# Non-perturbative states in string theory

Some useful references for this lecture are [1, 2, ?].

## 1 Motivation

We have studied the main properties of string theory within the framework of perturbation theory. We have uncovered very interesting formal properties of the theory, and potential applications for model building of unified theories of gauge and gravitational interactions.

In the following lectures we start reviewing several results of the recent years on the structure of string theory beyond perturbation theory. This is important i) to obtain information perhaps eventually leading to a non-perturbative formulation of string theory, and ii) to determine non-perturbative effects which may be important even at weak coupling.

In particular in this lecture we describe certain important non-perturbative states in string theory (the so-called $p$-branes), their properties, and their implications for string theory at the non-perturbative level (for instance, duality properties, etc).

## 2 $p$-branes in string theory

Non-perturbative states are states in the theory which do not have a perturbative description, i.e. they do not correspond to oscillation states of the string. Given that there is no definition of string theory beyond perturbation theory, the main question is how to look for non-perturbative states.

The main tool to do so is to use the low energy effective theory to construct them. The form of the supergravity effective actions, for large enough number of supersymmetries, is fixed by supersymmetry up to some order in the number of derivatives. Therefore it is valid even at finite coupling, if the energy densities involved are not too large (low energies). We can thus construct field configurations solving the supergravity equations of motion, with the structure of a localized core and asymptoting to flat space. These solutions describe classical excitations over the vacuum of the theory, which is given by flat space. It is useful to regard them as the field background created by a source sitting at the core of the solution. Unfortunately, supergravity is just an effective theory, and is clearly not enough to provide us with a microscopic description of these objects.

First there is the approximation of taking the lowest order in $alpha'$. Solutions will be reliable when the curvature lengths are larger than the string length. Second, there is the approximation of describing the solutions at leading order in $g_s$. However, some reliable information can be extracted from supergravity for some particular classes of solutions. This is the topic of this lecture.

In particular we will center on solutions which preserve some supersymmetry (and correspond to the so-called BPS states), and on properties of the solutions which are protected by supersymmetry. Before entering this discussion, let us describe the different kinds of objects we will deal with.

**Detour on $q$-form gauge fields and charges**

To describe them in a unified way, it will be useful to introduce, for each $(p+1)$-form field $C_{p+1}$ in the theory, with field strength $(p+2)$-form $H_{p+2}$, the corresponding dual $(7-p)$-form $C_{7-p}$ with field strength $(8-p)$-form $H_{8-p}$, defined by $H_{8-p} = *H_{p+2}$.

An object with $p$ spatial dimensions sweeps out a $(p+1)$-dimensional subspace $W_{p+1}$ of spacetime as it evolves in time. Such object is said to be electrically charged under $C_{p+1}$ if the theory contains a coupling $Q \int_{W_{p+1}} C_{p+1}$. The terms containing $C_{p+1}$ in the action are

$$\int_{10d} H_{p+1} \wedge *H_{p+1} + Q \int_{W_{p+1}} C_{p+1} = \int_{10d} C_{p+1} \wedge d * H_{p+1} + Q \int_{10d} C_{p+1} \wedge \delta(W_{p+1}) \quad (1)$$

where $\delta(W_{p+1})$ is the Poincare dual to the cycle $W_{p+1}$, bump $(9-p)$-form with support on $W_{p+1}$. The equation of motion reads

$$dH_{8-p} = Q\delta(W_{p+1}) \quad (2)$$

This implies that the flux of $H_{8-p}$ around a $(8-p)$-sphere surrounding the object in the transverse $(9-p)$-dimensional space is

$$\int_{S^{8-p}} H_{8-p} = \int_{B^{9-p}} dH_{8-p} = Q \int_{B^{8-p}} \delta(W_{p+1}) = Q \quad (3)$$

where $B^{9-p}$ is the interior of the $(8-p)$-dimensional sphere. Similarly, an object with $(7-p)$-dimensional volume $W_{7-p}$ is charged magentically under $C_{p+1}$ if it satisfies

$$\int_{S^{p+2}} H_{p+2} = Q' \quad (4)$$

Notice that this implies that the object couples electically to the dual potentical $C_{7-p}$.

## 2.1 $p$-brane solutions

The main examples of elementary [1] are the D-branes, the NS fivebranes, and the fundamental strings.

**The D$p$-brane**

[1]in the sense that they carry charge under just one $p$-form field

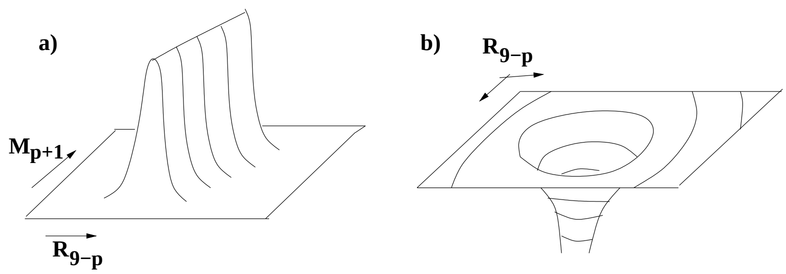



Figure 1: Two pictures of the $p$-brane as a lump of energy. The second picture shows only the transverse directions, where the $p$-brane looks like point-like.

This solution exists in type IIB theory for $p$ odd, in type IIA theory for $p$ even, and in type I theory for $p = 1, 5$; this kind of solution does not exist for heterotic theories.

