D-branes and gauge field theories

1 Motivation

String theory in the presence of D-branes contains sectors of gauge interactions (open string sectors). The strength of gauge and gravitational interactions in these setups is different [1], making it possible to switch off gravitational (and other closed string) interactions, while keeping the gauge sectors interacting. This can be done essentially by taking a low energy limit in the configuration. In the limit, the dynamics of the open string sector of the theory reduces to a gauge field theory.

Hence, string theory is able to reproduce the richness of gauge field theory. The idea is to use string theory to explore the dynamics of gauge field theories; for instance, study non-perturbative effects in gauge theories by exploiting what we already know about the non-perturbative dynamics in string theory. In order to do so, we must center on theories with enough supersymmetry. In this talk we center on theories with 16 supersymmetries, and four-dimensional gauge sectors (i.e. we center on configurations of parallel D3-brane in type IIB string theory).

[1] This is not true in heterotic models, for instance.

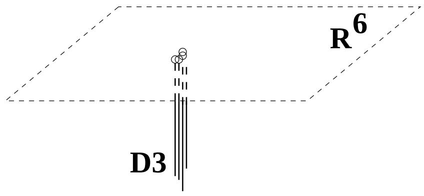

Figure 1: Stack of coincident D3-branes in flat space..

2 D3-branes and 4d $\mathcal{N} = 1$ $U(N)$ super Yang-Mills

2.1 The configuration

Consider a stack of N coincident type IIB D3-branes in flat 10d space, see figure 1. The open string spectrum contains massless modes corresponding to 4d $\mathcal{N} = 4$ $U(N)$ super Yang-Mills, propagating on the 4d D3-brane world-volume. This includes $U(N)$ gauge bosons, four Majorana fermions λ_r, $r = 1, \ldots, 4$ in the adjoint representation, and six real scalars ϕ_m, $m = 1, \ldots, 6$ in the adjoint representation. The configuration also contains massive open string modes, and massless and massive closed string modes.

Let us consider the limit of very low energies, or equivalentely of very large string scale (i.e. we take the limit $E/M_s \to 0$). In this limit, all massive string modes (open or closed) decouple, and moreover all interactions of massless closed string modes (which are controlled by 10d Newton's constant $\simeq M_s^{-8}$ go to zero. Interactions for massless open string modes, however, are controlled by the dilaton vev g_s and remain non-trivial. The whole dynamics of the configuration reduces to $\mathcal{N} = 4$ $U(N)$ gauge field theory. In $\mathcal{N} = 1$ supermultiplet language we have a vector multiplet V and three chiral multiplets Φ_i, $i = 1, 2, 3$, with action

$$S_{YM} = \int d^4x \left[\int d^2\theta \tau \mathrm{tr}\,(W_\alpha W_\alpha) + \int d^4\theta \sum_i \mathrm{tr}\, \Phi_i e^V \Phi_i^\dagger + \int d^2\theta \mathrm{tr}\,(\epsilon_{ijk}\Phi_i\Phi_j\Phi_k) \right] \quad (1)$$

where W_α is the field strength chiral multiplet, and $\tau = \theta + i/g_{YM}^2 = a + i/g_s$. This leads to the kinetic terms for gauge bosons, matter fields, and to the scalar potential proportional to square of the modulus of the commutators of scalar fields. For most of the discussion we center on $\theta = 0$, $a = 0$.

2.2 The dictionary

We can now establish a dictionary between properties of the gauge field theory and properties of the D3-brane configuration in string theory. A first example already described is that the complex gauge coupling constant corresponds in the underlying string theory to the complex IIB coupling constant. Also, for instance, the $SU(4) = SO(6)$ R-symmetry of $\mathcal{N} = 4$ super Yang-Mills theory, which acts on the four 4d spinor supercharges, corresponds in the underlying string theory to the $SO(6)$ rotational symmetry in the $\mathbf{R^6}$ transverse to the D3-branes.

The dictionary become particularly interesting in discussing the so-called Coulomb branch. The $\mathcal{N} = 4$ $U(N)$ gauge theory has a moduli space of vacua, parametrized by the vevs of the scalar fields. For these vevs to minimize the scalar potential, the vevs for the real scalar fields $\langle\phi_m\rangle$ should be $N \times N$ commuting matrices. Then they can be diagonalized simultaneously, with real eigenvalues $v_{m,a}$, $a = 1, \ldots, N$, namely

$$\langle\phi_m\rangle = \begin{pmatrix} v_{m,1} & & \\ & \cdots & \\ & & v_{m,N} \end{pmatrix} \quad (2)$$

The gauge symmetry in this vacuum is broken to $U(1)^N$, if the vevs are generic. In all cases, $\mathcal{N} = 4$ supersymmetry is unbroken in these vacua, so we have full $\mathcal{N} = 4$ vector multiplets of $U(1)^N$. In fact, each $U(1)$ gauge boson (refered to as the a^{th} $U(1)$) is associated with six massless scalars, which correspond to the moduli associated to the a^{th} set of vevs $v_{m,a}$.

States of the theory with electric charges $+1$, -1 under the a^{th} and b^{th} $U(1)$'s acquire a mass

$$M_{ab} = g_{YM} |\vec{v}_a - \vec{v}_b| \tag{3}$$

where $\vec{v}_a$ is a 6d vector with components $v_{m,a}$. This arises from the Higgs mechanism for gauge bosons, from the scalar potential for scalars, and from Yukawa couplings for fermions.

There are enhanced $U(n)$ gauge symmetries in the non-generic case when n of the $\vec{v}_a$ are equal. That is, the corresponding charged vector multiplets become massless.

