Orbifold compactification

The basic reference for this lecture is [1]. See also [2].

1 Introduction

1.1 Motivation

We have seen that compactification on smooth Calabi-Yau spaces leads to very interesting 4d theories. However, they require quite a lot of geometrical tools, and the information one can extract is, in a sense, limited (because of the need to use the supergravity approximation (lowest order in α' expansion), and the difficulty in constructing explicit metrics, only topological quantities can be reliably obtained).

In this lecture we discuss orbifold compactifications. They share many of the features of compactification on smooth Calabi-Yau spaces (they can be regarded as compactifications on singular Calabi-Yau's), but are described by free 2d worldsheet theories. Hence, the quantization of the string can be carried out exactly in the α' expansion, and one can compute quantities explicitly, and including the stringy corrections. In this sense, orbifolds are (almost) as simple as toroidal compactifications, but have the advantage of leading to models with reduced supersymmetry. In this lecture we center on 6d orbifolds preserving 1/8 of the supersymmetries; namely i.e. leading to 4d $\mathcal{N} = 2$ supersymmetry for type II theories or to 4d $\mathcal{N} = 1$ supersymmetry for heterotic theories. The description of orbifolds of type I theory (also known as type IIB orientifolds) is more technical and is not discussed (left for the final projects).

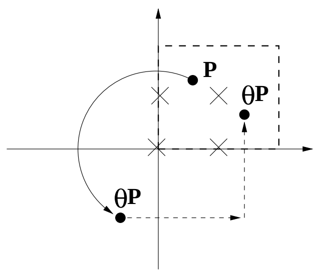


Figure 1: $\mathbf{T^2/IZ_2}$ orbifold. The 2-torus is shown as the 2-plane modded by discrete translations; hence the sides of the unit cell, shown in dashed lines, are identified. The rotation θ maps each point to its symmetric with respect to the origin. The action on the 2-torus is obtained by translating the points into the unit cell. Crosses represent points fixed under the action of θ on $\mathbf{T^2}$.

1.2 The geometry of orbifolds

A toroidal orbifold (or just orbifold, for short) $\mathbf{T^6/\Gamma}$ is the quotient space of $\mathbf{T^6}$ by a finite isometry group Γ, which acts with fixed points.

One simple example, before going to the 6d case, is the 2d orbifold $\mathbf{T^2/Z_2}$. Consider a $\mathbf{T^2}$ parametrized by two coordinates x_1, x_2, with periodic identifications $x_i \simeq x_i + 1$, and consider the $\mathbf{Z_2}$ action generated by the symmetry $\theta : x_i \to -x_i$. The orbifold $\mathbf{T^2/Z_2}$ is $\mathbf{T^2}$ with the identification of points related by the action of θ. This is shown in figure 1.

The action θ has fixed points, namely points with coordinates (x_1, x_2) equivalent to $(-x_1, -x_2)$ up to periodicities. Namely obeying

$$(-x_1, -x_2) = (x_1, x_2) + n(1,0) + m(0,1) \tag{1}$$

for some $n, m \in \mathbf{Z}$. There are four such points, with coordinates $(0,0)$, $(0,1/2)$, $(1/2,0)$ and $(1/2,1/2)$. These fixed points of the orbifold action descend to conical singularities in the quotient space. This can be seen by

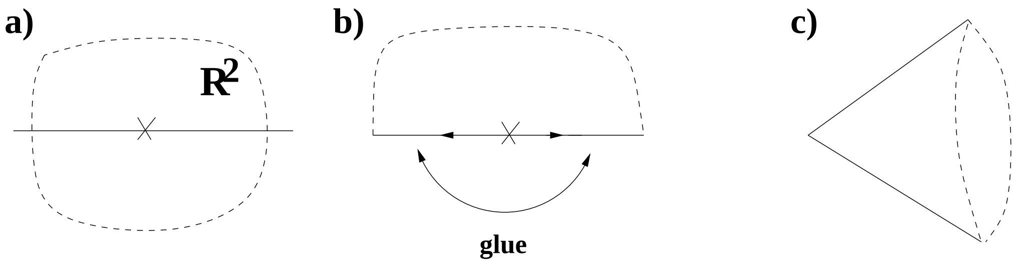


Figure 2: The quotient $\mathbf{R^2}/\mathbf{Z_2}$ has a conical singularity at the origin. This can be seen by starting with the 2-plane (a), keeping points in the upper half (b) (points in the lower half are their θ images, and performing the remaining θ identification in the horizontal boundary (c).

studying the local geometry near one of this points, which is a quotient space $\mathbf{R^2}/\mathbf{Z_2}$, and can be regarded as the space obtained by taking a piece of paper, cutting half of it, and glueing the two halves of the boundary to obtain a cone. This is shown in figure 2. The idea generalizes to more complicated higher-dimensional orbifolds.

Notice that to obtain a well-defined quotient, the discrete group must be a symmetry of the torus. This is most easily checked by regarding the d-dimensional torus as $\mathbf{R^d}$ modded out by translations in a lattice. The group Γ should be a symmetry of the lattice. Such groups are said to act crystallographically on the lattice, by analogy with crystallographic groups in solid state physics. An example of a 2d lattice is shown in figure 3.

A very popular example is the 4d orbifold $\mathbf{T^4}/\mathbf{Z_2}$, with the generator θ of $\mathbf{Z_2}$ acting by $x_i \to -x_i$ on the four coordinates of $\mathbf{T^4}$. The resulting quotient space is a singular limit of the Calabi-Yau space K3, with 16 singular points, locally of the form $\mathbf{R^4}/\mathbf{Z_2}$.