The solutions (see section 14.8 in [4]) have the form (for $p \leq 6$, so as to have flat space asymptotics)

$$\begin{aligned} ds^2 &= Z(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z(r)^{1/2}dx^m dx^m \\ e^{2\phi} &= Z(r)^{(3-p)/2} \\ Z(r) &= 1 + \frac{\rho^{7-p}}{r^{7-p}} \quad ; \quad \rho^{7-p} = g_s Q \alpha'^{(7-p)/2} \\ H_{8-p} &= \frac{Q}{r^{(8-p)}} d(vol)_{S^{8-p}} \end{aligned} \tag{5}$$

where $\mu = 0, \ldots, p$, $m = p+1, \ldots, 9$, $r = \sum_m |x^m|^2$, and $d(vol)_{S^{8-p}}$ is the volume form of the $(8-p)$-sphere of unit volume.

The above solution has a core given by a flat $(p+1)$ dimensional plane at $r = 0$ and asymptotes to flat 10d space. See figure 1. The core describes an object electrically charged under the RR field $C_{p+1}$, with charge proportional to $Q$. This is very remarkable, since there is no perturbative state in string theory charged under RR fields.

It is possible to compute the tension and charge using standard ADM techniques in gravitational systems, and get the result

$$T_p^2 = \frac{\pi}{g_s^2 \kappa_{10}^2} (4\pi^2\alpha')^{3-p} \quad ; \quad \mu_p^2 = \frac{\pi}{\kappa_{10}^2} (4\pi^2\alpha')^{3-p} \tag{6}$$

Notice that the tension is inversely proportional to the string coupling, so the state is non-perturbative, and is often referred to as soliton.

The solution is invariant under half of the supersymmetries of the vacuum of the theory. It described a so-called BPS state. This implies the particular relation between the tension and charge of the object, as we discuss below.

The fluctuations of the supergravity fields around the soliton background contain a sector of fluctuations which are localized on the $(p+1)$-dimensional volume of the soliton core. Since the soliton leaves 16 unbroken supersymmetries, these fluctuations must arrange into supermultiplets of the corresponding $(p+1)$-dimensional supersymmetry. In fact, for D$p$-branes in type II theory, they form a $U(1)$ vector multiplet of 16 susys in $(p+1)$-dimensions (e.g. for a type IIB D3-brane, a vector multiplet of 4d $\mathcal{N} = 4$ supersymmetry); this contains a $U(1)$ gauge boson, $(9-p)$ real scalars, and a set of fermion superpartners. On the other hand, for type I D-branes, the spectrum of fluctuations is more complidated and will be discussed in later lectures, using a simpler microscopic description.

These fluctuations localized on the soliton volume can be thought of as field living on the brane world-volume. Moreover, their dynamics is related to the dynamics of the soliton. For instance, the scalars on the brane volume are goldstone bosons of translational symmetries of the vacuum, broken by the presence of the soliton. As such, the vevs of these $(9-p)$ scalars parametrize the location of the brane in transverse $(9-p)$-dimensional space. A fluctuation leadint to non-constant profile for these scalars describes a fluctuation where the brane volume is no longer flat. The low energy effective action of these $(p+1)$-dimensional fields (which is basically the Maxwell action and kinetic terms for the scalars and fermions) is an effective action for the dynamics of the brane.

There exist also multi-soliton solutions, where the field configuration has several cores, localized at different positions $x_a^m$ in the transverse space. The interactions between the different soliton cores cancel as a consequence of the BPS conditions, namely the gravitational attraction cancels agains their 'Coulomb' repulsion due to their (equal sign) RR charges. Thus these static configurations are solutions of supergravity. They are described by a bakcground (5), with

$$Z(r) \;=\; 1+\sum_a \frac{\rho^{7-p}}{|x^m - x_a^m|^{7-p}} \tag{7}$$

and a more complicated form for $H_{8-p}$, with the property that integrated over any $(8-p)$ sphere surrounding $x^m = x_a^m$ gives $Q$.

The analysis of certain properties (e.g. the analysis of fluctuations around the soliton background) of these multisoliton configurations is reliable only if the inter-soliton distances are larger than the string length.

We would like to conclude by emphasizing that at weak coupling there exists a microscopic description for D$p$-branes, which will be the topic of next lecture. The above facts and many other will be derived from this microscopic description.

**The NS5-brane**

This 5-brane solution exists for type IIA and type IIB theories, and also for heterotic theories; type I theory does not contain such states.

For type II theories, the solution (see page 182 in [4]) is of the form

$$ds^2 \;=\; \eta_{\mu\nu} dx^\mu dx^\nu + Z(r) dx^m dx^m$$

$$\begin{aligned} e^{2\phi} &= Z(r) = g_s^2 + \frac{Q}{2\pi^2 r^2} \\ H_3^{NSNS} &= *_{6789} d\phi \end{aligned} \tag{8}$$

For heterotic theories, it has a similar expression, since the background does not excite the 10d gauge fields.

The solution describes a $(5+1)$-dimensional core, namely a 5-brane. It is electrically charged under the NSNS 6-form dual to the NSNS 2-form. Namely it is magnetically charged under the latter. The tension and charge of the object can be computed to be

$$T_{NS5} = \frac{2\pi^2\alpha'}{g_s^2\kappa_{10}^2} Q \quad ; \quad Q_{NS5} = \frac{2\pi^2\alpha'}{\kappa_{10}^2} Q \tag{9}$$

The solution is invariant under half of the supersymmetries of the vacuum, and so describes a BPS state. This implies the above manifest relation between the tension and charge of the object.

