Calabi-Yau compactification of superstrings. Heterotic string phenomenology

1 Motivation

We have seen that toroidal compactification leads to 4d theories at low energies. However, it is too simple to lead to anything realistic, similar to the Standard Model of Particle Physics. The fact that toroidal compactification does not break any of the supersymmetries of string theory implies the 4d theories are non-chiral. We are missing an essential ingredient of Particle Physics.

Thus we have to consider more general compactifications with background geometry $\mathbf{M_4} \times \mathbf{X_6}$, where $\mathbf{X_6}$ is a compact curved manifold [1]. Since the background metric is not flat, the worldsheet 2d theory is interacting, and not exactly solvable. Hence one usually works at leading order in the 2d expansion prameter, which is α'/r^2, where r is a curvature length scale in spacetime. This corresponds to working at low energies, in the supergravity limit, and is a good approximation if all curvature length scales of $\mathbf{X_6}$ are large compared with the string length. This is essentially a point particle limit, and the stringy physics will be hidden in the α' corrections, which are very difficult to obtain.

It should be pointed out, though, that there exist some abstract exactly solvable 2d conformal field theories, known as Gepner models, which are proposed to describe (exactly in α') the physics of string theories on spaces of

[1] We also include in our ansatz that backgrounds for other bosonic fields are trivial, e.g. we do not consider compactifications with field strength fluxes for p-form fields, which only very recently have been considered in the literature.

stringy size. Also, in next lecture we will study orbifolds, which are in a sense, simple versions of non-trivial spaces, which still lead to free 2d worldsheet theories (with sectors of non-trivial boundary conditions).

1.1 Supersymmetry and holonomy

We are interested in compactifications which preserve some 4d supersymmetry. Compactifications breaking all the supersymmetries would be very interesting but

- often contain instabilities, appearing as tachyonic fields in 4d.
- lead to a too large 4d cosmological constant to be of any phenomenological use to describe the real world.

Nevertheless, it is important to realize that assuming supersymmetry is also an oversimplification if one is interested in describing the real world, which is not exactly supersymmetric. Upon breaking supersymmetry (by some of the mechanisms in the market) the above two problems rearise [2].

Finally, it is possible to see that the conditions imposed on $\mathbf{X_6}$ by supersymmetry ensure that the background satisfies the supergravity equations of motion, it is a good vacuum of the theory. This can be found in the main reference for this lecture [1].

What are the conditions on $\mathbf{X_6}$ in order to have some unbroken 4d supersymmetry? Recall from our discussion of Kaluza-Klein reduction that 4d fields visible at low energies are zero modes, constant in the internal space. Similarly, gauge symmetries visible at low energies correspond to gauge transformations constant over the internal space. Analogously, supersymmetries unbroken in the low energy 4d physics correspond to (local) supersymme-

[2] Yes, it is a bit disappointing that for the moment string theory has not given a strong proposal to solve the cosmological constant problem, despite many attempts.

try parameters (which are spinors $\xi(x^\mu, x^i)$ in $\mathbf{M_4} \times \mathbf{X_6}$) which are covariantly constant in $\mathbf{X_6}$ (with the connection inherited from the metric), i.e. $\nabla_{\mathbf{X_6}} \xi(x^i) = 0$.

Recalling now the discussion of the holonomy group of a Riemannian manifold, we can obtain a conditions on $\mathbf{X_6}$ to admit covariantly constant spinors. Clearly, a covariantly constant spinor is a singlet under the holonomy group (of the spinor bundle with the spin connection), since it does not change under parallel transport around a closed loop. This implies that the holonomy group of a Riemannian manifold IX_6 leaving some 4d susy unbroken is not generic. The generic holonomy for a metric in a 6d manigold is $SO(6)$, and spinors transform in the representation 4 or $\overline{4}$ under it (depending on their chirality [3]), hence there is no singlet, and no covariantly constant spinor. For metrics of $SU(3)$ holonomy, spinors transform as $3+1$ or $\overline{3}+1$, hence there are components which are singlets under the holonomy group, corresponding to covariantly constant spinors. The decomposition of a susy parameter in 10d under the holonomy and 4d Lorentz groups follows from the following chain

$$\begin{array}{ccccc} SO(10) & \rightarrow & SO(6)\times SO(4) & \rightarrow & SU(3)\times SO(4) \\ 16 & & (4,2)+(4',2') & & (3,2)+(\overline{3},2')+(1,2)+(1,2') \end{array}$$

In the last column only the $SU(3)$ singlet components lead to 4d supersymmetries, while the others are broken by the compactification.