Clearly, one can form orbifold using other discrete groups. For instance,

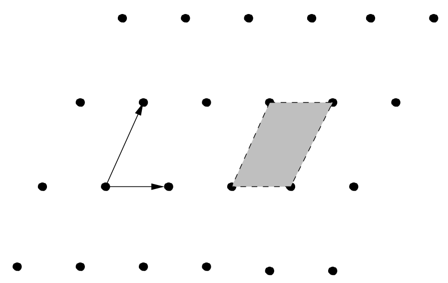


Figure 3: A 2d lattice, admitting a $\mathbf{Z_2}$ symmetry (reflection with respect to any point in the lattice. It is easy to cook up other 2d lattices with $\mathbf{Z_3}$ or $\mathbf{Z_4}$ symmetry.

we will later on center on a 6d orbifold $\mathbf{T^6}/\mathbf{Z_3}$, where $\mathbf{T^6}$ is described by three complex coordinates z_i, with the periodic identifications $z_i \simeq z_i + 1$ and $z_i \simeq z_i + e^{2\pi i/3}$. The generator θ of $\mathbf{Z_3}$ is an order three action given by

$$\theta \ : \ (z_1, z_2, z_3) \to (e^{2\pi i/3} z_1, e^{2\pi i/3} z_2, e^{-4\pi i/3} z_3) \tag{2}$$

We have used $e^{-4\pi i/3}$ instead of $e^{2\pi i/3}$ for z_3 in order to stick to the convention (useful in later purposes) that the sum of the phases in the rotations add up to zero. The orbifold action is a simultaneous rotaion by 120 degrees in all three complex planes, as shown in figure 4. The action has 27 fixed points which are points where the coordinates z_i are either of the values 0, $(1 + e2\pi i/3)/3$, $(e2\pi i/3 + e^{4\pi i/3})/3$. Each point is locally of the form $\mathbf{C^3}/\mathbf{Z_3}$.

Although it is possible to construct orbifolds where Γ is a non-abelian discrete group, these are technically more involved and not specially illuminating. So in this lecture we center on abelian Γ, and in particular to cases $\Gamma = \mathbf{Z_N}$, generated by an action θ acting on three complex coordinates by

$$\theta \ : \ (z_1, z_2, z_3) \to (e^{2\pi i v_1} z_1, e^{2\pi i v_2} z_2, e^{2\pi i v_3} z_3) \tag{3}$$

with $(v_1, v_2, v_3) = (a_1, a_2, a_3)/N$ and $a_i \in \mathbf{Z}$ [1].

[1] An additional condition $\sum_i a_i = even$, is required for the quotient space to be spin.

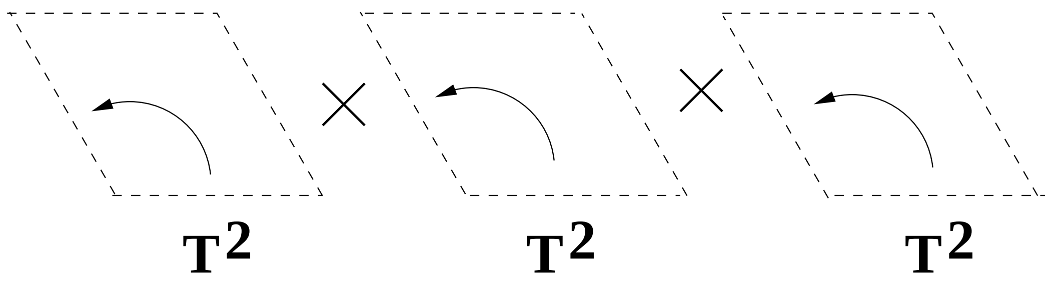


Figure 4: The $\mathbf{T^6}/\mathbf{Z_3}$ orbifold.

Orbifolds are not smooth manifolds, but are similar in many respects to manifolds. Indeed, removing the singular points they are manifolds. In fact one can define the holonomy group, and will be related to the amount of supersymmetry preserved by the compactification, just like for smooth manifolds. By parallel transporting a vector around closed loops which were closed in the torus, the holonomies generated are trivial, because the metric on the torus is flat. However, there are loops in the quotient space that surround the singular points, and are closed in the quotient altough they are not closed in the 'parent' torus. The holonomies around those loops are non-trivial, and generate a holonomy group which is precisely Γ. This is shown for $\Gamma = \mathbf{Z_3}$ in figure 5.

This suggests that $2n$-dimensional orbifold preserving some supersymmetry should be defined by discrete groups Γ whose geometric action is in a subgroup of $SU(n)$. For 6d orbifolds with $\Gamma = \mathbf{Z_N}$ generated by the action (3), the condition is $v_1 \pm v_2 \pm v_3 = 0 \bmod N$, for some choice of signs (the choice determines *which* susy (out of the many susys of the torus) is preserved). We will stick to orbifolds obeying the condition

$$v_1 + v_2 + v_3 = 0 \bmod N \tag{4}$$

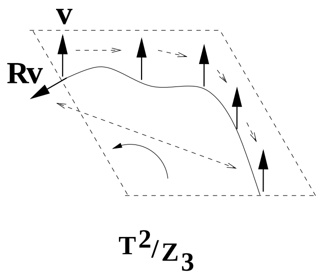


Figure 5: Holonomy on an orbifold: we start with the vector v and parallel transport it along a loop (closed up to the θ action); the vector ends up rotated by an action R which is isomorphic to θ.

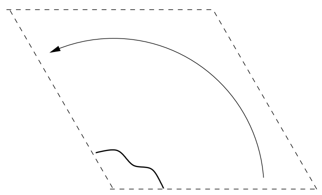
These orbifolds are simple versions of Calabi-Yau manifolds.

One easily checks that the $\mathbf{T^4/Z_2}$ and $\mathbf{T^6/Z_3}$ examples above are supersymmetry preserving, while $\mathbf{T^2/Z_2}$ is non-supersymmetric.

1.3 Generalities of string theory on orbifolds

One might think that a physical theory defined on an orbifold space could be singular, due to the bad geometric behaviour at the singular points. Interestingly, string theory on orbifold spaces is completely non-singular and well-behaved. This result follows from a very special set of states in string theory (twisted states), which arise due to the extended nature of strings (and would be absent in a theory of point particles).