The spectrum of fluctuations localized on the brane volume fill out supermultiplets under the unbroken supersymmetries. For the type IIA NS5-brane, they form a 6d $\mathcal{N}=(2,0)$ tensor multiplet (containing a 2-form with 6d self-dual field strength, 5 real scalars, and 2 Weyl fermions); for the type IIB NS5-brane, they form a 6d $\mathcal{N}=(1,1)$ vector multiplet (containing a gauge boson, 4 real scalars and 2 Weyl fermions); for the $E_8 \times E_8$ heterotic, they form a 6d $\mathcal{N}=1$ tensor multiplet (containing a self-dual 2-form, 1 scalar and 1 Weyl fermion) and hypermultiplet (containing 4 scalars and one Weyl fermion); for the $SO(32)$ heterotic, one 6d $\mathcal{N}=1$ vector multiplet (with one gauge boson, and one Weyl fermion), one neutral hypermultiplet and 29 hypermultiplets charged under the 10d gauge group (this will more easily determined in later lectures).

Other properties of the solution are analogous to those of D-brane. For instance, the existence of multi soliton solutions, or the interpretation of

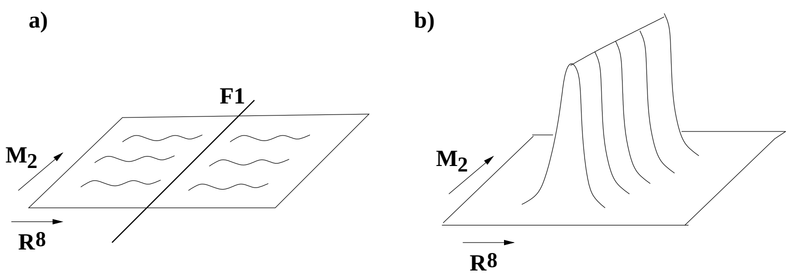



Figure 2: An infinitely extended fundamental string is a source for supergravity fields. The field configuration it excites is a solution of the supergravity equations of motion, which corresponds to the 1-brane like configuration. The two are simply different descriptions of the same object.

fluctuations as 6d fields describing the dynamics of the brane. An important difference, however, is that there is no known microscopic description for NS5-branes at weak coupling. One intuitive explanation of this is that the effective couplig constant $g_{eff} = e^{\phi}$ grows at the core of the soliton, no matter how small the asymptotic coupling $g_s$ is.

**Fundamental string**

In addition to the above objects, there exist supergravity solutions preserving half of the supersymmetries, and describing 1-branes electrically charged under the NSNS 2-form, and with tension $T_{F1} = (2\alpha')^{-1}$. This object is not non-perturbative, and has the same properties as a fundamental string with infinitely extended flat worldsheet. The natural proposal is that the supergravity solution is providing the field configuration excited by a large macroscopic fundamental string, so does not correspond to a new object. In this sense, the fundamental string is providing a microscopic description of the object we found in the 'rough' approximation of supergravity. See fig 2.

This object exists for type IIA, type IIB and heterotic theories. The reason why type I theory does not have a fundamental string sugra solution is that the type I string is not a BPS state. In fact, BPS states are necessarily stable, while the type I string can break.

## 2.2 Dirac charge quantization condition

Following an analysis similar to the discussion in section A.1, we can show that in a quantum theory the electric and magnetic charges under a $p+1$ form $C_{p+1}$ must satisfy a Dirac quantization condition.

Consider a $p$-brane charged electrically under $C_{p+1}$, i.e. the theory contains a term $Q_e \int_{W_{p+1}} C_{p+1}$ in the action. In the presence of a $(6-p)$-brane coupling magnetically under $C_{p+1}$, the flux of the dual field strength $H_{p+2}$ over an $(p+2)$-sphere surrounding the $(6-p)$-brane in the transverse $(p+3)$-dimensional space is

$$\int_{\mathbf{S^{p+2}}} H_{p+2} = Q'_m \tag{10}$$

Wrapping the $p$-brane over a $\mathbf{S^{p+1}}$ in the equator of the above $\mathbf{S^{p+2}}$, see figure 2.2, the phase in the path integral can be written as an intergral of $H_{p+2}$ over a hemisphere. The change in the phase depending on which hemisphere one chooses is

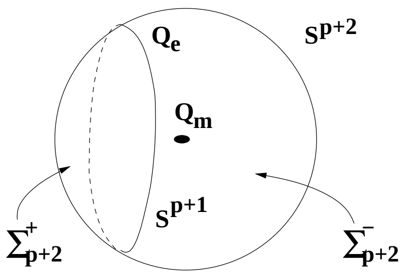
$$Q_e \Delta \int_{\mathbf{S^{p+1}}} C_{p+1} = Q_e\left(\int_{\Sigma^+_{p+2}} H_{p+2} - \int_{\Sigma^-_{p+2}} H_{p+2}\right) = Q_e \int_{\mathbf{S^{p+2}}} H_{p+2} = Q_e Q'_m \tag{11}$$

In order to have a well-defined phase, we then need

$$Q_e Q'_m \in 2\pi \mathbf{Z} \tag{12}$$

If the theory contains dyonic objects, carrying electric and magnetic charges at the same time, consistency requires

$$Q_e Q'_m - Q_m Q'_e \in 2\pi \mathbf{Z} \tag{13}$$




At the level of supergravity these conditions are not visible. However, they should follow form any consistent microscopic description of these solitons (see lecture on D-branes). And they should hold in any consistent quantum theory, so we explicitly require them to hold in our theories.

## 2.3 BPS property

In analogy with the discussion in the field theory setup in section A.2, the 10d supersymmetry algebras of the different string theories can be seen to admit extensions by central charges, which is this case are tensorial. The supersymmetry algebras have the structure

$$\{Q_\alpha^A, Q_\beta^{B\,\dagger}\} = -2\delta^{AB} P_\mu \Gamma^\mu_{\alpha\beta} - 2i Z^{AB}_{\mu_1 \ldots \mu_{p+1}} (\Gamma^{\mu_1} \ldots \Gamma^{\mu_{p+1}})_{\alpha\beta} \tag{14}$$

The operators $Z_{\mu_1 \ldots \mu_{p+1}}$ are central charges, in the sense that they commute with the $Q$'s and $P_\mu$'s, but behave as tensors with respect to the generators of the Lorentz group. They commute with the hamiltonian, hence are moduli-dependent multiples of the $(p+1)$-brane charge.

