Type II Superstrings

We are already familiar with bosonic string theory, and have learned how to solve the issue of reducing it to lower dimensions via compactification. However, we have been unable to construct a theory with fermions in spacetime.

In this and coming lectures we study string theories whose massless spectrum contains spacetime spinor particles. These are the superstring theories, and today we will center on a particular kind of them: type II superstrings (leaving other superstrings, like heterotic strings and type I strings, for later lectures).

Before getting started, let us mention that in order to identify the quantum numbers of states with respect to the spacetime Lorentz group, it is quite crucial to have in mind the representation theory of $SO(2n)$ Lie algebras, which can be found in section 6 of the appendix on group theory.

1 Superstrings

1.1 Fermions on the worldsheet

To describe a new string theory we have to modify the worldsheet theory. Clearly, if we keep the same field content as in the bosonic string and simply add interactions, the spectrum in spacetime will not be very different from that in the bosonic theory, and in particular it will not contain spacetime fermions. Addint interactions is more similar to just curving the background on which the string is propagating.

Instead, we propose to change the field content of the 2d theory describing the worldsheet. A simple possibility which preserves D-dimensional Poincare invariance is to make the 2d worlsheet theory supersymmetric [1]. Namely, to add 2d fermion fields $\psi^\mu(\sigma,t)$, partners of the 2d bosonic fields $X^\mu(\sigma,t)$, and gravitino partners for the worldsheet metric $g_{ab}(\sigma,t)$ (notice that since supersymmetry commutes with global symmetries, the 2d fermionic fields should transform in the vector representation of the D-dimensional spacetime Lorentz group, just like the 2d bosonic fields). It is important to emphasize that at this stage it is not obvious at all that such theory will lead to spacetime fermions or spacetime supersymmetry; in fact, the 2d fermion fields are bosons with respect to the spacetime Lorentz group!

Two-dimensional theories of this kind are sometimes refered to as 'fermionic strings'. We will *not* write down the 2d action for those fields, etc, but instead use the simple practical rules to give the final result of physical fields and hamiltonian after light-cone quantization. Recall that upon light-cone quantization of the bosonic theory the physical fields where the bosonic fields associated to the transverse coordianates $X^i(\sigma,t)$, $i=2,\ldots,D-1$, with hamiltonian given by an infinite set of decoupled harmonic oscillators.

The light-cone quantization for the fermionic sector also leaves the transverse fermionic coordinates $\psi^i(\sigma,t)$, $i=2,\ldots,D-1$ as the only remaining physical fields. Their hamiltonian corresponds to an infinite set of fermionic

[1] One may wonder if 2d susy is really necessary to achieve spacetime fermions. In our discussion it would seem that we are emphasizing just the need of worldsheet fermions, and that 2d susy appears as an accidental symmetry in the system of decoupled fermionic and bosonic harmonic oscillators; however it is possible to argue as in the first section of chapter 10 in [1] that the equation of motion for spacetime spinors arises from the conserved supercurrent of the 2d theory. From this viewpoint 2d susy is quite crucial. In fact, even in our simplified discussion spatime fermions are seen to arise from *fermionic zero modes* in the R sector, where the zero point energy exactly vanishes due to 2d susy; hence susy turns out to be crucial as well in our description, although not in a very explicit way.

harmonic oscillators.

In closed string theories it is possible to carry out the quantization etc independently for left- and right-moving degrees of freedom. This is quite convenient for us, so we split our degrees of freedom in $X^i_L(\sigma+t)$, $\psi^i_L(\sigma+t)$, $X^i_R(\sigma-t)$, $\psi^i_R(\sigma-t)$, and work with just the left moving piece. The level matching constraints etc will be discussed at a later stage.

1.2 Boundary conditions

We are interested in discussing closed fermionic strings in flat D-dimensional Minkowski space. To have closed string in flat space, the 2d bosocic fields must be periodic in σ

$$X^i_L(\sigma+t+\ell) = X^i_L(\sigma+t) \tag{1}$$

and we have the oscillator expansion

$$X^i_L(\sigma+t) \;=\; \frac{x^i}{2} + \frac{p_i}{2p^+}(t+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbf{Z}-\{\mathbf{0}\}} \frac{\alpha^i_n}{n}\, e^{-2\pi i\, n\,(\sigma+t)/\ell} \tag{2}$$

with modes having commutation relations

$$[x^i, p_j] = i\delta_{ij} \quad ; \quad [\alpha^i_n, \alpha^j_m] = m\,\delta_{ij}\,\delta_{m,-n} \tag{3}$$

and hamiltonian

$$\begin{aligned} H_B &= \frac{\sum_i p_i p_i}{4p^+} + \frac{1}{\alpha' p^+}\left[\sum_{n>0} \alpha^i_{-n}\alpha^i_n \;+\; E^B_0\right] \\ E^B_0 &= -\frac{D-2}{24} \end{aligned} \tag{4}$$

For fermions, there is a subtlety in discussing boundary conditions. In the two-dimensional worldsheet field theory, as in any quantum field theory, the only observables are expressions that go like products of two fermion fields. That means thae periodicity in σ of observables is consistent with antiperiodicity of the fermion fields . Hence there are two consistent boundary conditions

$$
\begin{array}{lll}
\text{Neveu} - \text{Schwarz} & \text{NS} & \psi_L^i(\sigma + t + \ell) = -\psi_L^i(\sigma + t) \\
\text{Ramond} & \text{R} & \psi_L^i(\sigma + t + \ell) = \psi_L^i(\sigma + t)
\end{array}
\tag{5}
$$

These can be chosen independently for left and right sectors. It is important to notice that consistency, e.g. Lorentz invariance, already requires that in a given sector, fermions fields ψ_L^i for all i are all periodic or all antiperiodic.