The surviving supersymmetries can also be verified by looking at the KK reduction of 10d gravitinos under the holonomy group. This is described by the following chain. The 10d gravitinos are in the say 56_S of $SO(8)$, which arises from a product $8_V \times 8_C$. Decomposing with respect to $SO(6)\times SO(2)$,

[3]The Lie algebras of $SO(6)$ and $SU(4)$ are the same, and the spinor representations of $SO(6)$ are the fundamental and antifundamental of $SU(4)$, so they are often written 4 and $\overline{4}$.

we have $8_V \to 6_0 + 1_{\pm 1}$ and $8_C = 4_{1/2} + \overline{4}_{-1/2}$, where subindices denote the $SO(2)$ charges. We are interested in 4d gravitinos, which have spin 3/2 with respect to $SO(2)$; these fields are obtained from the product $1_{\pm 1} \times (4_{1/2} + \overline{4}_{-1/2}$, and decompose under $SU(3) \times SO(2)$ as $1_{\pm 1} \times (3_{1/2} + \overline{3}_{-1/2} + 1_{1/2} + 1_{-1/2})$. Clearly only the latter lead to 4d gravitinos unbroken by the compactification. It is possible to check that one 10d gravitino leads to one 4d gravitino if the holonomy group of the compactification manifold is $SU(3)$.

The generalization of the above statements to other dimensions is that compactification on a $2n$-dimensional manifold with $SU(n) \subset SO(2n)$ holonomy preserves some supersymmetry.

1.2 Calabi-Yau manifolds

A $2n$-dimensional manifold admitting a metric with spin connection of $SU(n)$ holonomy is a Calabi-Yau manifold.

This definition is difficult to use in order to determine whether a manifold is Calabi-Yau, since in principle one needs an explicit construction of the metric. This is very difficult: in fact there is no known explicit metric for any (non-trivial) compact Calabi-Yau, explicit metrics are knonw only for a few examples of non-compact spaces. Happily the existence of a metric with this property is guaranteed for manifolds satisfying the following (simplest to check) topological conditions: the manifold must be Kahler and (its tangent bundle must) have vanishing first Chern class.

To understand better the meaning of these conditions, we need some background information on complex differential geometry.

An n-dimensional complex manifold is a topological space M, together with a holomporphic atlas, i.e. a set of charts $(U_\alpha, z_{(\alpha)})$ where $z_{(\alpha)}$ are maps from U_α to some open set in $\mathbf{C^n}$, such that i) the U_α cover M, ii) on $U_\alpha \bigcap U_\beta$,

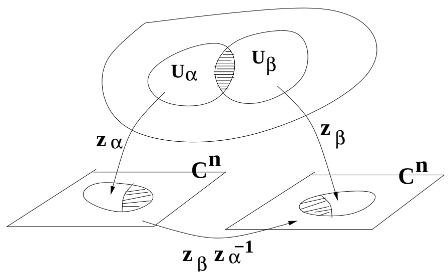

Figure 1: Charts covering a complex manifold.

the map

$$z_{(\beta)} \circ z_{(\alpha)}^{-1} : z_{(\alpha)}(U_\alpha \bigcap U_\beta) \longrightarrow z_{(\beta)}(U_\alpha \bigcap U_\beta) \tag{1}$$

is holomorphic (namely $\partial z_{(\beta)}/\partial \overline{z}_{(\alpha)} = 0$. See figure 1.