In a sector where just one of these central charges is non-zero, one can go to the rest frame of the corresponding state and derive a BPS bound for the tension of the corresponding $p$-brane object. Also, BPS states, i.e. states saturating the bound, belong to short representations of the supersymmetry algebra. This implies that they cannot cease to be BPS under continuous deformations of the theory, and also that the dependence of their tension with the moduli is exactly determined from the classical result (does not change by quantum corrections or otherwise).

The $p$-brane states studied in section (2.1) are BPS states, in this sense. This guarantees that, although they were constructed in the supergravity approximation, they exist in the complete theory (once $\alpha'$ and $g_s$ corrections are included), and their properties, charge and tension are exactly knonw as function of the moduli.

Going through the list of string theories and brane states, the conclusion is that for any string theory, the theory contains states charged under all $p$-form gauge fields and their duals. These states have tension controlled by their charges, and are guaranteed to be stable (since there is no lighter state carrying those charges (it would violate the BPS bound)).

# 3 Duality for type II string theories

In this section we scratch the surface of the implications of the existence of these states in string theory. The main implication we would like to explore here is the existence of duality relations in string theory, which are analogous to the field theory duality in section A.3. Our discussion is not complete, but just inspirational. We will return to the issue of duality in latter lectures.

## 3.1 Type IIB $SL(2,\mathbf{Z})$ duality

Ten-dimensional Type IIB supergravity has a classical $SL(2,\mathbf{R})$ invariance. It acts on the NSNS and RR 2-forms $B$, $\tilde{B}$ and the complex coupling $\tau =$

$a + ie^{-\phi}$ (which takes values in the coset $SL(2, \mathbf{R})/\mathbf{U(1)}$ as

$$\begin{aligned} \tau &\rightarrow \frac{a\tau + b}{c\tau + d} \\ \begin{pmatrix} B \\ \tilde{B} \end{pmatrix} &\rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ \tilde{B} \end{pmatrix} \end{aligned} \tag{15}$$

leaving the metric $G$ (in the Einstein frame) and the 4-form $A_4$ fields invariant.

Clearly this continous symmetry cannot be a symmetry of the complete quantum theory, since it would rotate the charges continously, in contradiction with the fact that they must lie in a lattice by Dirac quantization condition. There is however plenty of evidence for the conjecture that a discrete $SL(2, \mathbf{Z})$ subgroup (defined by $a, b, c, d \in \mathbf{Z}$) is an exact symmetry of the complete string theory.

This remarkable proposal has the implication that there is a strong-weak duality between the theory at coupling $g_s$, $a = 0$ and the theory at coupling $1/g_s$, $a = 0$. Namely, the strong coupling regime of type IIB theory is equivalent to the perturbative weak coupling regime of a dual type IIB theory. Following the dependence of brane tensions as $g_s$ changes it is possiblo to match the BPS states in both theories. For instance

| IIB at $g_s$ | | IIB at $1/g_s$ |
|---|---|---|
| F1 | $\leftrightarrow$ | D1 |
| D1 | $\leftrightarrow$ | F1 |
| NS5 | $\leftrightarrow$ | D5 |
| D5 | $\leftrightarrow$ | NS5 |
| D3 | $\leftrightarrow$ | D3 |

We see that starting at $g_s \simeq 0$, as $g_s$ increases at goes to infinity the initial fundamental string becomes a D1-brane in the dual description, while the original D1 becomes light and turns into the fundamental, perturbative

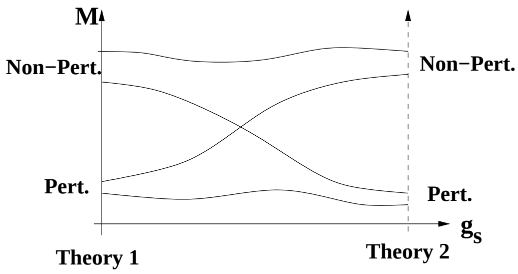



Figure 3: As a modulus (the dilaton vev) is changed, the original weakly coupled string theory becomes strongly interacting, and at infinite coupling it can be described as a weakly interacting *dual* theory. Perturbative and non-perturbative states are reshuffled in this interpolation.

string in the dual description. The flow of BPS states is illustrated in figure 3.

This has the striking implication that the fundamental string is 'fundamental' only at weak coupling, while at finite coupling both the D1 and the F1 are both simply two BPS string-like objects, and at strong coupling the D1 is the one becoming the fundamental, perturbative object.

Indeed the situation is even more intriguing. The $SL(2,\mathbf{Z})$ symmetry predicts the existence of BPS strings with charges $(p,q)$ under the two type IIB 2-forms, all forming an orbit of $SL(2,\mathbf{Z})$. These are easily constructed as supergravity solutions, by applying $SL(2,\mathbf{Z})$ transformations to the known F1 or D1 solutions (which correspond to $(p,q) = (1,0)$, $(0,1)$). At different points in the moduli space of the coupling $\tau$, related to the perturbative limit by an $SL(2,\mathbf{Z})$ transformation, it is a different $(p,q)$ string which becomes the perturbative object in the dual ($SL(2,\mathbf{Z})$ transformed) theory.

Since the symmetry relates theories which are equivalent, up to (very non-trivial) field redefinitions, the moduli space of physically distinct theories is

$SL(2,\mathbf{R})/(\mathbf{U(1)} \times \mathbf{SL(2,Z)})$,

Duality relations in other 10d string theories will be studied in later lectures. We conclude this lecture by pointing out that the picture for type II theories is even more intricate as one lowers the dimension.