To define string theory on an orbifold, we should regard the orbifold as a quotient of the torus by a symmetry. Therefore, string theory on the orbifold can be constructed by starting with string theory on the 'parent' torus, and imposing invariance under the discrete symmetry group, i.e. keeping only states which are invariant under the action of Γ (on the Hilbert space of string states). This sector is inherited from the spectrum of states in the



T^2/Z_3

Figure 6: Open string in a twisted sector in a $\mathbf{Z_3}$ orbifold.

toroidal compactification, and is called untwisted. Clearly it is described by a free 2d theory, because the metric is locally flat.

However, this is not the complete story. There exist additional closed string sectors arising from strings which are closed in the orbifold, but do not correspond to closed string in the 'parent' torus. They correspond to strings whose 2d fields have boundary conditions periodic, up to the action of some element $g \in \Gamma$, for instance

$$X(\sigma + \ell, t) = (gX)(\sigma, t) \tag{5}$$

this is shown in figure 6.

These sectors/states are known as twisted sectors/states. Notice that, these sectors are localized in the neighbourhood of fixed points, so in a sense are the sectors that carry the information that the orbifold space is not a torus, but has some curvature concentrated at those points. Note however, that the local 2d dynamics on the string is still the same as in the torus (since the inside of these strings still propagates in a flat metric), and all the non-triviality of the geometry enters simply in boundary conditions like (5). This remarkable feature allows to quantize the 2d theory exactly in α', although it describes propagation of strings in a non-trivial geometry. Note finally that twisted states exist because strings are extended objects, they would be absent in a theory of point particles.

The complete spectrum of the string theory on the orbifold is given by the untwisted sector (states in the torus, projected onto Γ-invariant states), and twisted sectors (one per element of Γ and per fixed point of the element).

Modular invariance

We would like to make a short and qualitative comment (although the argument is also quantitatively correct) showing that twisted sectors are absolutely crucial in order to have a consistent modular invariant theory, i.e. a consistent worldsheet geometry. Hence, twisted states are crucial in maintaining the good properties of string theory (finiteness, unitarity, anomaly cancellation, etc), and making it smooth even in the presence of the singular geometry. In a sense, we may say that α' stringy effects (the very existence of twisted states) corrects the singular behaviour of the geometry and leads to smooth physics.

Let us describe the 1-loop partition function for the theory on $\mathbf{T^6}$ as a torus, parametrized by σ, t, as in figure 7a. In order to construct the theory on $\mathbf{T^6}/\mathbf{Z_N}$, let us insert a projector operator

$$P = \frac{1}{N}(1 + \theta + \ldots \theta^{N-1}) \tag{6}$$

in the t direction, which forces that only $\mathbf{Z_N}$-invariant states give a non-zero contribution to the partition function. See fig 7b. Since only $\mathbf{Z_N}$-invariant states propagate, this describes the partition function for the untwisted sector.

Now we can see that this contribution is not modular invariant. Let us rewrite it as a sum of contributions with insertions of θ^k in the t direction, and perform a modular transformation $\tau \to -1/\tau$, which exchanges σ and t. We obtain a sum of amplitudes with insertions of θ^k in the sigma direction,

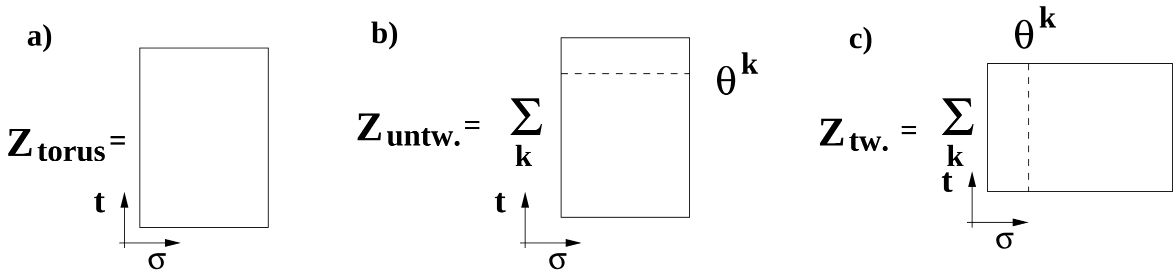


Figure 7: Modular invariance of string theory on orbifolds requires the existence of twisted sectors.

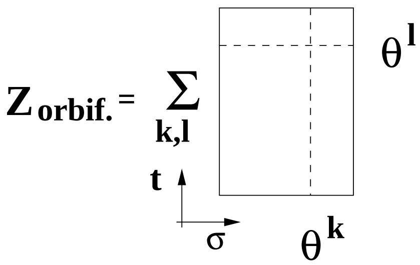


Figure 8: Modular invariant partition function for an orbifold.

see figure 7c. They correspond to closed strings which are periodic in σ, up to the action of θ^k; that is, they are twisted strings. Clearly, the complete modular invariant amplitude is as in figure 8, a sum over the untwisted and twisted sectors, with projector insertions in t to ensure that only $\mathbf{Z_N}$-invariant states propagate.

2 Type II string theory on $\mathbf{T^6/Z_3}$

Let us consider the above described $\mathbf{T^6/Z_3}$ orbifold, where the underlying $\mathbf{T^6}$ background is described by three complex coordinates $z_i \simeq z_i + R_i \simeq z_i + R_i e^{2\pi i/3}$, and zero NSNS B-field. Recall that the generator θ of $\mathbf{Z_3}$ acts by $\theta : z_i \to e^{2\pi i v_i}$ with $v = (1,1,-2)/3$.