In the underlying string picture, the moduli space of vacua corresponds to the moduli space of D3-branes. There exists a continuous set of configurations, corresponding to the choice of locations of the D3-branes in the transverse space $\mathbf{R^6}$ [2]. Labelling the D3-branes by a (Chan-Paton) index $a = 1, \ldots, N$, the configurations are described by the locations $r_{m,a}$ of the a^{th} D3-branes in the coordinate x^m in $\mathbf{R^6}$. All these configurations are $\mathcal{N} = 4$ supersymmetric. As will become clear in a moment, they correspond precisely the vacua of the $\mathcal{N} = 4$ gauge theory, via the relation $v_{i,a} = r_{i,a}/\alpha'$.

In this configuration, the gauge symmetry on the D3-branes is broken generically to $U(1)^N$, since only aa open strings are massless. If n D3-branes are located at coindicent positions in $\mathbf{R^6}$, their gauge symmetry is enhanced

[2]In order to allow for the possibility of branes separated in transverse space without decoupling them from each other in the low energy limit discussed above, the limit should also rescale the distance in transverse space $\mathbf{R^6}$. This will be discussed more carefully in section 2.

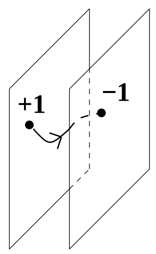

Figure 2: In a system of parallel D3-branes, there are BPS states obtained as minimal length 1-branes (fundamental strings or otherwise) suspended between the a^{th} and the b^{th} D3-brane.

to $U(n)$. Furthermore, states with charges $+1$, -1 under the a^{th} and b^{th} $U(1)$'s correspond to ab open strings, see figure 2. Their lightest modes have a mass

$$\alpha' M_{ab}^2 = \frac{|\vec{r}_a - \vec{r}_b|^2}{\alpha'} \tag{4}$$

in the string frame. Going to the Einstein frame, there is a rescaling of energies by by $\sqrt{g_s}$, so we get

$$M_{ab} = g_s^{1/2} \frac{|\vec{r}_a - \vec{r}_b|}{\alpha'} \tag{5}$$

which is in precise agreement with (3). That is, we can match the spectrum of electrically charges states in the gauge theory from the set of fundamental open strings stretched between the D3-branes.

As we discussed in the $\mathcal{N} = 4$ field theory appendix on the lecture of non-perturbative states, there are other BPS states in the $\mathcal{N} = 4$ gauge theory. In particular, each $SU(2)$ subgroup of the $U(N)$ is spontaneously broken to $U(1)$ in the Coulomb branch. Within each $SU(2) \to U(1)$ sector one can construct non-perturbative 't Hooft-Polyakov monopole states, carrying a magnetic charge under the corresponding $U(1)$, and mass proportional to the gauge symmetry breaking vev. More precisely, for the pair given by the a^{th} and b^{th} $U(1)$'s we have monopole states with charges $\pm(1,-1)$ under $U(a)_a \times U(1)_b$, and mass given by

$$M_{ab} = \frac{|\vec{v}_a - \vec{v}_b|}{g_{YM}} \tag{6}$$

These states are BPS, etc, which guarantees that the abvove formula is exact quantum mechanically. Note that they are not charged under the diagonal $U(1)$, which is the extra $U(1)$ factor of $U(2)$ not in $SU(2)$, so we recover the $SU(2) \to U(1)$ monopole.

In the underlying string picture, there are states with these properties, corresponding to (open) D1-branes suspended between the a^{th} and b^{th} D3-branes. It is possible to see that they have the correct charges [3]. Their mass is given by its length times the D1-brane tension. In the Einstein frame,

$$M_{ab} = g_s^{1/2} \frac{|\vec{r}_a - \vec{r}_b|}{\alpha'^{1/2}} \frac{\alpha^{-1/2}}{g_s} = \frac{1}{g_s^{1/2}} \frac{|\vec{r}_a - \vec{r}_b|}{\alpha} \tag{7}$$

in agreement with (6).

Some comments are in order:

• It is possible to understand that the D1-brane states are supersymmetric, by analyzing the directions along which the D3- and D1-branes stretch. For instance, for D3-brane separated just along x^4, we have

[3]This would require describing the effect of the D1-brane pulling on the D3-brane. Such configurations are described by the so-called BIon solutions of the Dirac-Born-Infeld action on the D3-brane worldvolume: Intuitively, the D1-brane pulls the D3, so that the coordinate of the D3-brane vary as one moves away from the D1-brane endpoint. To keep energy finite, one must switch on the world-volume gauge field. The final configuration is such that there is net flux of F around a 2-sphere surrounding, at spatial infinity, the D1-brane endpoint, which thus corresponds to a magnetic monopole.

	0	1	2	3	4	5	6	7	8	9
D3	-	-	-	-	×	×	×	×	×	×
D3	-	×	×	×	-	×	×	×	×	×

The number of DN (Dirichlet Neumann) and ND directions is four, which corresponds to a supersymmetric situation (note that this would be a proof if the D1-branes were infinitely extended; since they are of finite extent, the argument is just heuristic, but gives the right answer).

• Open strings with endpoints on the D1-branes give rise to fields localized on the latter. The massless sector indeeed corresponds to the zero modes of the gauge theory monopole: bosons associated to the monopole position, and fermions due to the supersymmetries broken by the monopole state.

• In fact, the string theory configuration tells us that there are infinitely many BPS states associated to each pair of D3-branes, corresponding to (p,q) strings suspended between them. They must also exist in the gauge field theory, where they are known as dyons, which carry p and q units of electric and magnetic charge. They can be directly searched in the $\mathcal{N}=4$ field theory, and have been constructed for particular values of (p,q). The masses of these states, (obtained in string theory language and transaled) is (for general τ)

$$M^2 = |\vec{v}_a - \vec{v}_b| \frac{1}{\Im \tau} |p + \tau q|^2 \tag{8}$$

• Note that in the limit of coincident D3-brane positions / coincident vevs (this is known as the origin in the Coulomb branch), the theory has massless electrically charged states, but also massless monopoles, and masless dyons. The $\mathcal{N}=4$ theory at the origin in the Coulomb branch is highly non-trivial! It is only simple in perturbation theory, where all non-perturbative states are infinitely massive.

