that the gauge variation of $I^{(0)}$ under any symmetry transformation is closed, hence it is also locally exact and we can write $\delta_\lambda I^{(0)} = dI^{(1)}$, where λ is the gauge parameter and $I^{(1)}$ is the above n-form. Hence we have

$$I = dI^{(0)} \quad ; \quad \delta_\lambda I^{(0)} = dI^{(1)} \tag{44}$$

To give one simple example, consider a 4d $U(1)$ gauge theory with n chiral fermions carrying charge $+1$. The anomaly polynomial is given by $I = n\, F^3$. We then have $I^{(0)} = n\, AF^2$ and $\delta_\lambda I^{(0)} = n\, d\lambda\, F^2$, hence $I^{(1)} = n\lambda F^2$, leading to the familiar form of the 4d anomaly.

Notice that the fact that the anomaly is a topological quantity is related to the fact that it is determined by the spectrum of chiral fermions. The latter is unchanged by continuous changes of the parameters of the theory, like coupling constants, etc, hence so is the anomaly, i.e. it is a topological quantity.

The fact that all anomalies in a theory can be derived from a unique anomaly polynomial implies that the anomalies for diverse symmetries (and for diagrams involving different kinds of gauge fields) obey the so-called Wess-Zumino consistency conditions. Roughly speaking, they imply that if a gauge variation wrt a symmetry 'a' generates a term involving the gauge curvature of a symmetry 'b', then a gauge variation of 'b' should generate terms involving the curvature of 'a'. This is clear from the fact that the diagram mediating the anomalies contains external legs of both 'a' and 'b'.

4.2 Anomalies in string theory and Green-Schwarz mechanism

In string theory, the spacetime theory is often chiral, for instance type IIB or heterotic superstrings in 10d (also type I, see next lectures).

From the string theory viewpoint, the theory is however finite and gauge invariant. This implies that the underlying string theory is providing a regularization of the corresponding effective field theory containing the chiral fields. From this viewpoint it is clear that string theory should lead to theories free of gauge and gravitational anomalies (In fact, the relation between modular invariance (ultimately responsible for finiteness of string theory) and absence of anomalies has been explored in the literature [4]).

In type IIB theory, the fields contributing to the gravitational anomalies are the 8_S, 56_S and 35_C, i.e. the fermions and the self-dual 4-form. With this matter content there is a miraculous cancellation of all terms in the anomaly polynomial, which then automatically vanishes. The theory is therefore non-anomalous [5].

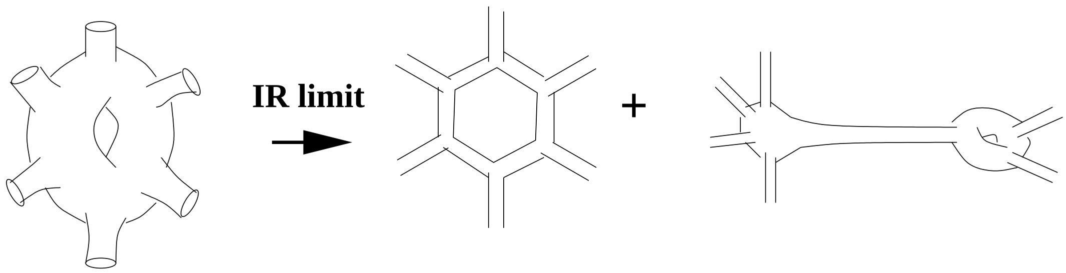

Figure 2: The low-energy limit of the six-point function for gravitons and gauge bosons contains two contributions, the familiar field theory hexagon, and a diagram of exchange of closed string modes at tree level with tree and one-loop level couplings to external legs.

In heterotic theories, the field content also leads to some miraculous cancellations of terms in the anomaly polynomial. For instance, the fact that the gauge group has 496 generators leads to the absence of $\mathrm{tr}\, F^6$ terms. This is called cancellation of the irreducible anomaly. However, even after these miracles, the anomaly polynomaly still is non-vanishing, but has a special structure, it is of the form

$$I \simeq \mathrm{tr}\, F^4 \,(\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2) \tag{45}$$

This residual anomaly, known as reducible anomaly, is cancelled by a special contribution to the six-point function of gauge bosons and gravitons, which does not have the standard field theory hexagon interpretation. As is shown in figure **??**, the contribution to the 1-loop amplitude with six external legs lead to two kinds of low-energy contributions. One of them is the familiar field thery hexagon diagram, of massless particles running in a loop. The second is however of the form of an exchange of massless modes along a tree level diagram, and a subsequent 1-loop coupling to some gauge fields.

The existence of the second contribution was noticed by Green and Schwarz [4], who provided the right field theory interpretation for it. The massless mode propagating along the tube is the 2-form B_2 (or its dual B_6) which has couplings to the curvatures as follows

$$\int_{10d} B_2 \wedge \mathrm{tr}\, F^4 \quad ; \quad \int_{10d} B_6 \wedge (\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2) \quad ; \tag{46}$$

which arise at tree level and 1-loop respectively. The last coupling is often expressed by saying that B_2 obeys the modified Bianchi identity

$$dH_3 = \mathrm{tr}\, F^2 - \mathrm{tr}\, R^2 \tag{47}$$

Using these couplings, the gauge variation of the effective action is

$$\delta \textstyle\int_{10d} H_3 \wedge (\mathrm{tr}\, F^4)^{(0)} \; = \; \int_{10d} H_3 \wedge \delta(\mathrm{tr}\, F^4)^{(0)} \; = \; \int_{10d} H_3 \wedge d(\mathrm{tr}\, F^4)^{(1)} \; = \tag{48}$$
$$\textstyle\int_{10d} dH_3 \wedge (\mathrm{tr}\, F^4)^{(1)} \; = \; \int_{10d} (\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2)(\mathrm{tr}\, F^4)^{(1)} \simeq \int_{10d} [\, (\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2)(\mathrm{tr}\, F^4)\,]^{(1)}$$

The total anomalous variation therefore vanishes. This is the so-called Green-Schwarz mechanism. This is very remarkable, indeed so remarkable that triggered a lot of interest in string theory since the mid 80's.

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