## 3.2 Toroidal compactification and U-duality

Let us consider compactification of type IIB theories on e.g. $\mathbf{T^6}$. The results for type IIA on $\mathbf{T^6}$ would be equivalent via T-duality, but the interpretation in terms of the original 10d theory is clearly different. It will be better understood in later lectures.

We are interested in studying non-perturbative states and duality properties of this theory (the case of other toroidal compactification is similar in many respects, see [1, 2]. We are interested in i) the moduli space of scalars ii) the 4d gauge fields, in particular 1-form gauge bosons iii) the BPS states preserving 1/2 of the supersymmetries iv) the duality group.

i) Let us determine the structure of the moduli space of scalars. In $\mathbf{T^6}$ compactifications of type IIB theory we have 36 scalars from the moduli $G_{ij}$, $B_{ij}$. These are known from the Narain lattice description to take values in the coset

$$\frac{SO(6,6)}{SO(6)\times SO(6)\times SO(6,6;\mathbf{Z})} \tag{16}$$

In addition, we have the scalars $a$, $\phi$ inherited from 10d, and which parametrize the coset

$$\frac{SL(2,\mathbf{R})}{U(1)\times SL(2,\mathbf{Z})} \tag{17}$$

In addition, we have 15 scalars $\tilde{B}_{ij}$, 15 scalars $A^{+}_{ijkl}$ and two scalars, dual to the 4d 2-forms $B_{\mu\nu}$, $\tilde{B}_{\mu\nu}$. Overall we have 70 scalars, which in the supergravity approximation live in a coset locally of the form

$$E_7/SU(8) \tag{18}$$

where $E_7$ denotes the (non-compact) group generated by exponentiating the Lie algebra generated by generators of $SO(6,6)$ and $SL(2)$.

The supergravity effective action has a continuous symmetry $E_7$ acting non-trivially on the moduli space of scalars. As usual, classical supergravity is not sensitive to quantization conditions, and it will be only a subgroup of this which will be proposed to correspond to a full symmetry of the theory. This will come later on.

ii) The theory contains 56 4d 1-form fields. 24 of them are given by $B_{\mu i}$, $\tilde{B}_{\mu i}$ and their 4d duals; these transform in the representation $(12,2)$ of the classical global symmetry $SO(6,6)\times SL(2,\mathbf{R})$. The remaining 32 are given by 12 from $G_{\mu i}$ and their duals and 20 from [2] $A^+_{ijk\mu}$; these transform in the representation $(32,1)$ of $SO(6,6)\times SL(2,\mathbf{R})$. In total the 56 gauge bosons transform in the representation 56 of the classical symmetry $E_7$.

iii) The elementary (in the sense that they carry at most one charge) BPS states carrying charged under gauge bosons are of different kinds

• We can have fundamental strings winding along any of the 6 directions in $\mathbf{T^6}$. We can also have D1-strings winding along any of these directions. These are charged under the fields $B_{\mu i}$, $\tilde{B}_{\mu i}$

• We can have 6 particle-like states in 4d from NS5-branes wrapped in all dimensions of $\mathbf{T^6}$ except one [3] Similarly we get 6 additional states from D5-branes wrapped in all dimensions of $\mathbf{T^6}$ except one. These are charged

[2]Notice that $A_4^+$ has self-dual field strength in 10d.

[3]To consider branes with some transverse compact circle, we can consider starting with an infinite transverse dimension, on which we place and infinite periodic array of branes (this is possible and static due to the BPS no-force condition), and then modding by discrete translations to obtain a circle.

under the duals of $B_{\mu i}$, $\tilde{B}_{\mu i}$. The above 12 states plus these 12 transform in the $(12,2)$ representation of the global symmetry $SO(6,6) \times SL(2,\mathbf{R})$.

• KK momentum states. These are described by fundamental string states with momentum along some internal direction in $\mathbf{T^6}$. There are 6 basic states, charged under the 4d gauge fields $G_{\mu i}$.

• The corresponding states charged magnetically under $G_{\mu i}$ (i.e. charged electrically under their 4d duals) are Kaluza-Klein monopoles (also known as KK5-branes). The KK monopole configurations are discussed in appendix B. These 6 states are labelled by $i = 1, \ldots, 6$ and have their isometrical direction along the $i^{th}$ direction in $\mathbf{T^6}$ and volume spanning the remaining 5 directions in $\mathbf{T^6}$.

• Finally we have 20 additional states given by D3-branes wrapped on three internal directions in $\mathbf{T^6}$. The above 12 states plus these 20 transform in the representation $(32,1)$ of $SO(6,6) \times SL(2,\mathbf{R})$.

In total, these states transform in the representaion 56 of the classical symmetry group $E_7$

iv) These states must have quantized charges, so clearly the full continuous $E_7$ symmetry cannot be an exact symmetry of the complete theory. Rather, the proposal is that the discrete subroup of $E_7$ which leaves the 56-dimensional lattice of charges invariant is an exact symmetry of the quantum theory.

This is a simple generalization of thing we already know. In fact, the discrete duality group, denoted $E_7(\mathbf{Z})$, is the also the group of discrete transformations containing the T-duality group $SO(6,6;\mathbf{Z})$ and the S-duality group $SL(2,\mathbf{Z})$. The global structure of the moduli space is

$$\frac{E_7}{SU(8) \times E_7(\mathbf{Z})} \tag{19}$$

All BPS states in the theory transform in representaions of the duality group

$E_7(\mathbf{Z})$ (known as U-duality group).