Notice that a complex n-dimensional manifold can always be regarded as a real $2n$-dimensional differential manifold, by simply splitting the complex coordinates into its real and imaginary parts. On the other hand,a real $2n$-dimensional manifold M can be regarded as an n-dimensional complex manifold only if it admits a globally defined tensor of type $(1,1)$, $J_m^n dx^m \otimes \partial_n$ satisfying

i) $J_m^n J_n^l = -\delta_m^l$

(this is used to define local complex coordinates $dz^i = dx^i + iJ_l^i dy^l$ and $d\overline{z}^i = dx^i - iJ_l^i dy^l$)

ii) The Niejenhuis tensor vanishes

$$N_{ij}^k = \partial_{[j} J_{i]}^k - J_{[i}^p J_{j]}^q \partial_q J_p^k = 0 \tag{2}$$

which ensures that the local complex coordinates have holomorphic transition functions. Such a J is called a complex structure [4].

[4]Manifolds with tensors J satisfying i) but not ii) are called almost complex, and J is called almost complex structure.

Notice that a given real differential manifold can admit many complex structures. A familiar example is provided by the 2-torus, which admits a one (complex) dimensional family of complex structures, parametrized by a complex number τ; the two real coordinates x, y can be combined to form a complex coordinates via $dz = dx + \tau dy$.

In a complex manifold, p-forms and their cohomology classes (and p-chains and their homology classes) can be refined according to their number of holomorphic and antiholomorphic indices [5]. For instance, the 3-cohomology group splits as

$$H^3(M) = H^{(3,0)}(M) + H^{(2,1)}(M) + H^{(1,2)}(M) + H^{(0,3)}(M) \tag{3}$$

where $H^{(p,q)}$ corresponds to forms with p holomorphic and q antiholomorphic indices (spanned by a basis $dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \ldots \wedge dz^{j_p}$. The dimensions of the $H^{(p,q)}$ are denoted $h_{p,q}$ and known as Hodge numbers; although to define them we have introduced a complex structure, they do not depend on the particular complex structure chosen, so they are topological invariants of M.

A metric in a complex manifold is called hermitian if it is of the form

$$ds^2 = g_{i\overline{j}} dz^i d\overline{z}^{\overline{j}} \tag{4}$$

namely has non-zero components only for mixed indices. Such metric can be used to lower one index of the complex structure tensor and thus define the $(1,1)$ form

$$J = g_{i\overline{j}} dz^i \wedge d\overline{z}^{\overline{j}} \tag{5}$$

A hermitian metric is called Kahler metric if the associated $(1,1)$-form satisfies

$$dJ = 0 \tag{6}$$

[5]In fact, this can be done even for almost complex manifolds.

The $(1,1)$-form is known as Kahler form. A manifold which admits a Kahler metric is called a Kahler manifold; this is a topological property of the manifold.

Notice that the Kahler form defines a non-trivial cohomology class in $H^{(1,1)}(M)$. It defines a cohomology class because it is closed. We can show that the class is non-trivial because (5) implies

$$\int_M J \wedge \ldots \wedge J = \int_M \sqrt{\det g}\, dz^1 d\overline{z}^1 \ldots dz^n d\overline{z}^n = \mathrm{Vol}(M) \tag{7}$$

which would be vanishing if J is exact (since $J = dA$ would imply $\int J \ldots J = \int d(AJ \ldots J) = 0$).

The Kahler form is very interesting since it characterizes the overall volume of the manifold M. In particular, α' corrections are in fact weighted by the adimensional parameter α'/r^2, where r is an overall size determined by the Kahler form.

Returning to the issue of holonomy, the crucial property of Kahler manifolds is that the Christoffel connection induced by the Kahler metric leads to a parallel transport that does not mix holomorphic and antiholomorphic indices. This implies that the holonomy group is in a $U(n)$ subgroup of $SO(2n)$, as is manifest e.g. by splitting the basis of tangent space in holomorphic and antiholomorphic elements

$$(\partial_{z^1}, \ldots, \partial_{z^n}; \partial_{\overline{z}^1}, \ldots, \partial_{\overline{z}^n}) \tag{8}$$

The $U(1)$ part of the holonomy can be seen to be associated to the Ricci tensor, so the manifold must admit a Kahler and Ricci-flat metric to have $SU(n)$ holonmy. A necessary topological condition for this is that the first Chern class $c_1(R)$ of the tangent bundle is trivial. Calabi conjectured this to be also a sufficient condition, as was finally proved by Yau (hence the name Calabi-Yau for such spaces).