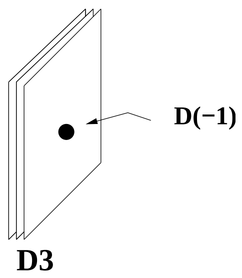

Figure 3: A D(−1)-brane on a D3-brane corresponds to an instanton on the 4d gauge field theory on the D3-brane world-volume.

Finally, we would like to point out another small piece of the dictionary. The $\mathcal{N}=4$ gauge theory has instantons, which are field configurations in the euclidean version of the theory. In the underlying string language, they are described by D(−1)-branes, which are D-branes localized in all directions in the euclidean version of the string theory. See figure 3. This agrees with the familiar fact that a D$(p-4)$-brane on the volume of a Dp-brane behaves as an instanton (recall that an instanton carries D$(p-4)$-brane charge due to the WZ worldvolume coupling on the Dp-brane).

2.3 Montonen-Olive duality

As we discussed for $SU(2)$ in the $\mathcal{N}=4$ field theory appendix on the lecture of non-perturbative states, there is a non-perturbative exact $SL(2,\mathbf{Z})$ symmetry of $\mathcal{N}=4$ $U(N)$ super Yang-Mills theory, acting non-trivially on the complex gauge coupling, and exchanging the roles of perturbative and non-perturbative states.

This is easily derived from the underlying string picture. The D3-brane configuration is invariant under the exact non-perturbative $SL(2,\mathbf{Z})$ symmetry of type IIB theory, exchanging the roles of the different (p,q)-strings. This implies that the 4d $\mathcal{N}=4$ gauge field theory inherits this as an exact symmetry, which exchanges the roles of the different electrically charged states, monopoles and dyons of the theory. Namely

Type IIB D3-branes	$\mathcal{N}=4$ **gauge theory**
$\tau \to \frac{a\tau+b}{c\tau+d}$	$\tau \to \frac{a\tau+b}{c\tau+d}$
(p,q)-string $\to$ (p',q')-string	(p,q)-dyon $\to$ (p',q')-dyon

with $\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$.

As usual, this implies that e.g. the strong coupling dynamics of $\mathcal{N}=4$ $U(N)$ super Yang-Mills is described by a dual weakly coupled $U(N)$ gauge theory where the perturbative degrees of freedom (electrically charged states in dual theory) are the original magnetic monopoles.

Montonen-Olive $SL(2,\mathbf{Z})$ duality arises from type IIB $SL(2,\mathbf{Z})$ duality in this setup.

2.4 Generalizations

There have been many generalizations of the possibility to study gauge theory phenomena by embedding them in the worldvolume of D-brane configurations in string theory. For a review see [1]. Some further examples and results one can show using string theory tools are

• Montonen-Olive dualities for $\mathcal{N}=4$ gauge field theories with $SO(N)$ or $Sp(N)$ gauge groups, from configurations of D3-branes (and O3-planes) in type IIB string theory.

• For theories with 16, 8 supersymmetries in dimensions $d=5,6$, construction of interacting field theories which correspond to ultraviolet fixed points of the renormalization group (superconformal field theories).

• For 4d theories with 8 supersymmetries (4d $\mathcal{N} = 2$), exact computation of the low energy effective action (up to two derivatives) exactly in g_{YM} (including non-perturbative effects), in agreement with the Seiberg-Witten solution [2].

• A non-perturbative duality for 3d theories with 8 supersymmetries, known as mirror symmetry [4].

• For 4d theories with 4 supersymmetries (4d $\mathcal{N} = 1$), a non-perturbative equivalence in the infrared of theories which are different in the ultraviolet, knonw as Seiberg duality. Also, some qualitative features of $\mathcal{N} = 1$ pure Yang-Mills, like number of vacua, etc.

3 The Maldacena correspondence

In a sense, this is a more precise version of the relation between string theory and gauge field theory. It even allows the quantitative computation of quantities in gauge field theory from the string theory / supergravity point of view. An extensive review is [3].

3.1 Maldacena's argument

This section follows [4].

Consider a stack of N coincident type IIB D3-branes in flat 10d space [5]. The dynamics of the configuration is described by closed strings, and open string ending on the D3-branes. Let us take a careful low energy limit, where we send the string scale to infinity, but keep the energies of 4d field theory excitations finite. One example of such states are open strings stretched

[4]The name is due to some relation with mirror symmetry in type II string theory.

[5]Similar arguments can be repreated for the M theory M2 and M5-branes. For other branes, non-trivial varying dilatons modify the argument substantially.

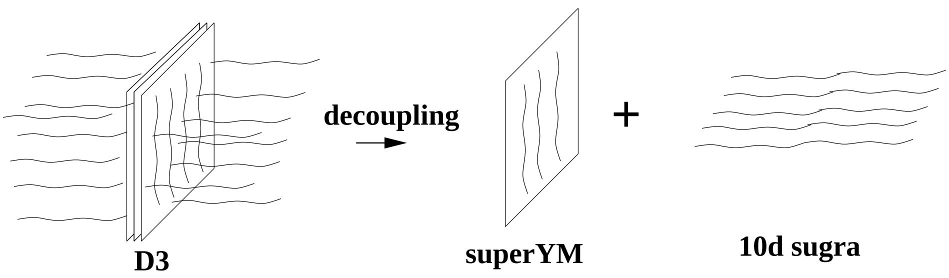

Figure 4: Maldacena's low energy limit in the system of N coincident D3-branes, described as hypersurfaces on which open strings end. In the limit, we obtain $\mathcal{N}=4$ super Yang-Mills gauge field theory, decoupled from free 10d supergravity modes.

between D3-branes separated by a distance r in transverse space (i.e. electrically charged states in the Coulomb branch), which have a mass

$$M^2 = r^2/\alpha'^2 \tag{9}$$

Hence we need to take the limit $\alpha' \to 0$ and $r \to 0$, keeping r/α' finite. In this limit, the theory reduces to two decoupled sectors, one of them is 4d $\mathcal{N}=4$ super Yang-Mills theory, and the other is free 10d gravitons (or better free fields corresponding to the massless closed string modes). See figure 4.