Hence it would seem that we can define four different kinds of closed strings, according to whether the left and right sectors have NS or R fermions; namely we would have NS-NS, NS-R, R-NS and R-R strings. Very surprisingly, we will see that modular invariance requires these different boundary conditions to coexist within the same theory. In a sense, in the same way that a consistent string theory requires us to sum over different worlsheet topologies (topological sectors of the embedding functions X^i), it also requires us to sum over different topological sectors (boundary conditions) for the 2d fermion fields, in a precise way dictated by the requirement to get a modular invariant partition function. This has been formulated very precisely as a sum over spin structures on the worldsheet [2].

1.3 Spectrum of states for NS and R fermions

Before going further, it will be useful to compute the oscillator expansion, hamiltonian and spectrum of states for 2d fermions with NS and R boundary conditions. We describe this for the left-moving sector, being analogous (and independent) for the right-moving one.

1.3.1 NS sector

Antiperiodic boundary conditions require the oscillator modding to be half-integer. We have the oscillator expansion

$$\psi^i_L(\sigma + t) \;=\; i\sqrt{\frac{\alpha'}{2}}\,\sum_{r\in\mathbf{Z}} \psi^i_{r+1/2}\, e^{-2\pi i(r+1/2)(\sigma+t)/\ell} \tag{6}$$

Notice that there are no zero modes in the expansion. The oscillators have anticommutation relations

$$\{\psi^i_{n+1/2}, \psi^j_{m+1/2}\} \;=\; \delta^{ij}\,\delta_{m+1/2,-(n+1/2)} \tag{7}$$

The hamiltonian for the fermionic degrees of freedom is

$$H_{F,NS} \;=\; \frac{1}{\alpha' p^+}\,\left[\sum_{r=0}^{\infty}(r+\frac{1}{2})\,\psi^i_{-r-1/2}\,\psi^i_{r+1/2} \;+\; E_0^{F_{NS}}\right] \tag{8}$$

where the zero point energy for NS fermionic oscillators is

$$E_0^{F_{NS}} \;=\; -\frac{1}{2}\sum_{n=0}^{\infty}(n+\frac{1}{2}) \tag{9}$$

evaluated with the exponential regularization. It is useful to compute in general (for $\alpha > 0$)

$$Z_\alpha \;=\; \frac{1}{2}\sum_{n=0}^{\infty}(n+\alpha) \tag{10}$$

as the $\epsilon \to 0$ limit of the finite part of

$$\begin{aligned}
Z_\alpha(\epsilon) \;&=\; \frac{1}{2}\sum_{n=0}^{\infty}(n+\alpha)\,e^{-(n+\alpha)\epsilon} \;=\; -\frac{1}{2}\frac{\partial}{\partial\epsilon}\sum_{n=0}^{\infty} e^{-(n+\alpha)\epsilon} \;=\; -\frac{1}{2}\frac{\partial}{\partial\epsilon}\left(\frac{e^{-\alpha\epsilon}}{1-e^{-\epsilon}}\right) \;=\\
&=\; -\frac{1}{2}\frac{\partial}{\partial\epsilon}\left[(1-\alpha\epsilon+\alpha^2/2\epsilon^2+\mathcal{O}(\epsilon^3))\,(\frac{1}{\epsilon}+\frac{1}{2}+\frac{1}{12}\epsilon+\mathcal{O}(\epsilon^2))\right] \;=\\
&=\; -\frac{1}{2}\frac{\partial}{\partial\epsilon}\left[\frac{1}{\epsilon}+\frac{1}{2}+\frac{1}{12}\epsilon-\alpha-\frac{1}{2}\alpha\epsilon+\frac{1}{2}\alpha^2\epsilon+\mathcal{O}(\epsilon^2)\right] \;=\\
&=\; \frac{1}{2\epsilon^2}-\frac{1}{24}+\frac{1}{4}\alpha(1-\alpha)+\mathcal{O}(\epsilon)
\end{aligned} \tag{11}$$

so we get

$$Z_\alpha = -\frac{1}{24} + \frac{1}{4}\alpha(1-\alpha) \tag{12}$$

and

$$E_0^{F_{NS}} = -\frac{1}{48}(D-2) \tag{13}$$

The total bosonic and fermionic hamiltonian for the 2d theory in the NS sector is

$$H_L = \frac{\sum_i p_i p_i}{4p^+} + \frac{1}{\alpha' p^+}\left[\sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{r=0}^{\infty}(r+\frac{1}{2})\,\psi^i_{-r-1/2}\,\psi^i_{r+1/2} + (D-2)\frac{-1}{16}\right] \tag{14}$$

The contribution of the left-moving sector to the spacetime mass is

$$m_L^2 = 2p^+ H_L - \frac{1}{2}\sum_i p_i^2 \tag{15}$$

namely

$$\alpha' m_L^2/2 = \left[\sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{r=0}^{\infty}(r+\frac{1}{2})\,\psi^i_{-r-1/2}\,\psi^i_{r+1/2} - \frac{(D-2)}{16}\right] \tag{16}$$

The spectrum in the NS sector is obtained by defining a groundstate $|k\rangle_{NS}$ with spacetime momenta k_i, and annihilated by all positive modding oscillators

$$\begin{aligned} \psi^i_{n+1/2}|k\rangle_{NS} = 0 \quad &; \quad \forall n \geq 0 \\ \alpha^i_n|k\rangle_{NS} = 0 \quad &; \quad \forall n > 0 \end{aligned} \tag{17}$$

and applying negative modding oscillators in all possible ways.

The lightest left moving states (for zero spacetime momentum) are

State	$\alpha' m_L^2/2$
$\|0\rangle_{NS}$	$-\frac{(D-2)}{16}$
$\psi^i_{-1/2}\|0\rangle_{NS}$	$\frac{1}{2} - \frac{(D-2)}{16}$

(18)

Now we realize that the first excited state is a vector with respect to spactime Lorentz transformations, and that it only has $D-2$ components. So it forms a representation of the group $SO(D-2)$, which is the little group of a massless particle in a Lorentz invariant D-dimensional theory. This means that in order to be consistent with Lorentz invariance, the state should be massless, and this requires $(D-2)/16 = 1/2$, namely $D = 10$. Namely we obtain the result that the string theory at hand propagates consistently only in a spacetime of *ten* dimensions.