We describe the 2d worldsheet theory (in the light-cone gauge) by the following fields: Along the two real non-compact coordinates, we have 2d bosons X^2, X^3 and 2d fermions ψ^2, ψ^3; to describe the three complex dimensions in $\mathbf{T^6}$, we have 2d bosons Z^1, Z^2, Z^3 (and their conjugates $Z^{\bar{\imath}}$) and 2d fermions Ψ^1, Ψ^2, Ψ^3 (and their conjugates $\Psi^{\bar{\imath}}$). The action of θ on these 2d fields is

$$Z^i \to e^{2\pi i v_i}\, Z_i \quad ; \quad \Psi^i \to e^{2\pi i v_i}\, \Psi^i \tag{7}$$

Let us consider the untwisted sector. The spectrum is obtained by simply taking the spectrum of the theory on $\mathbf{T^6}$ and keeping states invariant under the $\mathbf{Z_3}$ action. In the theory on $\mathbf{T^6}$, different sectors are labelled by the momentum and winding along the internal dimensions. For the corresponding 2d fields we have the following expansion

$$\begin{aligned} Z^i(\sigma,t) &= z_o^i + \frac{k_i}{R_i p^+}\, t + \frac{2\pi R_i}{\ell}\, w^i\, \sigma + \\ &= i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left[\frac{\alpha_n^i}{n}\, e^{-2\pi i\, n(\sigma+t)/\ell} + \frac{\tilde{\alpha}_n^i}{n}\, e^{2\pi i\, n(\sigma-t)/\ell} \right] \end{aligned} \tag{8}$$

where all the coefficients in the mode expansion (z_0, k, w, α's, $\tilde{\alpha}$'s) are complex, and the expansion for $Z^{\bar{i}}$ involve the complex conjugates.

The action of θ on the coefficient of the mode expansion are

$$\begin{aligned} z_0^i \to e^{2\pi i v_i} z_0^i \quad ; \quad k_i \to e^{2\pi i v_i} k_i \quad ; \quad w^i \to e^{2\pi i v_i} w^i \\ \alpha_n^i \to e^{2\pi i v_i} \alpha_i^n \quad ; \quad \tilde{\alpha}_n^i \to e^{2\pi i v_i}\, \tilde{\alpha}_n^i \end{aligned} \tag{9}$$

ans similarly for the 2d fermionic coordinates.

Untwisted states in the orbifold are obtained by taking suitable $\mathbf{Z_3}$ invariant linear combinations. For sectors of non-zero momentum and/or winding, such states are roughly of the form

$$\mathcal{O}|k,w\rangle + (\mathcal{O}^{\theta})|\theta k, \theta w\rangle + (\mathcal{O}^{\theta^2})|\theta^2 k, \theta^2 w\rangle \tag{10}$$

where $\mathcal{O}$ is a generic sausage of operators, and superscript θ^k implies taking its image under θ^k. The zero momentum and winding sectoris not mixed with other sector by θ, so one is constrained to use only operators $\mathcal{O}$ which are directly $\mathbf{Z_3}$ invariant. The mass formula for all these states is given by the same expression as for $\mathbf{T^6}$.

We will be interested in massless states. As usual, they arise from the sector of zero momentum and winding, so the spectrum is obtained by constructing the left and right vacua, and applying left and right moving operators whose phase transformation under θ cancel each other.

Consider the massless states in the left moving NS sector. They are

State	$SO(8)$ weight	$\mathbf{Z_3}$ phase
$\psi^2_{-1/2}\|0\rangle$, $\psi^3_{-1/2}\|0\rangle$	$(0,0,0,\pm)$	1
$\Psi^i_{-1/2}\|0\rangle$	$(\underline{+},0,0,0)$	$e^{2\pi i/3}$
$\Psi^{\bar{i}}_{-1/2}\|0\rangle$	$(\underline{-},0,0,0)$	$e^{-2\pi i/3}$

The phase picked up by the different states [2] can be also described as $e^{2\pi i\, r\cdot v}$, where r is the above $SO(8)$ weight and $v=(v_1,v_2,v_3,0)$.

For left handed states in the R sector (with GSO projection selecting the 8_C as vacuum), we have

[2]This arises naturally if one bosonizes the internal 2d fermions into 2d bosons ϕ^i compactified on a lattice of $SO(8)$ weights. The phase rotation of the 2d fermions becomes a translation of the corresponding bosons, which carry a lattice momentum r.

$SO(8)$ weight	$\mathbf{Z_3}$ phase
$\frac{1}{2}(+,+,+,-)$	1
$\frac{1}{2}(-,-,-,+)$	1
$\frac{1}{2}(\underline{-,+,+},+)$	$e^{2\pi i/3}$
$\frac{1}{2}(\underline{+,-,-},-)$	$e^{-2\pi i/3}$

Performing the same computation for the right movers (with opposite GSO on the R sector, since we are working on type IIA), the massless untwisted states are