On the other hand, the configuration has an equivalent description, as type IIB string theory in the background created by the stack of D3-branes

$$\begin{aligned} ds^2 &= Z(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z(r)^{1/2}dx^m dx^m \\ e^{2\phi} &= 1/g_s \\ F_5 &= (1+*)dtdx^1dx^2dx^3dZ^{-1} \end{aligned} \tag{10}$$

where $\mu=0,\ldots,3$, $m=4,\ldots,9$, and

$$r=\sum_m (x^m)^2 \quad , \quad Z(r) = 1+\frac{R^4}{r^4} \quad , \quad R^4 = 4\pi g_s \alpha'^2 N \tag{11}$$

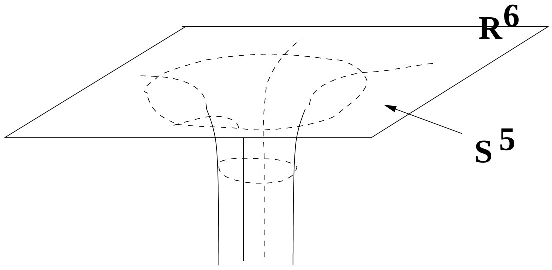

Figure 5: Geometry of the background created by N coincident D3-branes.

We have a gravitational background, pictorially shown in figure 5, and a RR 5-form field strength background, such that there are N units of flux piercing through a 5-sphere surrounding the origin in the transverse 6d space, $\int_{\mathbf{S}^5} F_5 = N$.

Notice that we say that this description is given by *full* string theory in this background, namely we assume that we include all stringy (i.e. α' corrections) and quantum corrections of the background.

We now would like to take the same kind of limit as above. First, it is a low energy limit; this corresponds to sending the string scale to infinity (i.e. $\alpha' \to 0$), keeping energies, as measured by an asymptotic observer in the above spacetime geometry, fixed. Second, we want to take the limit keeping energies of excitations in the near core region $r \simeq 0$ finite. Due to the non-trivial g_{tt} metric component, an excitation of proper energy $E_{(r)}$ localized at r in the radial direction, has an energy

$$E_\infty = E_{(r)} Z(r)^{-1/4} \tag{12}$$

as measured in the reference frame of an observer at infinity. That is, as an excitation approaches $r \simeq 0$, its energies measured in the reference frame of the asymptotic observer suffers a large redshift. For excitations near $r \simeq 0$, the above relation reads

$$E_\infty \simeq E_{(r)} \frac{r}{R} = (E_{(r)} \alpha'^{1/2}) \frac{r}{(g_s N)^{1/4} \alpha'} \tag{13}$$

We want to take $\alpha' \to 0$ keeping E_∞ fixed (large string scale keeping fixed energy) and $E_{(r\to 0)} \alpha'^{1/2}$ finite (finite energy for excitations in the near core region). This corresponds to taking $r \to 0$, $\alpha' \to 0$ keeping r/α' finite.

There are two decoupled sector that survive in this low energy limit. There is a sector of fields propagating in the asymptotically flat region, which suffer no redshift; so the only fields surviving in the low energy limit are the massless 10d supergravity fields, which are free fields in this limit. A second sector corresponds to modes localized in the $r \simeq 0$ region; these fields suffer an infinite redshift, hence modes of arbitrarily large proper energy have small energy measured in the asymptotic reference frame, and survive in the low energy limit. This second sector is described by the full type IIB string theory on the background

$$\begin{gathered} ds^2 = \tfrac{r^2}{R^2} (\eta_{\mu\nu} dx^\mu dx^\nu) + \tfrac{R^2}{r^2} dr^2 \; + \; R^2 d\Omega_5^2 \\ \textstyle\int_{\mathbf{S^5}} F_5 = N \end{gathered} \tag{14}$$

The first two pieces of the metric describe a 5d anti de Sitter space AdS_5, of radius (or rather, length scale) R, while the last terms describes a 5-sphere. Through the latter there are N units of RR 5-form flux. See figure 6.

Since the limit involves $r \to 0$, it is useful to rewrite the above metric in terms of the quantity $U = r/\alpha'$, which remains finite in the limit. We have

$$ds^2 = \alpha' \left[\frac{U^2}{(4\pi g_s N)^{1/2}} (\eta_{\mu\nu} dx^\mu dx^\nu) + (4\pi g_s N)^{1/2} \frac{dU^2}{U^2} \; + \; (4\pi g_s N)^{1/2} d\Omega_5^2 \right] \tag{15}$$

The overall factor of α' simply encodes the fact that we are zooming into the region of small r.

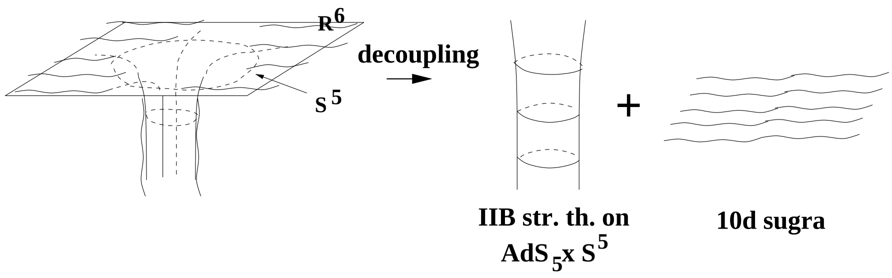

Figure 6: Maldacena's low energy limit of the system of N coincident D3-branes, described by type IIB theory on the D3-brane background. In the limit we obtain full type IIB string theory on the near core limit $\mathrm{AdS}_5 \times \mathbf{S^5}$, decoupled from free 10d supergravity modes.