The states we have transform under the $SO(8)$ group as

State	$\alpha' m_L^2/2$	$SO(8)$
$\|0\rangle_{NS}$	$-1/2$	$\mathbf{1}$
$\psi^i_{-1/2}\|0\rangle_{NS}$	0	$\mathbf{8_V}$

where 8_V is the vector representation of $SO(8)$.

1.3.2 Ramond sector

Periodic boundary conditions require integer modding for fermionic oscillators

$$\psi^i_L(\sigma+t) \;=\; i\sqrt{\frac{\alpha'}{2}}\sum_{r\in\mathbf{Z}} \psi^i_r\, e^{-2\pi i r(\sigma+t)/\ell} \tag{19}$$

An important difference with respect to the NS sector is the existence of fermion zero modes ψ^i_0.

The anticommutation relations read

$$\{\psi^i_n,\psi^j_m\} \;=\; \delta^{ij}\,\delta_{m,-n} \tag{20}$$

The hamiltonian for the fermionic degrees of freedom is

$$H_{F,R} \;=\; \frac{1}{\alpha' p^+}\left[\sum_{r=1}^{\infty} r\,\psi^i_{-r}\,\psi^i_r \;+\; E_0^{F_R}\right] \tag{21}$$

with $E_0^{F_R} = (D-2) \times (-1/2) \sum_{r=1}^{\infty} r$, which for $D = 10$ equals $E_0^{F_R} = 8 \times \frac{1}{24}$. The total bosonic plus fermionic zero point energies cancel in the R sector [2]

The total bosonic and fermionic hamiltonian for the 2d theory in the NS sector is

$$H_L \;=\; \frac{\sum_i p_i p_i}{4p^+} \;+\; \frac{1}{\alpha' p^+} \left[\sum_{n>0} \alpha^i_{-n} \alpha^i_n \;+\; \sum_{r=1}^{\infty} r\, \psi^i_{-r}\, \psi^i_r \right] \tag{22}$$

The contribution of the left-moving sector to the spacetime mass is

$$m_L^2 \;=\; 2p^+ H_L - \frac{1}{2} \sum_i p_i^2 \tag{23}$$

namely

$$\alpha' m_L^2/2 \;=\; \left[\sum_{n>0} \alpha^i_{-n} \alpha^i_n \;+\; \sum_{r=0}^{\infty} r\, \psi^i_{-r}\, \psi^i_r \right] \tag{24}$$

To compute the spectrum we have to be careful with the definition of the ground state, because of fermion zero modes. Given a groundstate, application of some ψ^i_0 costs no energy and we get another groundstate. The system has a degenerate set of groundstates, and we have to find how the fermionic operators act on them. Clearly we can require that positive modding operators annihilate it; however we cannot require that all fermionic zero modes annihilate it, since this is not consistent with the zero mode anticommutators

$$\{\psi^i_0, \psi^j_0\} \;=\; \delta^{ij} \tag{25}$$

which is a Clifford algebra (see section 6 of the lesson on group theory). In fact, defining the action of the ψ^i_0 on the set of groundstates is constructing a representation of the corresponding Cliffor algebra

[2]In the NS sector the local 2d susy is broken by the different boundary conditions betwen bosons and fermions, leaving a finite zero point energy contribution; in the R sector the 2d susy is globally preserved by the boundary conditions, so the zero point energies cancel.

By now we know that to construct such a representation we should define the operators

$$A_a^{\pm} = \psi_0^{2a} \pm i\psi_0^{2a-1} \quad \text{for} \;\; a = 1, \ldots, 4 \tag{26}$$

define a lowest weight state by $A_a^-|0\rangle = 0$, and build the set of states by application of the A_a^+ operators

$$\begin{array}{cc} |0\rangle & A_{a_1}^+|0\rangle \\ A_{a_1}^+A_{a_2}^+|0\rangle & A_{a_1}^+A_{a_2}^+A_{a_3}^+|0\rangle \\ A_1^+A_2^+A_3^+A_4^+|0\rangle & \end{array} \tag{27}$$

A representation of the Clifford algebra splits into two spinor representations, of different chiralities, of the $SO(8)$ Lie algebra. These correspond to the two above columns; we denote the corresponding states by $\mathbf{8_S}$ and $\mathbf{8_C}$, or equivalently by the corresponding weights $\frac{1}{2}(\pm,\pm,\pm,\pm)$ with the number of $-$'s even for $\mathbf{8_S}$ and odd for $\mathbf{8_C}$.

The Hilbert space in the R sector is obtained by applying the negative modding operators to these groundstates in all possible ways. At the massless level, the only states are the groundstates, transforming under $SO(8)$ as

$$\mathbf{8_S} + \mathbf{8_C} \tag{28}$$

Our results, to summarize, are that the light modes in the NS and R sectors are

NS	State	$\alpha' m_L^2/2$	$SO(8)$
	$\|0\rangle_{NS}$	$-1/2$	$\mathbf{1}$
	$\psi^i_{-1/2}\|0\rangle_{NS}$	0	$\mathbf{8_V}$
R	$\frac{1}{2}(\pm,\pm,\pm,\pm)$ $\#- =$ even	0	$\mathbf{8_S}$
	$\frac{1}{2}(\pm,\pm,\pm,\pm)$ $\#- =$ odd	0	$\mathbf{8_C}$

We can choose these states independently for left and right movers. We now need to discuss how to glue them together to form physical states. One conditions is the level matching constraint, which amounts to

$$m_L^2 = m_R^2 \tag{29}$$

The glueing is also constrained from modular invariance. Namely, a string in one of these sectors, namely NS for left movers and NS for right movers, is *not* modular invariant.

The real, physical, string theories are formed by combining NS and R sectors in a way consisntent with modular invariance. In a sense we need to sum over boundary conditions for the fermions, i.e. combine the spectra of different sectors.