Left $\otimes$ Right	$e^{2\pi i\, r\cdot v}$	$e^{2\pi i\, \tilde{r}\cdot} v$	4d field
NSNS			
Left $\otimes$ Right	$e^{2\pi i\, r\cdot v}$	$e^{2\pi i\, \tilde{r}\cdot} v$	4d field
$(0,0,0,\pm)\otimes(0,0,0,\pm)$	1	1	$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ
$(\underline{+,0,0},0)\otimes(\underline{-,0,0},0)$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	$G_{i\bar{j}}$, $B_{i\bar{j}}$ =
$(\underline{-,0,0},0)\otimes(\underline{+,0,0},0)$	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	9 cmplx scalars
NS-R			
Left $\otimes$ Right	$e^{2\pi i\, r\cdot v}$	$e^{2\pi i\, \tilde{r}\cdot} v$	4d field
$(0,0,0,\pm)\otimes\frac{1}{2}(+,+,+,+)$	1	1	$\psi_{\mu\alpha}$, ψ_α
$(0,0,0,\pm)\otimes\frac{1}{2}(-,-,-,-)$	1	1	4d gravitino and Weyl fermion
$(\underline{+,0,0},0)\otimes\frac{1}{2}(\underline{+,-,-},+)$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	9 spin 1/2 ...
$(\underline{-,0,0},0)\otimes\frac{1}{2}(\underline{-,+,+},-)$	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	...Weyl fermions
R-NS			
Left $\otimes$ Right	$e^{2\pi i\, r\cdot v}$	$e^{2\pi i\, \tilde{r}\cdot} v$	4d field
$\frac{1}{2}(+,+,+,-)\otimes(0,0,0,\pm)$	1	1	$\psi_{\mu\alpha}$, ψ_α
$\frac{1}{2}(-,-,-,+)\otimes(0,0,0,\pm)$	1	1	4d gravitino and Weyl fermion
$\frac{1}{2}(\underline{-,+,+},+)\otimes(\underline{-,0,0},0)$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	9 spin 1/2
$\frac{1}{2}(\underline{+,-,-},-)\otimes(\underline{+,0,0},0)$	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	Weyl fermions
RR			
Left $\otimes$ Right	$e^{2\pi i\, r\cdot v}$	$e^{2\pi i\, \tilde{r}\cdot} v$	4d field
$\frac{1}{2}(+,+,+,-)\otimes\frac{(}{+},+,+,+)$	1	1	Gauge boson
$\frac{1}{2}(+,+,+,-)\otimes\frac{(}{-},-,-,-)$	1	1	A_μ and
$\frac{1}{2}(-,-,-,+)\otimes\frac{(}{+},+,+,+)$	1	1	cmplx scalar
$\frac{1}{2}(-,-,-,+)\otimes\frac{(}{-},-,-,-)$	1	1	C_{123}, $C_{\overline{123}}$
$\frac{1}{2}(\underline{-,+,+},+)\otimes\frac{1}{2}(\underline{+,-,-},+)$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	9 Gauge bosons
$\frac{1}{2}(\underline{+,-,-},-)\otimes\frac{1}{2}(\underline{-,+,+},-)$	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	9 $C_{i\bar{j}\mu}$

Notice that there are two 4d gravitinos, signalling $\mathcal{N}=2$ 4d supersymmetry. Recalling the structure of the corresponding supermultiplet, the above fields are easily seen to gather into the supergravity multiplet ($G_{\mu\nu}$, the two

$\psi_{\mu\alpha}$ and A_μ), one hypermultiplet (the two ψ_α, ϕ and the scalar dual to $B_{\mu\nu}$), and 9 vector multiplets (scalars $G_{i\bar{j}}$, $B_{i\bar{j}}$, Weyl fermions in RNS and NSR, gauge bosons $C_{i\bar{j}\mu}$).

Let us now consider the twisted sector. As mentioned above, there is one such sector per non-trivial element in $\mathbf{Z_3}$ and per fixed point. The twisted states at each fixed point are similar, so we simply obtain 27 replicas of the content in one of them. Finally one can check that states in the θ^2 twisted sector correspond to the antiparticles of states in the θ twisted sector (it is easy to see graphically that states in oppositely twisted sectors can annihilate into the vacuum). So we just compute the latter.

In the θ twisted sector, we impose boundary conditions of the kind

$$Z^i(\sigma+\ell,t) \;=\; e^{2\pi i v_i}\, Z^i(\sigma,t) \;+\; 2\pi R_i\, n^i \tag{11}$$

(where n^i is a vector in the two-torus lattice Λ_i). That is, the string is closed up to the rotational and translational identification in the toroidal orbifold. Similarly for the 2d fermions. Using the general mode expansion

$$\begin{aligned} Z^i(\sigma,t) \;&=\; z_0^i + \frac{p^i}{p^+}\, t \,+\, \frac{2\pi R_i}{\ell}\, n^i\, \sigma \,+ \\ &+ \sum_{\nu_i} \frac{\alpha^i_{\nu_i}}{\nu_i}\, e^{-2\pi i\,\nu_i(\sigma+t)/\ell} + \sum_{\tilde{\nu}_i} \frac{\tilde{\alpha}^i_{\tilde{\nu}_i}}{\tilde{\nu}_i}\, e^{2\pi i\,\tilde{\nu}_i(\sigma-t)/\ell} \end{aligned} \tag{12}$$

(and similarly for 2d fermions) the boundary conditions impose that the zero mode sits at a fixed point

$$z_0^i = e^{2\pi i v_i} z_0^i \bmod 2\pi R_i\,\Lambda_i \tag{13}$$

that the momentum p^i and winding w^i vanish, and that the moddings of oscillators are shifted by $\pm v_i$. Indeed, we have the oscillators

$$\begin{array}{ccccccc} & \alpha^i_{n-v_i} & ; & \tilde{\alpha}^i_{n+v_i} & ; & \alpha^{\bar{i}}_{n+v_i} & ; \quad \tilde{\alpha}^{\bar{i}}_{n-v_i} \\ \Psi^i_{n+\rho-v_i} & ; & \tilde{\Psi}^i_{n+\rho+v_i} & ; & \Psi^{\bar{i}}_{n+\rho-v_i} & ; & \tilde{\Psi}^{\bar{i}}_{n+\rho+v_i} \end{array} \tag{14}$$

with $\rho = 1/2, 0$ for NS and R fermions.

The fractional modding of the oscillator modifies the vacuum energies. In the notes on type II superstring we used the familiar regularization by an exponential, and derived the relation

$$\frac{1}{2}\sum_{n=0}^{\infty}(n+\alpha) = -\frac{1}{24} + \frac{1}{4}\alpha(1-\alpha) \tag{15}$$

for $\alpha \geq 0$. Vacuum energies for orbifold follow from application of this formula.

We should now construct the Hilbert space of twisted states and impose the $\mathbf{Z_3}$ projection. Centering on left movers, the mass formula is given by

$$M_L^2 = \frac{2}{\alpha'}(N_B + N_F + E_0) \tag{16}$$

with $E_0 = -1/6$ in the NS sector and $E_0 = 0$ in the R sector.