Yau's theorem states that, given a complex manifold with $c_1(R) = 0$ and Kahler metric g with Kahler form J, there exists a unique Ricci-flat metric g' with Kahler form J' in the same cohomology class. It provides, as promised, a topological way of characterizing manifolds for which a $SU(n)$ holonomy metric exists (without constructing it explicitly). This facilitates the classification and study of Calabi-Yau spaces, in fact tables with many hundreds of such spaces exist in the literature.

Yau's theorem also provides a characterization of the parameters that determine the $SU(n)$ holonomy metric. For a given differential manifold M we should

i) specify the parameters that define a complex structure on this real manifold to make it a complex manifold. This set of parameters spans what is called the compelx structure moduli space, and can be computed to have (complex) dimension $h_{2,1}(M)$.

ii) for fixed complex structure, specify the parameters which define the Kahler class. This set of parameters is known as Kahler moduli space, and clearly has (real) dimension $h_{1,1}(M)$.

The complete moduli space of Calabi-Yau metrics in a given differential manifold M is (locally) the product of these.

We would like to point out that the condition for supersymmetry which we have used is valid to lowest order in α'. In particular, one can imagine that there could be higher order α' corrections that modify the 'equation of motion' condition Ricci=0. However, there are diverse arguments (see [2]) showing that in differential manifolds, satisfying the topological conditions of being Kahler and have zero first Chern class, there exists some underlying 2d interacting field theory which is conformal exactly in α'. In other words, the leading α' proposal for the metric can be consistently completed to an α' exact one.

The Calabi-Yau condition implies certain structure of Hodge number. For 6d manifolds admitting a metric of holonomy $SU(3)$ (and not in a proper subgroup like $SU(2)$), often referred to as Calabi-Yau threefolds, they read

$$
\begin{array}{ccccccc}
 & & & h_{0,0} & & & \\
 & & h_{1,0} & & h_{0,1} & & \\
 & h_{2,0} & & h_{1,1} & & h_{0,2} & \\
h_{3,0} & & h_{2,1} & & h_{1,2} & & h_{0,3} \\
 & h_{3,1} & & h_{2,2} & & h_{3,1} & \\
 & & h_{3,2} & & h_{2,3} & & \\
 & & & h_{3,3} & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h_{1,1} & & 0 & \\
1 & & h_{2,1} & & h_{2,1} & & 1 \\
 & 0 & & h_{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

where equality of some Hodge numbers is due to duality between $H^{(p,q)}$ and $H^{(3-p,3-q)}$. Due to its shape, this diagram is known as Hodge diamond.

We conclude with some examples. In one complex dimension, the only compact Calabi-Yau space actually has trivial holonomy, it is the 2-torus. In two complex dimensions, there is only one topological space admitting $SU(2)$ holonmy metrics, known as K3 (complex) surface. Although a lot is known about the topology of this space, no explicit metrics are known. In three complex dimensions, there exist many compact Calabi-Yau spaces. One of the simplest is the quintic, which can be described as the (complex) hypersurface

$$f_5(z_1, \ldots, z_5) = 0 \tag{9}$$

in $\mathbf{P_5}$, the (four) complex (dimensional) projective space [6]. Here $f_5(z_1, \ldots, z_5)$ denotes a degree 5 polynomial (so that it is homogeneous and well-defined in $\mathbf{P_5}$). The general such polynomial (up to redefinitions) depends on 101 complex parameters, which determine the complex structure of the Calabi-Yau.