This has remarkable implications. In particular there are infinite sets of points in moduli space which are equivalent to weakly coupled large volume compactifications of IIB on $\mathbf{T^6}$ once written in suitable dual terms. The perturbative parameter in these dual theories can be a complicated combination of the 70 scalars in the coset $E_7/SU(8)$, and not just a function of the dilaton. Moreover the string-like object which is becoming the fundamental string in this dual theory can be a complicated object, not just the F1 or the D1-string. In fact string-like objects also form a complicated representation (I think the 133) of $E_7(\mathbf{Z})$: we have the unwrapped F1, and D1, A D3-brane wrapped in two directions, D5-branes wrapped in four directions, etc. Any of these can become the fundamental string in one particualr corner of moduli space.

For the interested reader, let us simply point out that similar duality relations hold in toroidal compactifications of heterotic string theory. In fact, $\mathbf{T^6}$ comapctfifications lead to $\mathcal{N}=4$ 4d theories, whose gauge sector is a generalization of the kind of theories in appendix A, and have an $SL(2,\mathbf{Z})$ duality which corresponds to Montonen-Olive in the associated gauge field theory. We will rederive Montonen-Olive duality in later lectures, using D-branes to study gauge field theories.

# 4 Final comments

We have seen that string theory contains plenty of non-perturbative states. These are very important for the theory at finite coupling, and are in a sense on an equal footing with perturbative or fundamental objects in this regime ($p$-brane democracy). In fact, they can become the fundamental degrees of freedom in different corners in moduli space, and can be described as the fundamental strings in a suitable dual description.

We still do not have a microscopic description of string theory which is valid beyond perturbation theory, and which includes all these BPS states on an equal footing. What is clear anyway is that as soon as we go beyond the perturbative regime, string theory is no longer a theory of strings! and must also include other extended objects.

# A Some similar question in the simpler context of field theory

A more detailed reference for this section is [6].

## A.1 States in field theory

We consider a well studied and simple 4d field theory, which is $\mathcal{N} = 4$ supersymmetric $SU(2)$ gauge theory. The vector multiplets contain one gauge boson, four Majorana fermions and six real scalars in the adjoint. The scalar potential has the form $V(\phi) = |[phi^i, \phi^j]|$, so a generic vacuum is labelled by diagonal vevs of the form

$$\phi^i = \begin{pmatrix} v_i & 0 \\ 0 & -v_i \end{pmatrix} \tag{20}$$

We denote $v = \sum_i v_i^2$. A generic vev $v$ breaks spontaneously the gauge symmetry $SU(2) \to U(1)$.

At low energies in one of these vacua, $E \ll g_{YM} v$ the effective theory is $\mathcal{N} = 4$ susy $U(1)$ gauge theory, with action

$$S = \int_{4d} \frac{1}{g_{YM}^2} F \wedge *F + \theta \int_{4d} F \wedge F \tag{21}$$

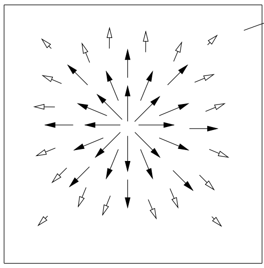



Figure 4: Picture of the hedgehodge configuration for the Higgs field.

The theory clearly contains states electrically charged under the gauge potential $A$; they are the massive gauge bosons. The mass of one such state with charge $n_e \in \mathbf{Z}$ is

$$M = |n_e| g_{YM} v \tag{22}$$

We can also look for non-perturbative states of the theory by constructing solutions to the equations of motion (see [5] for an introduction to solitons). Indeed the theory contains particle-like states known as 't Hooft-Polyakov monopoles, as we discussed in the introductory lectures. Such monopoles are described by field configurations asymptoting as

$$\begin{aligned} \phi^i(\vec{x},t) &\to \frac{v^i}{r}\, x^i \; + \mathcal{O}(1/r^2) \\ A^i(\vec{x},t) &\to \frac{1}{r^2}\, x^i \; + \mathcal{O}(1/r^2) \end{aligned} \tag{23}$$

This is the so-called hedgehdoge configuration, shown in figure 4. From the point of view of the low energy $U(1)$ theory, the field configurations are Wu-Yang monopoles of the kind studied in the differential geometry lecture.

These objects carry magnetic charge $n_m \in \mathbf{Z}$ under the gauge potential $A$, and their mass is

$$M = |n_m| v / g_{YM} \tag{24}$$

(if the $\theta$ parameter is non-zero, they also carry an electric charge proportional to $q_e \theta n_m$). The mass of a general state with electric and magnetic charges $(q_e, q_m)$ is given by

$$M^2 = v^2 \frac{1}{\Im \tau} |q_e + \tau q_m|^2 \tag{25}$$

where $\tau = \theta + i/g_{YM}^2$. For $\theta = 0$ this gives

$$M = |v| |g_{YM} q_e + \frac{1}{g_{YM}} q_m| \tag{26}$$

**Dirac charge quantization condition**

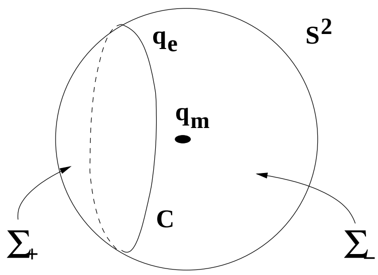
This is a consistency condition on the possible set of charges in a theory with electric and magnetic charges. A particle with electric charge $q_e$ moving in a circle worldline $C$ acquires a phase $\exp(iq_e \int_C A)$ in its path integral. In the presence of a particle carrying magnetic charge $q'_m$, the gauge potential is not globally well defined, so the above expression could be ambiguous, leading to an ill-defined wavefunction for the electric particle. Indeed, as shown in figure A.1, the integral $\int_C A$ can be computed via Stokes theorem as $\int_\Sigma F$ over some surface $\Sigma$ with $\partial\Sigma = C$. The result however can depend on the surface $\Sigma$ chosen. For the two surfaces in the picture, the difference in the exponent of the phase is

$$\Delta q_e \int_C A = q_e (\int_{\Sigma_+} F - \int_{\Sigma_-} F) = q_e \int_{\mathbf{S^2}} F = q_e q'_m \tag{27}$$

where $\mathbf{S^2}$ is a surface that encloses the magnetically charged particle. In order to have a well-defined phase, we then need

$$q_e q'_m \in 2\pi \mathbf{Z} \tag{28}$$

This is Dirac quantization conditions, which constrains the charges in a theory with electric and magnetic objects.