The Maldacena conjecture is that both descriptions, in terms of gauge field theory (plus 10d free gravitons) and in terms of string theory on the $\mathrm{AdS}_5 \times \mathbf{S^5}$ background (plus 10d free gravitons) are completely equivalent.

We thus propose the complete equivalence of 4d $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills [6] gauge field theory with full fledged type IIB string theory on $\mathrm{AdS}_5 \times \mathbf{S^5}$, with radius R^2/α' given above, and N units of 5-form flux through $\mathbf{S^5}$.

Let us emphasize once again that the equivalence involves full string theory, including all stringy modes, brane states, etc. Again, this is because arbitrarily high energy modes survive in the throat region in the limit.

This correspondence is very striking. It proposes that a string theory (in a particular background) is completely equivalent to a gauge field theory. It is very striking that a theory that includes gravity, and an infinite set

[6] A subtle issue is that the Maldacena gauge/string correspondence holds for the $SU(N)$ group, rather than for $U(N)$. The difference in the large N limit is of order $1/N^2$, and hence only detectable by computing loop corrections.

of fields, can be equivalent to a non-gravitational theory, which in principle looks much simpler. We will see later on how this correspondence works in more detail, although an extensive discussion is beyond the scope of this lecture. Let us also point out that this kind of relation, in the limit of large N, had been proposed by 't Hooft, see appendix.

The dictionary between the parameters of the gauge theory and the string theory are as follows

$\mathcal{N}=4$ $SU(N)$ super Yang-Mills		Type IIB on $\mathrm{AdS}_5 \times \mathbf{S^5}$
$\tau = \theta + i\frac{1}{g^2_{YM}}$		$\tau = a + i\frac{1}{g_s}$
$N =$ number of colors		$N =$ flux
$\lambda = g^2_{YM}N$	$\overset{R^2/\alpha' = 4\pi g^2_{YM}N}{\longleftrightarrow}$	R^2/α'

3.2 Some preliminary tests of the proposal

Some additional support for the above proposal is that the two systems have the same symmetry structure.

• The $SO(6)$ isometry group of $\mathbf{S^5}$ on the string theory side exactly reproduces the $SO(6)$ R-symmetry group of the $\mathcal{N}=4$ gauge field theory. This is analogous to the observation we made in previous section for systems of D3-branes.

• The isometry group of AdS_5 is $SO(4,2)$. This can be seen from the follwoing construction of AdS_5 space. Consider the hypersurface

$$(X^0)^2 + (X^5)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = R^2 \qquad (16)$$

in the 6d flat space with signature $(4,2)$ and metric

$$ds^2 = -(dX^0)^2 - (dX^5)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2 \qquad (17)$$

Clearly the above hyperboloid is a 5d space of signatures $(3,1)$ and isometry group $SO(4,2)$.

Performing the change of variables

$$\begin{aligned} X^0 &= \frac{1}{2u}\left[1 + u^2(R^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - t^2)\right] \\ X^i &= Rux^i \\ X^4 &= \frac{1}{2u}\left[1 - u^2(R^2 - (X^1)^2 - (X^2)^2) - (X^3)^2 + t^2\right] \\ X^5 &= Rut \end{aligned} \tag{18}$$

the metric on the 5d space becomes

$$ds^2 = R^2u^2(-dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2) + \frac{R^2}{u^2}du^2 \tag{19}$$

And redefining $u = U\alpha'/R^2$, we get

$$ds^2 = \alpha'\left[\frac{U^2}{R^2/\alpha'}\left[-dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + \frac{R^2/\alpha'}{U^2}dU^2\right] \tag{20}$$

which is precisely (15). Hence AdS_5 has an isometry group $SO(4,2)$.

This corresponds exactly to the conformal group of the 4d gauge field theory. $\mathcal{N} = 4$ theories at the origin of the Coulomb branch are conformally invariant, even at the quantum level (that is, the beta functions which encode the running of couplings with the scale are exactly zero, so the theory is scale invariant). The $SO(4,2)$ conformal group has a $SO(3,1)$ Lorentz subgroup and a $SO(1,1)$ scale transformations subgroup.

Hence the isometry group of the AdS_5 theory reproduces the conformal group of the 4d gauge field theory [7]. An important fact in this context is that the scale transformations in the gauge field theory correspond to translations in the variable U on the AdS side. Namely, this subgroup acts on the AdS_5 geometry as

$$(t, x^1, x^2, x^3, u) \to (\lambda t, \lambda x^1, \lambda x^2, \lambda x^3, \lambda^{-1}u) \tag{21}$$

[7] Hence the familiar name of AdS/CFT correspondence.

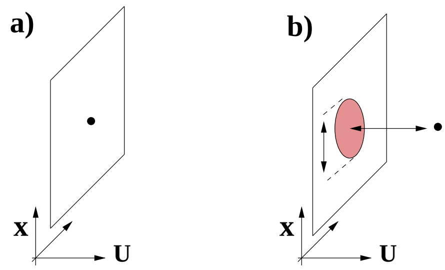

Figure 7: The spread on the boundary of the effect of an excitation in the bulk of AdS spacetime is smaller as the excitation are localized closer to infinity in the U direction (boundary). In terms of the dual gauge field theory, the near boundary region corresponds to the ultraviolet, while the interior corresponds to the infrared.

Moreover, going to small lengths in the gauge field theory corresponds to going to infinity in U in the AdS theory, and vice versa. This is known as the UV/IR correspondence. See figure 7.