[6] This is the set of points $(z_1, \ldots, z_5) \in \mathbf{C^5}$ with the equivalence relation $(z_1, \ldots, z_5) \simeq (\lambda z_1, \ldots, \lambda z_5)$ with $\lambda \in \mathbf{C} - \{\mathbf{0}\}$.

Also, there is one Kahler parameter determining the overall size of $\mathbf{P_5}$ and hence of the quintic. Its Hodge diamond has therefore $h_{2,1} = 101$, $h_{1,1} = 1$.

2 Type II string theories on Calabi-Yau spaces

We now study what kind of theories arise from compactification of type II string theories on $SU(3)$ holonomy spaces.

2.1 Supersymmetry

Type II theories have two 10d gravitinos. Upon compactification on Calabi-Yau threefolds we obtain two 4d gravitinos, which corresponds to 4d $\mathcal{N} = 2$ supersymmetry. This is a non-chiral supersymmetry, so it appears for both IIA and IIB theories. The massless supermultiplets that may appear are:

i) the gravity multiplet, containing a graviton, a gauge boson (graviphoton), and two gravitinos of opposite chiralities

ii) the vector multiplet, containing a gauge boson, a complex scalar and a Majorana fermion, all in the adjoint representaion of the gauge group

iii) the hypermultiplet, containing two complex scalars (in conjugate representations) and two Weyl fermions (in the same representation with opposite chiralities).

This structure makes it sufficient to determine the bosonic fields after compactification; the fermionic fields can be completed by using this multiplet structure.

2.2 KK reduction of p-forms

Since type II theories contain p-form fields in 10d, we need to know how to perform their KK reduction. A p-form in 10d $C_p(x^0, \ldots, x^9)$ can give rise to

4d q-forms via the ansatz

$$C_{\mu_1\ldots\mu_q m_1\ldots m_r}(x^0,\ldots,x^9) \;=\; C_{\mu_1\ldots\mu_q}(x^0,\ldots,x^3)\,A_{m_1\ldots m_r}(x^4,\ldots,x^9) \qquad (10)$$

with $q+r=p$. The 4d q-form has a 4d mass given by the laplacian acting on the internal piece. The laplacian is read off from the kinetic term of p-forms, which is

$$\int dC\wedge *dC = \int (dC,dC) = \int (C,\Delta C) \qquad (11)$$

and $\Delta = dd^\dagger + d^\dagger d$. Hence to get a massless 4d q-form we need to pick the internal r-form $A_{m_1\ldots m_r}(x^4,\ldots,x^9)$ to be a harmonic r-form in $\mathbf{X_6}$, namely $dA=0$, $d^\dagger A=0$.

Since the number of linearly independent harmonic r-forms in IX_6 is $b_r(\mathbf{X_6})$, the dimension of $H^r(\mathbf{X_6},\mathbf{R})$, we obtain b_r independent q-forms in the KK reduction of the 10d p-form C_p.

That is, the ansatz for the zero mode of C_p is

$$C_{\mu_1\ldots\mu_q m_1\ldots m_r}(x^0,\ldots,x^9) \;=\; \sum_{\alpha=1}^{b_r} C^a_{\mu_1\ldots\mu_q}(x^0,\ldots,x^3)\,A^a_{m_1\ldots m_r}(x^4,\ldots,x^9) \quad (12)$$

The 4d q-form is often written as $\int_{\Sigma_a} C_p$, where r of the indices of C_p are integrated along the r-cycle Σ_a, dual of the r-form A_a.

We would like to emphasize the fact that out of a unique 10d field we have obtained several 4d fields with same quantum numbers. This arises simply because of the existence of several zero modes for a kinetic operator in the internal space. That is, several zero energy resonance modes of a 10d field in the 6d 'cavity' given by the internal space. As we will see later on, this beautiful mechanism is a possible origin of family replication in heterotic models reproducing physics similar to the Standard Model.