If the theory contains dyonic particles, carrying electric and magnetic charges at the same time, consistency of the phase picked up by moving a particle of charges $(q_e, q_m)$ in the presence of a particle of charge $(q'_e, q'_m)$ requires

$$q_e q'_m - q_m q'_e \in 2\pi \mathbf{Z} \tag{29}$$

This implies that charges $(q_e, q_m)$ must lie in a 2d discrete lattice. One can check that the charges of the above theory, which are of the form $(q_e, q_m)$ with $q_e + iq_m = n_e + \tau n_m$, with $n_e, n_m \in \mathbf{Z}$, satisfy this constraint (zzz Warning: I was not careful about $2\pi$'s).

### A.2 BPS bounds

The general supersymmetry algebra for $\mathcal{N} = 4$ has the structure

$$\{Q^A_\alpha, Q^{B\,\dagger}_\beta\} = -2\delta^{AB} P_\mu \Gamma^\mu_{\alpha\beta} - 2iZ^{AB}\delta_{\alpha\beta} \tag{30}$$

where $Q^A_\alpha$, $A = 1, \ldots, \mathcal{N}$ are the $\mathcal{N}$ supercharges ($\mathcal{N} = 4$ in our case) with a (Majorana) spinor index $\alpha$. The $Z^{AB}$ are operators that commute with the $Q$'s, the $P$'s and hence with the Hamiltonian. Thus they are conserved charges of the system, known as central charges, which are combination of the conserved gauge charges of the theory.

In a given state, $Z^{AB}$ forms a real antisymmetric matrix, which can be brought to a block diagonal form with blocks

$$\begin{pmatrix} 0 & q_i \\ -q_i & 0 \end{pmatrix} \tag{31}$$

The supersymmetry algebra implies a bound on the mass of particle states in the sector of fixed (central) charges $q_i$. This is done as follows: take for simplicity a sector of equal charges $q_i = q$, we can go to the rest frame of the particle, where $(P_\mu) = (M, 0, 0, 0)$. Then the matrix $\{Q_\alpha^A, Q_\beta^{B\,\dagger}\}$, which is positive definite, is diagonal in blocks of the form

$$\begin{pmatrix} 2M & 2iq \\ -2iq & 2M \end{pmatrix} \tag{32}$$

This implies that the eigenvalues, which are $2(M \pm q)$ must be positive, so that we get a bound

$$M \geq |q| \tag{33}$$

This is known as BPS bound. States saturating this kind of bounds are called BPS states. They are special because they correspond to zero modes of the supercharge anticommutator matrix, and this implies that they are annihilated by some supercharges. This is equivalent to saying that BPS states are invariant under some supersymmetry transformations (generated by the corresponding supercharges). On the other hand, this implies that the supermultiplets to which these states belong are shorter than the generic supermultiplet.

This implies that upon continuous deformations of the theory (for instance including quantum corrections or threshold effects of the underlying high energy theory) BPS states cannot cease being BPS, since the number of fields in the supermultiplet cannot jump discontinuously. This also implies that, since the mass of the state is fixed by the supersymmetry algebra, it is exactly known, and does not suffer any correction form quantum loops or otherwise. Therefore, the classical result for the mass of a BPS state can be exactly extrapolated to strong coupling and other difficult regimes.

In our case above, it is possible to show that in a sector of electric and magnetic charges $(q_e, q_m)$ the central charge for the superalgebra is of the above form

$$q_i = q = v g_{YM}(q_e + \tau q_m) \tag{34}$$

This allows to claim that the above discussed states are BPS and the masses (25) is exact.

## A.3 Montonen-Olive duality

The equations of motion for the $U(1)$ gauge theory are (for $\theta = 0$)

$$\begin{aligned} dF &= j_m \\ d * F &= j_e \end{aligned} \tag{35}$$

where $j_e$, $j_m$ are the electric and magnetic charge currents. They have a global $SL(2,\mathbf{R})$ rotation invariance

$$\begin{pmatrix} *F \\ F \end{pmatrix} \to M \begin{pmatrix} *F \\ F \end{pmatrix} \quad ; \quad \begin{pmatrix} j_e \\ j_m \end{pmatrix} \to M \begin{pmatrix} j_e \\ j_m \end{pmatrix} \quad ; \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad ad - bc = 1 \tag{36}$$

This also acts by rotating the charges $(q_e, q_m)$, so it is able to exchange the roles of elementary electrically charged states and solitonic magnetic monopoles, i.e. of perturbative and non-perturbative states in the system. Indeed, for the theory (e.g. the energies of the states) to be invariant, $SL(2,\mathbf{R})$ must also act on the coupling constant $\tau$ by

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{37}$$

Since the charges must live in a discrete lattice due to the Dirac quantization condition, it is clear that the classical $SL(2, \mathbf{R})$ symmetry cannot be a symmetry of the full quantum theory. However, the subgroup $SL(2, \mathbf{Z})$ given by matrices $M$ with $a, b, c, d \in \mathbf{Z}$ leaves the charge lattice invariant as a whole, and also is a symmetry of the mass formula (25). The Montonen-Olive duality proposal is that this $SL(2, \mathbf{Z})$ is an exact symmetry of the full quantum theory.