In the NS sector, we define the vacuum as annihilated by all positive modding oscillators, and build the Hilbert space by applying negatively modded oscillators to it (and respecting the GSO projection). In the R sector, there are no fermion zero modes in the internal directions, just in the two non-compact ones. The vacuum is two-fold degenerate, and the GSO projection selects one of them as the only massless state. At the massless level, the states are

Sector	State	Mass	$r+v$	$e^{2\pi i(r+v)\cdot r}$
NS	$\Psi^3_{-1/6}\vert 0\rangle$	$m^2=0$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$	1
R	$A_1^+\vert 0\rangle$	$m^2=0$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{2})$	1

where we have labelled the states by a vector $r+v$, which is useful in determining the $\mathbf{Z_3}$ phase picked up by the state [3].

[3] In the bosonized formulation, twisted states have momentum in a shifted lattice, so the notation $r+v$ is more natural.

Working similarly with the right moving sector (with opposite GSO in the R sector, since we are in IIA), we can construct the massless physical states

Sector	$r+v \otimes \tilde{r}-v$	$SO(2)$
NSNS	$(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\otimes(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{3},0)$	0
NSR	$(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)\otimes(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2})$	$-1/2$
RNS	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{2})\otimes(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},0)$	$-1/2$
RR	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{2})\otimes(\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{2})$	-1

It is important to recall that right movers have an opposite shift in the modding of oscillators (hence we label the states are $\tilde{r}-v$.

Together with states in the θ^2 twisted sector (antiparticles), we obtain one 4d $\mathcal{N}=2$ vector multiplet per fixed point. They give rise to independent $U(1)$ gauge symmetries (no non-abelian enhancement).

The total spectrum of type IIA theory on the $\mathbf{T^6}/\mathbf{Z_3}$ orbifold is: the 4d $\mathcal{N}=2$ gravity multiplet, one hypermultiplet and $9+27=36$ (abelian) vector multiplets.

2.1 Geometric interpretation

This spectrum is very much like the spectrum on a compactification on a smooth Calabi-Yau with Hodge numbers $(h_{1,1},h_{2,1})=(36,0)$.

Indeed, mathematicians know that the singular space $\mathbf{T^6}/\mathbf{Z_3}$ can be regarded as a particular limit of a smooth Calabi-Yau, in the limit in which 27 4-cycles collaps to zero size (This is a singular limit in the geometric sense, but is completely smooth in string theory, due to twisted states, namely to α' effects).

In other words, the singular space $\mathbf{T^6}/\mathbf{Z_3}$ can be continuously smoothed to a non-singular space, preserving the Calabi-Yau property. This is done by

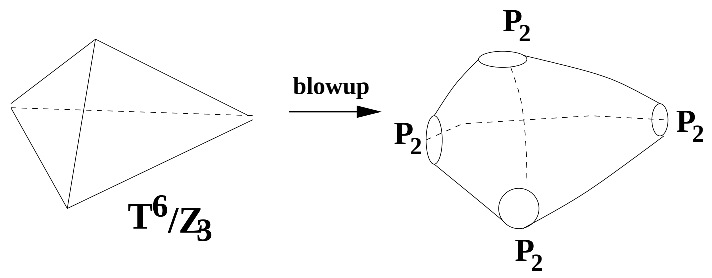


Figure 9: The singular orbifold $\mathbf{T^6/Z_3}$ is a particular limit of a smooth Calabi-Yau in the limit in which some $\mathbf{P_2}$ 4-cycles go to zero size. The smooth Calabi-Yau is called the blowup of the orbifold.

the procedure known as blowing-up the singular point; roughly, this amounts to removing the 27 singular points of the orbifold and replacing them by a suitable 4-cycle, which for the singularities at hand must be a $\mathbf{P_2}$, the two (complex) dimensional projective space [4]. see figure 9. The resulting space is Kahler and has vanishing first Chern class, so it admits a $SU(3)$ holonomy metric. The smooth spaces are characterized by moduli which control the size of the $\mathbf{P_2}$'s, so the singular orbifold is geometrically recovered at the point of moduli space corresponding to zero sizes. Of course this limit is beyond the reach of the supergravity approximation, which is not valid for so small lentgh scales. Happily, the singular limit is nice enough so that we can quantize string theory exactly in that regime.

The homology of the resulting smooth space can be computed as follows: Before blowing up the homology was given by the homology of cycles in $\mathbf{T^6}$ invariant under the $\mathbf{Z_3}$ (what mathematicians call the equivariant homology) which leads to Hodge numbers $(h_{1,1}, h_{2,1}) = (9, 0)$. To these we must add

[4]This is the set of points $(z_1, z_2, z_3) \in \mathbf{C^3}$ with the identification $(z_1, z_2, z_3) \simeq \lambda(z_1, z_2, z_3)$ with $\lambda IC - \{0\}$.

the homology of cycles associated to the $\mathbf{P_2}$'s, which appear after blowup. Each $\mathbf{P_2}$ has one 2-cycle and no 3-cycle inside it, so their contribution to Hodge numbers is $(27,0)$. Therefore we see that the homology of the smooth blowup $\tilde{\mathbf{T}}^6/\mathbf{Z_3}$ is $(36,0)$.

Thus, string theory is clever enough to 'know' that the singular orbifold belongs to a continuous family of smooth spaces with Hodge numbers $(36,0)$, and thus gives the right spectrum in the orbifold space.

The above geometric interpretation allows a geometric interpretation for the twisted sector fields in string theory. Indeed, denoting Σ the 2-cycle inside the collapsed $\mathbf{P_2}$ at each singularity, we interpret: the two real scalars correspond to the geometrical size of $\mathbf{P_2}$ (i.e. a metric modulus) and to $\int_\Sigma B$; the gauge boson corresponds to $\int_\Sigma C_3$.