1.4 Modular invariance

We would like to discuss the partition function

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}} \left(e^{-\tau_2 \ell H} \, e^{i\tau_1 \ell P} \right) \tag{30}$$

In order to keep discussion about left and right movers independently it is useful to recall that the trace over the physical level-matched Hilbert space of a string theory can be extended to a trace over an unconstrained Hilbert space, with independend left and right sectors, with level matching imposed upon integration of the τ_1 piece of the modular parameter (see lesson on modular invariance).

Using that

$$H = \frac{\sum_i p_i^2}{2\alpha' p^+} + H_L + H_R \quad ; \quad P = H_L - H_R \tag{31}$$

with $H_L = \frac{1}{\alpha' p^+}(N+E_0)$, $H_R = \frac{1}{\alpha' p^+}(\tilde{N}+\tilde{E}_0)$, the expression for the partition function can be written as

$$\begin{aligned} Z(\tau) &= \mathrm{tr}_{\mathcal{H}}\, e^{-\pi\alpha'\tau_2 \sum p_i^2}\, q^{N+E_0}\, \overline{q}^{\tilde{N}+\tilde{E}_0} = \mathrm{tr}_{\mathcal{H}_{c.m.}}\, e^{-\pi\alpha'\tau_2 \sum p_i^2}\, \mathrm{tr}_{\mathcal{H}_L}\, q^{N+E_0}\, \mathrm{tr}_{\mathcal{H}_R}\, \overline{q}^{\tilde{N}+\tilde{E}_0} = \\ &= (4\pi^2\alpha'\tau_2)^{-4}\, \mathrm{tr}_{\mathcal{H}_L}\, q^{N+E_0}\, \mathrm{tr}_{\mathcal{H}_R}\, \overline{q}^{\tilde{N}+\tilde{E}_0} \end{aligned} \quad (32)$$

where factorization follows from considering the left and right movers independently.

Within each sector we have such factorization. We would now like to compute the left movers partition functions for NS and R boundary conditions. At this point, it will be useful to recall some useful modular functions, (see appendix of the lesson on modular invariance), which we gather in the appendix.

The partition function in the left sector contains a trace over the bosonic oscillators, which is computed just like in bosonic string theory

$$\mathrm{tr}_{\mathcal{H}_{bos}}\, q^{N_B+E_0^B} = \eta(\tau)^{-8} \quad (33)$$

To obtain the partition function over the infinite set of fermionic oscillators, consider first the simplified situation of the partition function of a single fermionic harmonic oscillator. It has just two states, the vacuum $|0\rangle$ and $\psi_{-\nu}|0\rangle$, where ν denotes the oscillator moding. For this system we have

$$\mathrm{tr}_{\mathcal{H}}\, q^{N_F+E_0^F} = q^{E)^F}\,(1+q^{\nu}) \quad (34)$$

For several decoupled fermionic harmonic oscillators, we simply get the product of partition functions for the individual ones.

NS fermions

Using this, the partition function for 8 NS fermionic coordinates is the product of partiton functions for eight infinite sets of fermionic harmonic oscillators with half-integer moddings $n + 1/2$, namely

$$\mathrm{tr}\,_{\mathcal{H}_{NS}}\, q^{N_F + E_0^F} \;=\; \left[q^{-1/48} \prod_{n=1}^{\infty} (1 + q^{n-1/2}) \right]^8 \;=\; \frac{\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4}{\eta^4} \tag{35}$$

R fermions

This is the product over the partition function of eight infinite sets of fermionic harmonic oscillator with integer modding, times the multiplicity of 16 due to the degenerate ground state, namely

$$\mathrm{tr}\,_{\mathcal{H}_R}\, q^{N_F + E_0^R} \;=\; 16 \left[q^{1/24} \prod_{n=1}^{\infty} (1 + q^{n}) \right]^8 \;=\; \frac{\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4}{\eta^4} \tag{36}$$

Now we easily observe that modular transformations may mix different boundary conditions, and even require the introduction of new pieces in the partition function. For instance

$$\frac{\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4}{\eta^4} \stackrel{\tau \to \tau+1}{\longrightarrow} \frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^4}{\eta^4} \stackrel{\tau \to -1/\tau}{\longrightarrow} \frac{\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4}{\eta^4} \stackrel{\tau \to -1/\tau}{\longrightarrow} \tag{37}$$

Clearly a modular invariant partition function must be a sum over sectors with different boundary conditions.

1.5 Type II superstring partition function

Instead of working by trial and error, let us simply give the final result of a possible modular invariant partition function, and then interpret it in terms of the physical spectrum of the theory.

Consider the two partition functions for left movers

$$Z_\pm = \frac{1}{2}\,(4\pi^2\alpha'\tau_2)^{-2}\,\eta^{-8}\,\eta^{-4}\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4 - \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4 - \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4 \pm \vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\right) \quad (38)$$

The first piece is half of the contribution from spacetime momenta, then follows the piece from left bosonic oscillators, then the piece from left fermionic oscillators. Either of the two choices is invariant under $\tau \to -1/\tau$, and they transform as $Z_\pm \to -Z_\pm$ under $\tau- \to \tau+1$. Therefore, it is possible to cook up several modular invariant partition functions for the complete left times right theory. Namely we consider the partition functions

$$Z_+\overline{Z}_+ \quad ; \quad Z_-\overline{Z}_- \quad ; \quad Z_+\overline{Z}_- \quad ; \quad Z_-\overline{Z}_+ \qquad (39)$$

This means that there are four consistent string theories! (in fact, we will see later on that there are only two inequivalent ones).