This symmetry has very non-trivial implications:

• It implies that BPS solitons must appear in orbits of $SL(2, \mathbf{Z})$. In particular this implies the existence of BPS dyonic states with charges $q_e + iq_m = n_e + \tau n_m$ for coprime $n_e, n_m$; this is the orbit containint the elementary electically charged states $(n_e, n_m) = (\pm 1, 0)$ and the basic magnetic monopoles $(n_e, n_m) = (0, \pm 1)$. Some of these dyonic states have been explicitly constructed [7].

• It implies that the theory at coupling $g_{YM}$, $\theta = 0$ has a completely equivalent description in terms of a theory with coupling $g'_{YM} = 1/g_{YM}$, $\theta' = 0$. One says that it is a strong-weak coupling duality. This implies that the strong coupling of the first theory is described by a weakly coupled theory in the dual side. The theory simplifies enormously in the limit of very strong coupling, which in principle looked like a very difficult regime!. The theory becomes simply perturbative Maxwell theory in terms of the dual elementary fields, which are the solitons of the initial theory.

• In fact, there is an infinite number of limits where the dynamics reduces to perturbative Maxwell theory in terms of a dual theory, which is related via an $SL(2, \mathbf{Z})$ transformation to the original one.

• These properties are a good toy model for the dualities in string theory. This has been our motivation for discussing this field theory example. In fact, we will see in later lectures that duality in string theory implies duality in field theory.

# B The Kaluza-Klein monopole

Consider a $D$-dimensional theory with gravity, compactified on a circle, so that it corresponds to a vacuum of the form $M_{D-1} \times \mathbf{S^1}$. The Kaluza-Klein monopole is a purely metric configuration, which corresponds to an excited state of this theory, and exists if $D \geq 4$. It is described by a geometry $M_{D-4} \times X_{TN}$, where the so-called (multi)Taub-NUT space $X_{TN}$ has the following metric

$$ds^2 = V(\vec{x})^{-1} d\vec{x}^{\,2} + V(\vec{x})(d\tau + \vec{\omega} \cdot d\vec{x})^2 \tag{38}$$

with

$$\vec{\nabla} \times \vec{\omega} = V(\vec{x}) \quad ; \quad V(\vec{x}) = 1 + \sum \frac{1}{|\vec{x} - \vec{x}_a|} \tag{39}$$

The space $X_{TN}$ is a fibration of $\mathbf{S^1}$ (parametrized by $\tau$) over $\mathbf{R^3}$ (parametrized by $\vec{x}$), with the properties that (see figure 5)

i) the $\mathbf{S^1}$ in the fiber asymptotes to constant radius at infinity on the base $\mathbf{R^3}$. So it is a finite energy excitation of the vacuum $M_{D-1} \times \mathbf{S^1}$.

ii) the $\mathbf{S^1}$ denerates to zero radius at the location of the so-called centers $\vec{x} = \vec{x}_a$.

iii) The $\mathbf{S^1}$ fibered over an $\mathbf{S^2}$ in the base $\mathbf{R^3}$ surrounding a center, is a non-trivial $\mathbf{S^1}$ (or $U(1)$ )bundle over $\mathbf{S^2}$ with first Chern class equal to 1. If the $\mathbf{S^2}$ surrounds $k$ centers, the Chern class of the bundle of $\mathbf{S^1}$ over $\mathbf{S^2}$ is $k$. In fact, one can show that the mixed component of the Christoffel connection is exactly the gauge field of the Wu-Yang monopole studied in the lecture on differential geometry.

iv) This implies that the geometry carries a topological magnetic charge under the $D-1$ dimensional gauge boson $G_{\mu(\tau)}$. The sources of the charge

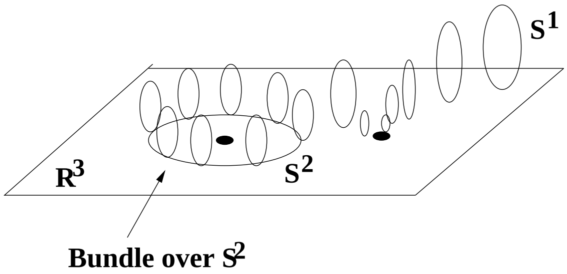



Figure 5: A picture of the multi-Taub-NUT space $X_{TN}$. It is a circle fibration over $\mathbf{R^3}$, with fiber asymptoting to constant radius at infinity, and degenerating to zero radius over the centers, shown as black dots. Around an $\mathbf{S^2}$ surrounding a center, the $\mathbf{S^1}$ fibrations defines a non-trivial $U(1)$ bundle with first Chern class 1.

are localized at the centers of the metric, which then behave as magnetic monopoles for this field. The configuration defined by Taub-NUT space is known as Kaluza-Klein monopole.

The above metric has $SU(2)$ holonomy (so can be though of as a non-compact Calabi-Yau in two complex coordinates) so it is invariant under half of the supersymmetries. It is a 1/2 BPS state. Its ADM tension is proportional to $R^2/g_s^2$, where $R$ is the radius of the isometrical direction $\mathbf{S^1}$ parametrized by $\tau$.

In circle compactifications of string theory, the resulting 9d object is Poincare invariant in six dimensions, and is localized in three dimensions. It is often called the Kaluza-Klein fivebrane. In toroidal compactficiations of several dimensions, one can have different BPS states given by the different choices of the circle in $\mathbf{T^d}$ chosen to correspond to the isometrical direction in $X_{TN}$.

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