• The supersymmetry structure is the same for both theories. The $AdS_5 \times \mathbf{S^5}$ background preserves 32 supercharges. Sixteen of them were present in the full D3-brane solution, but sixteen additional one appear (accidentally) in taking the near core limit. The gauge field theory has also 32 supercharges, sixteen of them are the familiar ones of $\mathcal{N} = 4$ theories, while sixteen additional ones, known as superconformal symmetries, are generated by the previous supersymmetries and conformal transformations.

• There is non-geometric non-perturbative symmetry, which also matches in the two theories. This is the $SL(2, \mathbf{Z})$ self-duality of type IIB string theory, which corresponds to the $SL(2, \mathbf{Z})$ Montonen-Olive self-duality of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills.

It would be interesting to test the proposal beyond a mere matching of the symmetries of the system. However, we do not know how to quantize type

IIB string theory on $AdS_5 \times \mathbf{S^5}$. This is difficult because there is curvature in spacetimes, hence the 2d worldsheet theory is not free (and not exactly solvable, for the moment). In addition, there are RR fields in the background, and this makes the worldsheet theory even more complicated [8].

Therefore we can analyze this system only in the supergravity approximation, i.e. keeping the leading behaviour in α'/R^2. This will be a good approximation for R^2/α' large, when all length scales of the geometry are large, and when the density of RR field strength is small. In the language of the corresponding gauge field theory, this corresponds to the limit of large $\lambda = g_{YM}^2 N$, also known as 't Hooft limit (where the so-called 't Hooft coupling λ is large), see appendix. We also need to restrict to classical supergravity, hence we ignore string loop corrections, and take g_s to be small. This implies that the AdS side is tractable when $N \to \infty$m, $g_s \to 0$, and λ is finite and large. In this limit the gauge field theory is not tractable. As we will see in a moment, although g_s is small the right parameter weighting loops is λ, so at large λ the perturbative expansion breaks down.

On the other hand, the gauge field theory is tractable in the perturbative regime, namely when g_s is small and N is small. In this limit, the string theory has a strongly coupled 2d worldsheet theory, and the supergravity approximation breaks down. Hence, the above correspondence is analogous to the duality relations studied in other lectures. There is an exact equivalence of two different descriptions, but when one of them is weakly coupled and tractable, the other is not.

The usual way in which the Maldacena correspondence is exploited is to consider the classical supergravity limit to compute certain quantities,

[8] It is interesting to point out that a subsector of this theory (corresponding to a so-called Penrose limit), given by type IIB string theory on $AdS_5 \times \mathbf{S^5}$ can be quantized exactly in α' in the light-cone gauge. In this situation it is possible to find stringy effects/states and try to identify them in the gauge theory description, with great success [5].

protected (or expected to be protected) by supersymmetry. These quantities can then be computed in perturbative gauge theory, extrapolated to the 't Hooft limit, and compared with the supergravity result. We will discuss some example in next section. For quantities not protected by any symmetry, the supergravity result need not agree with the perturbative gauge theory result. It can then be regarded as a prediction for the behaviour of that quantity in the 't Hooft limit.

In this setup, the mapping of systematic corrections beyond the 't Hooft limit / classical supergravity limit is as follows

gauge theory side	string theory side
λ corrections	α'/R^2 corrections
$\lambda/N = g_{YM}^2$ corrections	g_s loop corrections

This agrees very nicely with the picture of corrections to the 't Hooft limit in gauge field theories, see appendix.

3.3 AdS/CFT and holography

For these section, see [6] and [7]. These authors have proposed a precise recipe to obtain correlation functions in the super Yang-Mills theory, via a computation in type IIB theory in the $\mathrm{AdS}_5 \times \mathbf{S^5}$ background. Moreover, the proposal leads to a nice interpretation of 'where' the field theory is living, in the AdS picture, For most of the discussion in this section, the essential features arise from the geometry of AdS spaces. This suggests a generalization of the correspondence to a relation between type IIB string theory on $\mathrm{AdS}_5 \times \mathbf{X_5}$ (with $\mathbf{X_5}$ a compact Einstein space) and 4d conformal gauge field theories with lower or no supersymmetry.

AdS_5 space has a conformal boundary at $U \to \infty$, which is 4d Minkowski space M_4 (plus a point). That is, there is a conformally equivalent metric

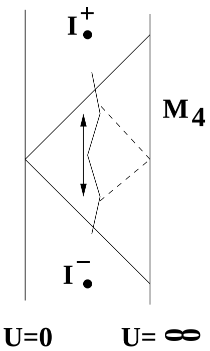

Figure 8: Penrose diagram from AdS spacetimes (only the directions U are shown). Light-rays travel at 45 degrees. The points $I^{\pm}$ are the infinite future/past of timelike lines. A timelike observer can send a light signal to the boundary and get it back in finite proper time.

($ds^2 \to e^{2f(t,x,U)} ds^2$) such that infinity is brought to a finite distance . The Penrose diagram, encoding the causal structure of the AdS space (light-like geodesics run at 45 degrees in the diagram) is shown in figure 8. A remarkable feature of AdS spacetime is that an timelike observer can send a light signal to the boundary of spacetime, and receive the reflected signal a finite amount of time after sending it, see figure 8. This means that, although the boundary of AdS spacetime is at infinity, information on the boundary can interact with information in the bulk within finite time. Hence, AdS_5 behaves as a box of finite size, and this makes it important to specify boundary conditions in order to define any theory on AdS_5.