1.6 GSO projection

It is now time to address the question of what is the meaning of pieces like $\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}$ or $\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}$ in the partition function. For NS fermions it is easy to realize that

$$\begin{aligned}
\eta^{-1}\,\vartheta\begin{bmatrix}0\\0\end{bmatrix} &= q^{-1/48}\prod_{n=1}^{\infty}(1+q^{n-1/2})^2 = \mathrm{tr}_{\mathcal{H}_{NS}}\, q^{N+E_0^F} \\
\eta^{-1}\,\vartheta\begin{bmatrix}0\\1/2\end{bmatrix} &= q^{-1/48}\prod_{n=1}^{\infty}(1-q^{n-1/2})^2 = \mathrm{tr}_{\mathcal{H}_{NS}}\, q^{N+E_0^F}\,(-1)^F
\end{aligned} \qquad (40)$$

On the second line we sum over NS fermions, weighting each fermionic oscillator mode by a minus sign; this can be implemented in the trace as the insertion of an operator $(-1)^F$ which anticommutes with all fermionic oscillator operators.

Using this, we are now ready to interpret the meaning of one of the pieces of the left partition functions $Z_\pm$. Namely

$$\eta^{-4}\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4-\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4\right)=\tfrac{1}{2}\,\mathrm{tr}_{\mathcal{H}_{NS}}\,q^{N+E_0^F}\;-\;\mathrm{tr}_{\mathcal{H}_{NS}}\,q^{N+E_0^F}\,(-)^F\;= \\ =\;\mathrm{tr}_{\mathcal{H}_{NS}}\,q^{N+E_0^F}\,\tfrac{1}{2}(1-(-)^F) \tag{41}$$

The operator $\frac{1}{2}(1-(-)^F)$ is a projector that allows to propagate only modes with an odd number of fermionic oscillators. This piece of the partition function traces over 8 fermions with NS boundary conditions, projecting out modes with an even number of fermionic oscillators. This is the GSO projection in the NS sector.

The effect on the light NS states is to remove the tachyonic groundstate $|0\rangle_{NS}$ from the physical spectrum, and leave the states $\psi^i_{-1/2}|0\rangle_{NS}$.

Similarly, the remaining pieces of the partition function correspond to

$$\eta^{-4}\left(\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4\pm\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\right)=\tfrac{1}{2}\,\mathrm{tr}_{\mathcal{H}_R}\,q^{N+E_0^F}\;\pm\;\mathrm{tr}_{\mathcal{H}_R}\,q^{N+E_0^F}\,(-)^F\;= \\ =\;\mathrm{tr}_{\mathcal{H}_R}\,q^{N+E_0^F}\,\tfrac{1}{2}(1\pm(-)^F) \tag{42}$$

which implements a GSO projection on the R sector. Namely, for the partition function Z_+ the GSO projection leaves states with even number of excitations over the groundstate 8_C and states with odd number of excitations over the groundstate 8_S (and projects out other possibilities); while Z_- leaves states with odd number of excitations over the groundstate 8_C and states with even number of excitations over the groundstate 8_S (and projects out other possibilities).

1.7 Light spectrum

The product form of the left times right partition function implies that left NS and R sectors can combine with right NS and R sectors. More explicitly, the fermionic piece of the partition function has the structure

$$\begin{aligned} Z_\psi(\tau) &= (\mathrm{tr}\,_{\mathcal{H}_{NS,GSO_-}} - \mathrm{tr}\,_{\mathcal{H}_{R,GSO_-}}) \times (\mathrm{tr}\,_{\mathcal{H}_{NS,GSO_-}} - \mathrm{tr}\,_{\mathcal{H}_{R,GSO_\pm}})^* = \\ &= \mathrm{tr}\,_{\mathcal{H}_{NS,GSO_-}} \mathrm{tr}\,^*_{\mathcal{H}_{NS,GSO_-}} - \mathrm{tr}\,_{\mathcal{H}_{NS,GSO_-}} \mathrm{tr}\,^*_{\mathcal{H}_{R,GSO_\pm}} - \\ &\quad - \mathrm{tr}\,_{\mathcal{H}_{R,GSO_-}} \mathrm{tr}\,^*_{\mathcal{H}_{NS,GSO_-}} + \mathrm{tr}\,_{\mathcal{H}_{R,GSO_-}} \mathrm{tr}\,^*_{\mathcal{H}_{R,GSO_\pm}} \end{aligned} \tag{43}$$

where the subindex $GSO_\pm$ implies we trace only over the states surviving the GSO projection $\frac{1}{2}(1 \pm (-)^F)$. Notice the minus sign in the contributions from the NS-R and R-NS partition function, which implies that loops of the corresponding spacetime fields are weighted with a minus sign, namely they are fermions. We will see that these states have half-integer spin, so these string theories automatically implement the spin-statistics relation.

We discuss the light (in fact massless) spectrum of the theories in what follows.

Type IIB superstring

Consider the theory described $Z_+\overline{Z}_+$. Using the above projections, it is easy to realize that (both for left and right sectors) the massless NS states are simply the $\psi^i_{-1/2}|0\rangle$, transforming in the 8_V, while in the R sector the states surviving the GSO projection transform as 8_C. These states can be glued together satisfying the level matching condition.

The $SO(8)$ representation of the complete states is obtained by tensoring the representations of the left and right pieces. Hence we have

NS-NS	$8_V \otimes 8_V$	$1 + 28_V + 35_V$
NS-R	$8_V \otimes 8_C$	$8_S + 56_S$
R-NS	$8_C \otimes 8_V$	$8_S + 56_S$
R-R	$8_C \otimes 8_C$	$1 + 28_C + 35_C$

The NS-NS sector contains an scalar (dilaton), a 2-index antisymmetric tensor (2-form $B_{\mu\nu}$), and a 2-index symmetric tensor (graviton $G_{\mu\nu}$).

The R-NS and NS-R sectors contain fermions, in fact the 56_S arising from a vector and a spinor under $SO(8)$ is a gravitino (a spin 3/2 particle).

The RR sector contains a bunch of p-forms, namely p-index completely antisymmetric tensors. In particular, a 0-form (scalar) a, a 2-form $\tilde{B}_2$, and a 4-form (of self-dual field strength) A_4^+. It is sometimes convenient to introduce the Hodge duals of these, which are a 6-form B_6, an 8-form C_8. Finally, it is also useful to introduce a 10-form C_{10}, which does not have any propagating degrees of freedom, since it has no spacetime kinetic term (since its field strength would be a 11-form in 10d spacetime).