It is important to emphasize that the philosophy of the geometric interpretation of the orbifold spaces also exists for other orbifolds (although the cycles arising upon blowing up are in general more involved). It is in this precise sense that orbifolds are very similar to Calabi-Yau spaces (in fact, they are CY's at a particular point in moduli space) but far more tractable.

3 Heterotic string compactification on $\mathbf{T^6/Z_3}$

3.1 Gauge bundles for orbifolds

Compactification of heterotic string on orbifolds is very similar to type II. The main difference is that noew we have the left moving internal bosons X^I, and we have the freedom of choosing a non-trivial action of Γ on them. For $\Gamma = \mathbf{Z_N}$ a simple choice is to require that the generator θ acts as a shift [5] $X^I \to X^I + V^I$, where NV is a vector in the internal 16d lattice $\Lambda_{\text{int.}}$.

[5] One may thing that it is more natural to use a rotation of the X^I instead of the above shift. In fact, both options are related by conjugation of the rotation to the Cartan

Using the relation of $\mathbf{T^6}/\mathbf{Z_3}$ with the singular limit of a smooth Calabi-Yau threefold, the above embedding of $\mathbf{Z_N}$ in the gauge degrees of freedom corresponds, from the geometric viewpoint, to using a non-trivial gauge bundle in the compactification. In fact, just as for Calabi-Yau compactification, it is not consistent to choose $V = 0$. Indeed, modular invariance imposes the constraint

$$N(V^2 - v^2) = \text{even} \tag{17}$$

(this arises from requiring invariance under $\tau \to \tau + N$, which imposes a constraint on the contributions from the unpaired right moving fermions and left moving internal bosons).

A natural choice of gauge shift, although there exist other consistent ones, is to take V to be a copy of v. For instance, we center on the $\mathbf{Z_3}$ orbifold of the $E_8 \times E_8$ heterotic string, so we take

$$V = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0, 0, 0, 0) \tag{18}$$

Clearly this is the equivalent of the standard embedding which we studied for smooth Calabi-Yau threefolds.

3.2 Computation of the spectrum

The computation of the spectrum is easy as for the type II theories. In the untwisted sector we need to take the states of the theroy on $\mathbf{T^6}$ and keep those invariant under $\mathbf{Z_3}$. In heterotic theory the only additional ingredient is to realize that states with internal momentum P^I pick up a phase $e^{2\pi i\, P\cdot V}$ under the action of θ. At the massless level, we have the following massless right and left moving states

subalgebra. More manifestly, the shift in the bosonic coordinates is equivalent to a rotation of the 32 internal fermions in the fermionic description of the heterotic.

Right

	r	$e^{2\pi i\, r\cdot v}$		r	$e^{2\pi i\, r\cdot v}$
NS	$(0,0,0,\pm)$	1	R	$\frac{1}{2}(+,+,+,-)$	1
	$(\underline{+,0,0},0)$	$e^{2\pi i/3}$		$\frac{1}{2}(-,-,-,+)$	1
	$(\underline{-,0,0},0)$	$e^{-2\pi i/3}$		$\frac{1}{2}(\underline{-,+,+},+)$	$e^{2\pi i/3}$
				$\frac{1}{2}(\underline{+,-,-},-)$	$e^{-2\pi i/3}$

Left

State	θ phase	State $\lvert P\rangle$	$e^{2\pi i\, P\cdot V}$
$\alpha^2_{-1}\lvert 0\rangle$	1	E_8'	1
$\alpha^3_{-1}\lvert 0\rangle$	1	$E_6\times SU(3)$	1
$\alpha^i_{-1}\lvert 0\rangle$	$e^{2\pi i/3}$	$(3,27)$	$e^{2\pi i/3}$
$\alpha^{\bar{i}}_{-1}\lvert 0\rangle$	$e^{-2\pi i/3}$	$(\overline{3}),\overline{2}7)$	$e^{-2\pi i/3}$
$\alpha^I_{-1}\lvert 0\rangle$			

The decomposition of the $E_8\times E_8$ roots with respect to the $E_6\times SU(3)\times E_8$ is exactly as in the lecture on Calabi-Yau compactification, from which the phases $e^{2\pi i\, P\cdot V}$ are easily obtained.

Glueing the left and right moving states in a $\mathbf{Z_3}$ invariant fashion we get

Sector	State	4d Field
NS	$(0,0,0,\pm)\otimes\alpha^{2,3}_{-1}\lvert 0\rangle$	$G_{\mu\nu}$, $B_{\mu\nu}$, ϕ
	$(0,0,0,\pm)\otimes[\,E_6\times SU(3)\times E_8'\,]$	Gauge bosons
	$(\underline{+,0,0},0)\otimes[(\overline{3},\overline{27})]$	Complex scalars
	$(\underline{-,0,0},0)\otimes[(3,27)]$	Complex scalars
R	$\pm\frac{1}{2}(+,+,+,-)\otimes\alpha^{2,3}_{-1}\lvert 0\rangle$	4d gravitino, Weyl spinor
	$\pm\frac{1}{2}(+,+,+,-)\otimes[\,E_6\times SU(3)\times E_8'\,]$	Gauginos
	$\frac{1}{2}(\underline{-,+,+},+)\otimes[(\overline{3},\overline{27})]$	Weyl spinors
	$\frac{1}{2}(\underline{+,-,-},-)\otimes[(3,27)]$	Weyl spinors

In total, we get the 4d $\mathcal{N}=1$ supergravity multiplet, vector multiplet with gauge group $E_6\times SU(3)\times E_8'$, one neutral chiral multiplet, and 3 chiral multiplets in the $(3,27)$. Note that the spinors in the conjugate representation have also opposite chirality, so they are their antiparticles.