For instance, the partition function of the theory (namely, the vacuum path integral over all the spacetime fields of the theory on AdS space) is in general a functional of the boundary values ϕ_0 for all the spacetime fields ϕ

of the theory

$$Z_{\text{part.funct.}}[\phi_0] = \int \mathcal{D}(\text{IIB fields})\, e^{-S_{\text{spacetime IIB}}[\text{fields}]} \tag{22}$$

The importance of boundary conditions, along with the fact that the boundary M_4 of AdS_5 spacetime is of the same form as the space on which the gauge field theory lives, motivates the following proposal. Quantities in the gauge field theory on M_4 provide the boundary conditions for fields (of the string theory) propagating on the AdS_5 spacetime. More precisely, the proposal is

• For each field ϕ propagating on AdS_5 there is an operator $\mathcal{O}_\phi$ in the gauge field theory. The field in AdS_5 can be *any* field associated to a 5d state of string theory in $\text{AdS}_5 \times \mathbf{S^5}$, for instance a massless 5d supergravity mode, any mode in the KK reduction of the massless 10d supergravity mode, any massive 10d string state, or even any state from the non-perturbative sectors of the type IIB string theory. The properties of $\mathcal{O}_\phi$, ϕ, like their behaviour under the symmetries of the systems, are related as we discuss a bit later.

• The value ϕ_0 of ϕ at the boundary at infinity

$$\phi_0(t,x) = \text{lim}_{U\to\infty}\phi(t,x,U) \tag{23}$$

is a function on M_4. This value acts as a source for the corresponding operator $\mathcal{O}_\phi$ in the field theory in M_4, namely, its lagrangian includes a term $\Delta\mathcal{L} = \phi_0\mathcal{O}_\phi$. Equivalently, a term in the lagrangian of the gauge field theory (given by a linear combination of operators, with some coefficients) corresponds to introducing specific boundary conditions for the corresponding fields in AdS space. Hence, the field theory data can be regarded as encoded in the boundary of AdS space, and as providing boundary conditions for string theory in AdS space.

• Correlation functions of operators $\mathcal{O}_\phi$ in the gauge field theory can be computed by taking functional derivatives of a generating functional

$Z_{\rm gauge}[\phi_0]$, which is a path integral with a source ϕ_0 for the operator

$$Z_{\rm gauge}[\phi_0] = \int \mathcal{D}(\text{gauge th. fields}) e^{-S_{YM}+\phi_0 \mathcal{O}_\phi} \qquad (24)$$

For instance, the two-point correlation function

$$\langle \mathcal{O}_\phi \mathcal{O}_\phi \rangle = \frac{\delta Z[\phi_0]}{\delta\phi_0 \delta\phi_0} \, |_{\phi_0=0} \qquad (25)$$

The proposal is that the partition function $Z_{\rm part.func.}[\phi_0]$ of IIB theory on AdS_5 with boundary conditions ϕ_0 for the 5d field ϕ (this for all fields of the theory), corresponds exactly to the generating functional $Z_{\rm gauge\, th}[\phi_0]$ of the gauge field theory, with ϕ_0 as source term for the corresponding operator $\mathcal{O}_\phi$. That is

$$Z_{\rm part.funct}[\phi_0] = Z_{\rm gauge\, th}[\phi_0] \qquad (26)$$

This is a precise correspondence that allows to encode all the dynamics of string theory on AdS in the dynamics of gauge field theory, and vice versa.

The above proposal can be used to obtain a relation between the mass m of a 5d field ϕ in string theory AdS_5 (which appears in the computaion of the partition function in the free field approximation) and the conformal dimension Δ of the corresponding operator $\mathcal{O}_\phi$ in the gauge field theory (which appears in the two-point correlation function). The relation, for a p-form field in AdS_5, reads

$$(\Delta + p)(\Delta + p - 4) = m^2 \qquad (27)$$

One can verify this matching by considering operators whose conformal dimensions are protected by supersymmetry. For instance, chiral operators are operators which belong to chiral multiplets when the $\mathcal{N} = 4$ theory is written in terms of the $\mathcal{N} = 1$ subalgebra. For instance, chiral operators are $\mathrm{Tr}\,(\Phi_{i_1} \ldots \Phi_{i_r})$, or $\mathrm{Tr}\,(W_\alpha W_\alpha \Phi_{i_1} \ldots \Phi_{i_r})$. Chiral operators are BPS like, in the sense that they belong to shorter multiplets, and their conformal dimensions is related to their R-charge. The conformal dimensions can then be computed in the perturbative Yang-Mills theory (small g_s, small λ), and then extrapolated and compared with the masses of (BPS) states in the string theory side. These states are easy to identify and correspond to the KK reduction on $\mathbf{S^5}$ of massless 10d supergravity modes. The perfect matching between towers of KK modes in AdS_5 and infinite sets of opertors in the gauge theory is a strong check of the correspondence.

Beyond these kind of checks, which have been extended in several directions, there are other qualitatively different checks that we would like to mention

• There is a precise recipe to compute Wilson loops in the gauge field theory (expectation values of operators given by path ordered integrals of the gauge field over a circuit C in 4d) from the string theory side, as the action of a minimal area worldsheet asymptoting to the circuit C as it approaches the boundary at infinity [8].

• Some D-brane states in string theory on $AdS_5 \times \mathbf{S^5}$ have been identified to operators in the gauge theory. For instance, a D5-brane wrapped on $\mathbf{S^5}$ has been shown to correspond to a baryonic operator in the gauge field theory [9].

• Taking a particular limit of the correspondence, which amounts to centering on a subsector of states/operators with large $SO(6)$ quantum numbers, a complete matching of stringy states and operators has been carried out [5]. On the string theory side, the limit reduces to string theory on a pp-wave background, which can be quantized exactly in α'.

3.4 Implications

We would like to conclude by mentioning some implications of this far-reacing correspondence

• It is a holographic relation! A theory with gravity in 5d is described in terms of a non-gravitational theory with degrees of freedom in 4d. This has deep implications for instance on question like the information problem in black holes in AdS space. The correspondence with gauge theory allows (in principle, although in practice it is not known how to do it) to describe the process of creation and evaporation of a black hole purely in terms of a manifestly unitary quantum field theory. Hence, no violation of the rules of quantum mechanics is involved.