The theory is invariant under spacetime coordinate reparametrization, and gauge transformations of the p-forms. It is also invariant under local supersymmetry. It is easy to verify from the tables in [4] that the massless spectrum is that of 10d $N = 2$ chiral supergravity. String theory is providing a finite ultraviolet completion of this supergravity theory, remarkable indeed!

Finally, this theory is chiral in 10d, and has potential gravitational anomalies. It was checked in [3] that the chiral sector of the theory is precisely such that all anomalies automatically cancel (in a very non-trivial, almost miraculous, way).

This is the TYPE IIB superstring.

Consider now the theory described by $Z_-\overline{Z}_-$. It is similar to the above by simply exchanging $C \leftrightarrow S$ in the $SO(8)$ representations. Hence, clearly the two theories are the same up to a redefinition of what we mean by left and right chirality in 10d (namely, up to a parity transformation). So we do not obtain a new theory from $Z_-\overline{Z}_-$. Similarly $Z_-\overline{Z}_+$ and $Z_+\overline{Z}_-$ are related, and is enough to study just one of them.

Type IIA superstring

Consider the theory described $Z_+\overline{Z}_-$. Using the above projections, the massless sector is

NS-NS	$8_V \otimes 8_V$	$1 + 28_V + 35_V$
NS-R	$8_V \otimes 8_S$	$8_C + 56_C$
R-NS	$8_C \otimes 8_V$	$8_S + 56_S$
R-R	$8_C \otimes 8_S$	$8_V + 56_V$

The NS-NS sector contains an scalar (dilaton), a 2-index antisymmetric tensor (2-form $B_{\mu\nu}$), and a 2-index symmetric tensor (graviton $G_{\mu\nu}$).

The R-NS and NS-R sectors contain fermions, in fact the 56_S, 56_C arising from a vector and a spinor under $SO(8)$ are gravitinos (a spin 3/2 particle).

The RR sector contains a bunch of p-forms, namely p-index completely antisymmetric tensors. In particular, a 1-form (scalar) A_1, and a 3-form C_3. It is sometimes convenient to introduce the Hodge duals of these, which are a 5-form C_5, a 7-form A_7. Finally, it is also useful to introduce a 9-form C_9, which does not contain much dynamics (and is related to Romans massive IIA supergravities [5]).

The theory is invariant under spacetime coordinate reparametrization, and gauge transformations of the p-forms. It is also invariant under local supersymmetry. It is easy to verify from the tables in [4] that the massless spectrum is that of 10d $N = 2$ non-chiral supergravity. String theory is providing a finite ultraviolet completion of this supergravity theory, remarkable indeed!

Finally, this theory is non-chiral in 10d, hence is automatically anomaly free.

This is the TYPE IIA superstring.

Some comments

• The construction we have described seems a bit intricate. However, it follows naturaly from the underlying worldsheet geometry of the string,

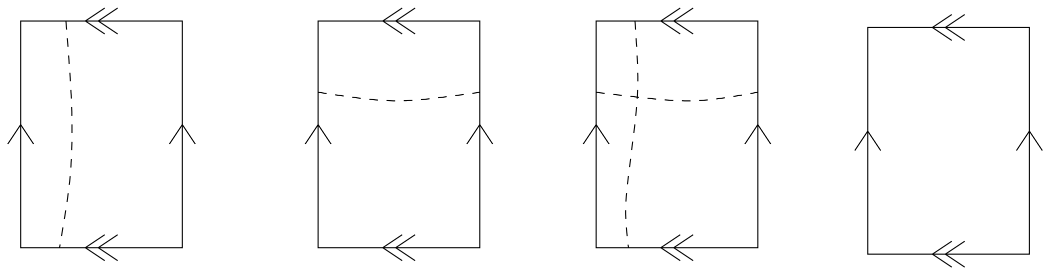

Figure 1: The four theta function contributions to the partition function can be understood as for possible boundary conditions in σ and t for fermions in a 2-torus. (Anti)periodicity in t is correlated with the presence of $(-)^F$ insertions in the trace, while (anti)periodicity in σ is correlated with the choice of NS or R fermions. Clearly modular transformations relate different contributions, so that a modular invariant theory needs to combine all of them.

namely from modular invariance, i.e. invariance under (large) coordinate transformations on the worldsheet. The reason why modular transformations mix different boundary conditions can be understood intuitively from figure 1: Starting with a GSO projected trace over NS states, the piece involving the $(-1)^F$ insertion implies that 2d fermions pick up a minus sign as they evolve in t; upon the modular transformation $\tau \to -1/\tau$, we obtain that fermions pick up an additional sign as σ varies, namely the boundary condition is not NS any longer, but is flipped to R in this sector. All contributions in the partition function may be understood in this language.

• We re-emphasize that the appearance of spacetime fermions is subtle, and is not automatically obtained from the existence of 2d fermions. Indeed, in the NS sector we have 2d fermions but no spacetime fermions. Similarly, the existence of spacetime supersymmetry does not automatically follow from 2d susy, rather it is implemented due to the GSO projection. This is one of the remarkable features of string theory, the deep relation between physics of the worldsheet (modular invariance, etc) and spacetime physics (spacetime susy).

• Spacetime supersymmetry is not manifest in the formalism we have described. It would be nice to find a formalism which describes type II superstring, and which makes spacetime supersymmetry manifest. Intuitively, we would like to describe the worldsheet theory by describing string configurations by an embedding of the worldsheet into 10d superspace, namely a set of embedding superfunctions $(X^\mu(\sigma,t), \Theta^\alpha(\sigma,t))$, where Θ^α transform in the spinor representation of the spacetime Lorentz group and parametrize the fermionic dimensions of superspace. Such a formulation exists and is known as the Green-Schwarz superstrings. For type II theories it is equivalent to the formulation we used (called the NSR formulation), but it is more difficult in some respects. Some useful comments on it may be found in section 12.6 in [1].