In the θ twisted sector, the only new ingredient is that the 16d internal momenta P are shifted by V. This follows from the boundary conditions for the internal coordinates in a twisted sector

$$X_L^I(\sigma + t + \ell) = X^I(\sigma + t) + P^I + V^I \tag{19}$$

(with P^I is a winding/momentum in $\Lambda_{\text{int.}}$. Upon imposing it on the corresponding mode expansion

$$X_L(\sigma + t) = \frac{P_\theta^I}{2p^+} + i\sqrt{\frac{\alpha'}{2}}\sum_\nu \alpha_n^I\, e^{-2\pi i\, n(\sigma+t)/\ell} \tag{20}$$

we obtain the promised relation $P_\theta^I = P^I + V^I$, and the oscillators are integer-modded.

The right moving sector behaves as in type II. The left-moving spacetime mass is

$$M_L^2 = \frac{2}{\alpha'}\,(N_B + \frac{(P+V)^2}{2} + E_0) \tag{21}$$

with $E_0 = -1 + 3 \times \frac{1}{2}\frac{1}{3}\frac{2}{3} = -\frac{2}{3}$.

We have the right moving massless states

Sector	$r+v$	$e^{2\pi i(r+v)\cdot r}$
NS	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$	1
R	$(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{2})$	1

The massless left moving massless states are

Osc.	P	$P+V$
$N_B=0$	$(-,-,0,0,0,0,0,0)$	$(-\frac{2}{3},-\frac{2}{3},-\frac{2}{3},0,0,0,0,0)$
	$(0,0,+,\underline{\pm,0,0,0,0})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3},\underline{\pm,0,0,0,0})$
	$\frac{1}{2}(-,-,+,\pm,\pm,\pm,\pm,\pm)$	$(-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$
$N_B=1/3$	$(0,0,0,0,0,0,0,0)$	$(\frac{1}{3},\frac{1}{3},-\frac{2}{3},0,0,0,0,0)$
	$(\underline{-,0},+,0,0,0,0,0)$	$(\underline{-\frac{2}{3},\frac{1}{3}},\frac{1}{3},0,0,0,0,0)$

where we have ignored the momentum in the second E_8 piece of $\Lambda_{\text{int.}}$ since it is zero for all these states.

The $N_B=0$ states transform in the representation $(1,27)$ under $SU(3)\times E_6$. All of them pick up a phase $e^{2\pi i(P+V)\cdot V}=1$. The states with $N_B=1/3$ transform in the representation $(\overline{3},1)$ under $SU(3)\times E_6$. There are three of them correspoding to the oscillators $\alpha^1_{-1/3}$, $\alpha^2_{-1/3}$ and $\alpha^{\overline{3}}_{-1/3}$. They also pick up a total phase 1 under θ, with the oscillator phase compensating the phase from the internal momentum.

Glueing the left and right moving states is now straightforward. The result is one chiral multiplet in the $(1,27)$ and three chiral multiplets in the $(\overline{3},1)$ per fixed point. Note that the θ^2 sector contains the antiparticles of these.

In total, the massless spectrum is given by the 4d $\mathcal{N}=1$ supergravity multiplet, $E_6\times SU(3)\times E_8'$ vector multiplets, the dilaton chiral multiplet, and the following charged chiral multiplets

$$3(3,27;1)+27(1,27;1)+27\times 3\,(\overline{3},1;1) \tag{22}$$

This is remarkable, since it corresponds to an E_6 grand unificiation theory with 36 fermion families. Although not realistic, it is remarkable that we can obtain an explicit construction of string theory models with features similar to those of the Standard Model.

There an important point we would like to mention. Notice that $SU(3)$ has potential chiral anomalies (E_6 is always automatically non-anomalous). The anomalies however vanish because the spectrum contains as many chiral multiplets in the 3 as in the $\overline{3}$. Note that for this to be true it is essential that twisted sectors are included in the theory! Hence this is a simple example where we see that string theory requires the presence of twisted sectors for consistency. Incidentally, we point out that the story of anomaly cancellation in 4d is even richer in models with $U(1)$ factors in the gauge group, since mixed anomalies involve a 4d version of the Green-Schwarz mechanism. we leave this discussion for the interested reader.

Notice that the above spectrum is roughly (looking just at the number of E_6 representations) that corresponding to compactification on a smooth Calabi-Yau with Hodge numbers $(36,0)$ and gauge bundle specified by the standard embedding. This agrees with the geometric interpretation of $\mathbf{T^6}/\mathbf{Z_3}$ we described in type II. It is interesting to notice that in this case the fields in twisted sectors that correspond to resolving the singularity are the states with $N_B = 1/3$. They not only blow up the singularities but also deform the gauge bundle (and break the gauge factor $SU(3)$). On the other hand, the states with $N_B = 0$ correspond to deformations of the gauge bundle (break the gauge group) preserving the singular geometry (these states do not carry any index of the internal space). See [3] for a nice discussion of moduli space of local versions of this orbifold.

3.3 Final comments

In conclusion, we see how easily and systematically one can construct compactifications on orbifolds. These have the advantage that they allow explicit string theory models, exact in α', while keeping the rich and interesting dynamics of reduced supersymmetry.

These constructions have many advantages:

• The low energy effective action is computable including α' corrections, which include the effects of massive string states. This kind of corrections can be important, for instance, in the computation of threshold effects to the unification of gauge coupling constants.

• The classification and construction of heterotic models is very systematic (and easy to program on a computer), hence allows for searching phenomenologically interesting models.

• There are many generalizations of the basic construction we have described: inclusion of Wilson lines, other orbifold groups. A less intuitive extension is that of asymmetric orbifolds [4], where one considers modding the left and right movers with different orbifold action, being careful to ensure modular invariance. These have the interesting feature that many moduli are frozen at fixed values (typically corresponding to self-dual points with respect to the T-duality group). They are however too technical to be discussed here.

The lesson to take home is that orbifolds allow to construct compactifications of full-fledged string theory (and not just supergravity) with interesting features, even close to those of Particle Physics.

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