• The correspondence provides a complete non-perturbative definition of string/M theory, in a particular background. This certainly changes the way to think about string theories. It is however difficult to extract the main physical principles to allow to develop a background independent definition.

• The correspondence and its generalizations provides a new powerful tool to analyze gauge field theories in the 't Hooft limit using supergravity duals. In particular it has been possible to describe non-conformal theories in these terms, for instance by finding the field profiles that must be introduced in supergravity to describe the introduction of mass terms for some of the matter fields of the $\mathcal{N} = 4$ theories. The gauge theories are a small perturbation of $\mathcal{N} = 4$ in the ultraviolet, and flow to interacting non-conformal theories in the infrared, sometimes showing interesting behaviour like confinement, etc. In the supergravity side, the solutions are asymptotically AdS near the boundary at infinity, and are deformed in the inside. The structure in the inside region reproduces the infrared features of the gauge field theory. See figure 9. For instance, confinement in the gauge field theory is usually associated to the presence of a black hole in the interior of the 5d space (see

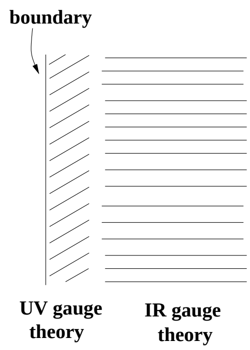

Figure 9: Rough holographic picture for non-conformal gauge field theories.

e.g. [10]).

The gauge/string correspondence is one of the deepest recent results in string theory and gauge theory. A lot of research is devoted to gaining a better understanding of the lessos it has for us concerning the nature of string theory, of holography, and of a new language to describe gauge field theory phenomena.

A Large N limit

The Maldacena correspondence fits well with 't Hooft's proposal that the large N limit of gauge field theories seems to be described by a string theory [9].

The main observation is that in a $SU(N)$ gauge theory, the effective

[9] 't Hooft was interested in QCD, and hence on non-supersymmetric and confining pure gauge theories, where the string is supposed to correspond to the confined gauge field fluxlines. The AdS/CFT has shown that similar ideas actually extend (often in a subtle way) to non-confining theories as well.

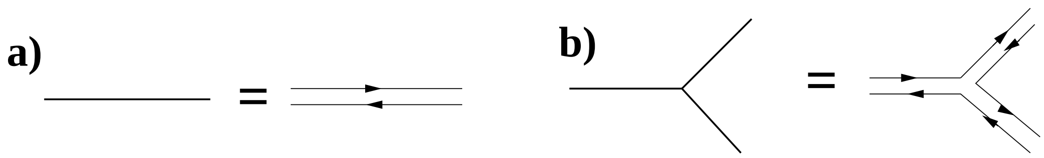

Figure 10: Propagator and 3-point interaction vertex in double line notation.

coupling constant is not g_{YM}^2, but $\lambda = g_{YM}^2 N$. The factor of N intuitively takes into account that the number of particles running in loops increases as we increase the number of colors. Hence in the large N limit perturabation theory breaks down, no matter how small g_{YM}^2 is. However, 't Hooft realized that there are additional simplifications in this limit, of large N keeping λ finite, that suggests it might have a simple description in terms of a dual string theory. Namely in this 't Hooft limit, for any amplitude the Feynmann expansion chan be recast as a double expansion in λ and $1/N$.

To understand this, let us introduce the double line notation, where in a Feynmann diagram a field in the adjoint representation is drawn as a pair of oppositely oriented arrows (can be thought of as representing degrees of freedom in the fundamental and antifundamental representations), see figure 10. One can classify diagrams according to its number of vertices V, external lines E, and closed loops of lines F. From (1) each vertex is weighted by N/λ, while each propagator is weighted by λ/N, while each loop of lines gives a factor of N. Each diagram is therefore weighted by a factor

$$N^{V-E+F}\,\lambda^{E-V} \tag{28}$$

The number $\xi = V - E + F$ is known as the Euler number of the diagram, and g, defined by $\xi = 2 - 2g$, is knons as the genus of the diagram. We have

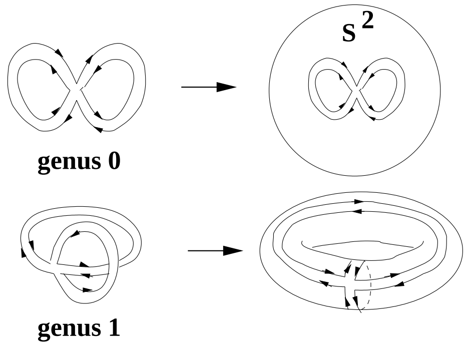

Figure 11: Two two-loop diagrams at genus 0 and 1.

the double expansion

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} \lambda^i \, c_{g,i} \tag{29}$$

The genus has a geometric intrepretation. A diagram of genus g if such that it can be drawn in double line notation without crossings on a Riemann surface of genus g, and cannot be drawn in a surface of genus $g-1$. In figure 11 we show two-loop diagrams of genus 0 and 1.

In the large N limit, keeping λ fixed, any amplitude has a genus expansion [10] in $1/N$. Hence, the large N limit is dominated by the so-called planar diagrams, which correspond to genus 0. This limit corresponds to a weakly coupled string theory, which is dominated by the genus 0 terms. The expansion in $1/N$ is supposed to reproduce the genus expansion of the dual string theory. Geometrically, this corresponds to 'filling the holes' of the gauge theory diagram in the double line notation to form the corresponding Rie-

[10]Note that, since λ is fixed, one can recast the series as an expansion in $\lambda/N = g_{YM^2}$, which becomes the coupling constant of the string theory (in fact, it is g_s in the AdS/CFT case).

mann surface. This has been physically understood in a related gauge/string duality context in [11].

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