• Recall that the partition function is the vacuum energy of the spacetime theory. Spacetime supersymmetry implies that the spectrum is fermion/boson degenerate, and that this vacuum energy vanishes. Indeed, the theta functions satisfy the 'abstruse identities'

$$\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 - \vartheta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^4 - \vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4 = 0 \quad ; \quad \vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 0 \qquad (44)$$

So the 1-loop cosmological constant vanishes in these theories.

• If the partition function is exactly zero, why should we bother about whether it is modular invariant or not?? The key observation is that modular invariance of the vacuum amplitude (without use of abstruse identities) guarantees that other more complicated amplitudes (with external legs) are also invariant under large coordinate reparametrizations on the worldsheet.

• Recall that the contribution $Z(\tau)$ must be integrated over the fundamental domain in τ to get the complete contribution. As discussed in the bosonic theory, the ultraviolet region is related, namely is equivalent geometrically, to the infrared region. A difference with the bosonic theory is that the type II superstrings do not contain tachyons, so there are no infrared divergences.

2 Type 0 superstrings

We would like to discuss (the only) other possible modular invariant partition functions that one can construct with the basic building blocks we have, namely the 2d fields of the (2d supersymmetric) strings. Interestingly enough, the theories we are about to construct, called type 0 theories, are *not* spacetime supersymmetric, and moreover do *not* contain spacetime fermions. So they clearly illustrate the fact that 2d fermions/susy do not guarantee spacetime fermions/susy.

The complete left times right partition function is given by

$$Z_{\pm} = \frac{1}{2}(4\pi^2\alpha'\tau_2)^{-4}\,|\eta|^{-16}|\eta|^{-8}\left(\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^8 + \left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^8 + \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^8 \pm \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^8\right) \quad (45)$$

We obtain two new inequivalent theories, whose structure in the fermionic partition function is

$$\begin{aligned} &\mathrm{tr}_{\mathcal{H}_{NS,GSO_+}}\mathrm{tr}^*_{\mathcal{H}_{NS,GSO_+}} + \mathrm{tr}_{\mathcal{H}_{NS,GSO_+}}\mathrm{tr}^*_{\mathcal{H}_{NS,GSO_-}} + \\ &+\mathrm{tr}_{\mathcal{H}_{R,GSO_+}}\mathrm{tr}^*_{\mathcal{H}_{NS,GSO_\pm}} + \mathrm{tr}_{\mathcal{H}_{R,GSO_-}}\mathrm{tr}^*_{\mathcal{H}_{R,GSO_\mp}} \end{aligned} \quad (46)$$

The lightest modes of the two theories are

Type 0A

Sector	States	$SO(8)$	$\alpha' m^2$	Fields
NS-NS	$1 \otimes 1$	1	-2	T
	$8_V \otimes 8_V$	$1 + 28_V + 35_V$	0	$\phi,\, B_2,\, G_{\mu\nu}$
R-R	$8_C \otimes 8_S$	$8_V + 56_V$	0	$A_1,\, C_3$
	$8_S \otimes 8_C$	$8_V + 56_V$	0	$A_1',\, C_3'$

Type 0A

Sector	States	$SO(8)$	$\alpha' m^2$	Fields
NS-NS	$1 \otimes 1$	1	-2	T
	$8_V \otimes 8_V$	$1 + 28_V + 35_V$	0	$\phi,\, B_2,\, G_{\mu\nu}$
R-R	$8_C \otimes 8_C$	$1 + 28_C + 35_C$	0	$a,\, \tilde{B}_2,\, A_4^+$
	$8_S \otimes 8_S$	$1 + 28_S + 35_S$	0	$a',\, \tilde{B}_2',\, A_4^-$

The theories contain a tachyon in the NS-NS sector. As usual, one interprets the tachyon as an instability of the theory, which is sitting at the top of some potential for the corresponding field. There are many speculations on what is the stable vacuum of type 0 theories, and even whether it exists or not. The issue remains for the moment as an open question.

Due to this feature, and to lack of fermions, most research is centered on type II strings, rather than type O.

3 Bosonization

We would like to finish with some comments on bosonization. Bosonization/fermionization is a phenomenon relating certain two-dimensional field theories; it is the complete physical equivalence of a 2d quantum field theory with bosonic degrees of freedom and one with fermionic degrees of freedom. This can happen two dimensions since all representations of the $SO(2)$ Lorentz group are one-dimensional, there is no real concept of spin.

For our simplified discussion, we will be interested in discussing simply the equivalences of partition functions of the corresponding 2d theories. But let us emphasize that bosonization/fermionization is complete equivalence of all physical quantities in both theories). Notice however that equivalence of partition functions implies a one-to-one map between states in the two Hilbert spaces, and agreement in their energies.

A simple example of bosonization/fermionization is that the 2d theory of two left-moving free fermions (with NS boundary conditions on the circle) is equivalent to the 2d theory of one left-moving boson compactified on a circle of radius $R = \sqrt{\alpha'}$. Indeed, let us compute the partition function of the theory with two fermions

$$Z_{2\psi} = \left[q^{-1/48} \prod_{n=1}^{\infty} (1 - q^{n-1/2}) \right]^2 = \frac{\vartheta \left[\begin{array}{c} 0 \\ 0 \end{array} \right] (\tau)}{\eta(\tau)} \qquad (47)$$

This final expression can be rewritten using (54) as

$$\frac{1}{\eta\tau} \sum_{n\in\mathbf{Z}} q^{n^2/2} \qquad (48)$$

which corresponds to the partition function of one left-moving boson parametrizing a compact direction of radius $\sqrt{\alpha'}$. The η corresponds to the trace over the oscillator degrees of freedom, while the sum over n corresponds to the sum over left-moving momentum p_L. Finally, purely left-moving bosons with no right-moving partner have no center of mass degrees of freedom, so there is no trace over center of mass momentum. Some of these issues will appear back in the study of the heterotic.

Using this kind of computations, it is possible to bosonize the complete left-moving sector of a type II superstring. Indeed it is possible to recast the left-moving fermion partition function in terms of a bosonic interpretation. In fact, starting with the GSO projected fermionic partition function

$$Z_{\pm} = \frac{1}{2} \eta^{-4} \left(\vartheta \left[\begin{array}{c} 0 \\ 0 \end{array} \right]^4 - \vartheta \left[\begin{array}{c} 0 \\ 1/2 \end{array} \right]^4 - \vartheta \left[\begin{array}{c} 1/2 \\ 0 \end{array} \right]^4 \pm \vartheta \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right]^4 \right) \qquad (49)$$

and writing the ϑ functions as infinite sums, we obtain

$$Z_\pm = \frac{1}{2}\eta^{-4}\left(\sum_{n_1,n_2,n_3,n_4} q^{\sum_i n_i^2} - \sum_{n_1,n_2,n_3,n_4} q^{\sum_i n_i^2} e^{\pi i \sum_i n_i} - \right.$$
$$\left. - \sum_{n_1,n_2,n_3,n_4} q^{\sum_i (n_i+1/2)^2} \pm \sum_{n_1,n_2,n_3,n_4} q^{\sum_i (n_i+1/2)^2} e^{\pi i \sum_i (n_i+1/2)}\right)$$

By gathering terms we may write

$$Z_\pm = \eta^{-4}\left(\sum_{\vec{r}=(n_1,n_2,n_3,n_4)} q^{\vec{r}^2}\frac{1}{2}(1-(-1)^{\sum_i n_i}) - \sum_{r=(n_1+1/2,\ldots,n_4+1/2)} q^{\vec{r}^2}\frac{1}{2}(1\pm(-1)^{\sum_i n_i})\right)$$

Defining lattices $\Lambda^\pm$ of vectors of the form

$$(n_1,n_2,n_3,n_4) \quad ; \quad n_i \in \mathbf{Z} \quad ; \quad \sum_{\mathbf{i}} \mathbf{n_i} = \text{odd} \tag{50}$$
$$(n_1+\tfrac{1}{2}, n_2+\tfrac{1}{2}, n_3+\tfrac{1}{2}, n_4+\tfrac{1}{2}) \quad ; \quad n_i \in \mathbf{Z} \quad ; \quad \sum_{\mathbf{i}} \mathbf{n_i} = \text{odd, even for } \mathbf{\Lambda^+},\ \mathbf{\Lambda^-}$$

we can write

$$Z_\pm = \eta^{-4} \sum_{r\in\Lambda^\pm} q^{r^2} \tag{51}$$

Which corresponds to the partition function of four left-moving bosons parametrizing a four-torus defined by the lattice $\Lambda^\pm$. Recall that this is not a fake trick, but a complete physical equivalence of 2d theories.

We will not use much this bosonic description. However, it is sometimes used in discussing more complicated models, like orbifolds, since it provides an easy bookeeping of the GSO projections in terms of a lattice.

A Appendix: some useful modular function

We introduce the eta and theta functions

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{52}$$

$$\vartheta\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau) = \eta(\tau)\, e^{2\pi i\,\theta\phi}\, q^{\frac{1}{2}\theta^2-\frac{1}{24}} \prod_{n=1}^{\infty}(1+q^{n+\theta-1/2}\, e^{2\pi i\phi})\,(1+q^{n-\theta-1/2}\, e^{-2\pi i\phi})$$

For particular interesting values of θ, ϕ we have

$$\begin{aligned}
\vartheta\left[\begin{array}{c}0\\ 0\end{array}\right](\tau) &= \prod_{n=1}^{\infty}(1-q^n)\,(1+q^{n-1/2})^2\\
\vartheta\left[\begin{array}{c}0\\ 1/2\end{array}\right](\tau) &= \prod_{n=1}^{\infty}(1-q^n)\,(1-q^{n-1/2})^2\\
\vartheta\left[\begin{array}{c}1/2\\ 0\end{array}\right](\tau) &= q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\,(1+q^n)\,(1+q^{n-1}) \;=\\
&= 2\,q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\,(1+q^n)^2\\
\vartheta\left[\begin{array}{c}1/2\\ 1/2\end{array}\right](\tau) &= i\,q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^2\,(1-q^{n-1}) \;=\; 0
\end{aligned} \tag{53}$$

They have easy modular transformation properties

$$\begin{aligned}
\vartheta\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau+1) &= e^{-\pi i(\theta^2-\theta)}\,\vartheta\left[\begin{array}{c}\theta\\ \theta+\phi-1/2\end{array}\right](\tau)\\
\vartheta\left[\begin{array}{c}\theta\\ \phi\end{array}\right](-1/\tau) &= (-i\tau)^{1/2}\,\vartheta\left[\begin{array}{c}\phi\\ -\theta\end{array}\right](\tau)\\
\eta(\tau+1) &= e^{\pi i/12}\,\eta(\tau)\\
\eta(-1/\tau) &= (-i\tau)^{1/2}\,\eta(\tau)
\end{aligned} \tag{54}$$

In particular

$$\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau+1)=\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau) \quad ; \quad \vartheta\begin{bmatrix}0\\0\end{bmatrix}(-1/\tau)=(-i\tau)^{1/2}\,\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau)$$

$$\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau+1)=\vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau) \quad ; \quad \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(-1/\tau)=(-i\tau)^{1/2}\,\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau)$$

$$\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau+1)=e^{-\pi i/4}\,\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(\tau) \quad ; \quad \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(-1/\tau)=(-i\tau)^{1/2}\,\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(\tau)$$

$$\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\tau+1)=e^{-\pi i/4}\,\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\tau) \quad ; \quad \vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(-1/\tau)=i(-i\tau)^{1/2}\,\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\tau)$$

Finally we will need the expression of the ϑ functions as infinite sums

$$\vartheta\begin{bmatrix}\theta\\\phi\end{bmatrix}(\tau) \;=\; \sum_{n\in\mathbf{Z}} q^{(n+\theta)^2/2}\,e^{2\pi i\,(n+\theta)\phi} \tag{55}$$

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