2.3 Spectrum

We now have enough tools to directly determine the spectrum of type IIA/B compactifications on Calabi-Yau threefolds with Hodge numbers $(h_{1,1}, h_{2,1})$. We just need to recall that the number of scalars obtained from the KK reduction of the metric is $h_{1,1}$ real scalars plus $h_{2,1}$ complex scalars. These arise because the metric depends on these numbers of complex and Kahler and complex structure parameters, so the internal kinetic operator for 10d gravitons should have the corresponding zero energy directions. It is important to note that Calabi-Yau threefolds do not have isometrical direction, thus the KK reduction of the 10d metric does not lead to 4d gauge bosons. Finally, p-forms are KK reduced as above. To simplify notation we denote Σ_a the non-trivial $(1,1,)$-cycles, $\tilde{\sigma}_a$ their dual $(2,2)$-cycles, Λ_b and $\tilde{\Lambda}_b$ the $(2,1)$- and $(1,2)$-cycles, and ω, $\tilde{o}mega$ the $(3,0)$- and $(0,3)$-cycles.

Recall that the bosonic fields for 10d type IIA are the graviton G, the NSNS 2-form B, the dilaton ϕ, and the RR 1-forms A_1 and 3-form C_3

IIA		Gravity	$h_{1,1}$ Vector	$h_{2,1}$ Hyper	Hyper
G	$\to$	$g_{\mu\nu}$	$h_{1,1}$	$2h_{2,1}$	
B	$\to$		$\int_{\Sigma_a} B$		c
ϕ	$\to$				ϕ
A_1	$\to$				
C_3	$\to$		$\int_{\Sigma_a} C_3$	$\int_{\Lambda_a} C_3$, $\int_{\tilde{\Lambda}_a} C_3$	$\int_{\omega} C_3$, $\int_{\tilde{\omega}} C_3$

Here c is the scalar dual to the 4d 2-form $b_{\mu\nu}$, i.e. $dc = *_{4d} db$. In total, we get the $\mathcal{N} = 2$ 4d supergravity multiplet, $h_{1,1}$ vector multiplets (with abelian group $U(1)^{h_{1,1}}$) and $h_{2,1}+1$ hypermultiplets (neutral under the gauge group).

The bosonic fields for 10d type IIA are the graviton G, the NSNS 2-form B, the dilaton ϕ, and the RR 0-form a, 2-form $\tilde{B}$ and 4-form C_4^+ (with self dual field strength).

IIB		Gravity	$h_{2,1}$ Vector	$h_{1,1}$ Hyper	Hyper
G	$\rightarrow$	$g_{\mu\nu}$	$2h_{2,1}$	$h_{1,1}$	
B	$\rightarrow$			$\int_{\Sigma_a} B$	c
ϕ	$\rightarrow$				ϕ
a					a
$\tilde{B}_2$	$\rightarrow$			$\int_{\Sigma_a} \tilde{B}$	$\tilde{c}$
C_4^+	$\rightarrow$	$\int_\omega C_4^+$	$\int_{\Lambda_b} C_4^+$	$\int_{\Sigma_a} C_4^+$	

Note that the self duality $dC_4 = *dC_4$ reduces the number of independent integrals of C_4^+ that can be taken.

In total, we obtain the $\mathcal{N} = 2$ 4d supergravity multiplet, $h_{2,1}$ vector multiplets (with abelian gauge group) and $(h_{1,1}+1)$ hypermultiplets (neutral under the gauge group).

2.4 Mirror symmetry

Consider two Calabi-Yau threefolds $\mathbf{X}$ and $\mathbf{Y}$, such that $(h_{1,1}, h_{2,1})_{\mathbf{X}} = (h_{2,1}, h_{1,1})_{\mathbf{Y}}$. Then the low energy spectrum of type IIA on $\mathbf{X}$ and type IIB on $\mathbf{Y}$ are the same.

This suggest more that a coincidence. The mirror symmetry proposal is that for each Calabi-Yau threefold $\mathbf{X}$ there exists a mirror threefold $\mathbf{Y}$ such that type IIA string theory on $\mathbf{X}$ is exactly equivalent to type IIB string theory on $\mathbf{Y}$. This of couse implies the above relation between their Hodge numbers, but much more, since the claim implies equivalence of the two theories to all orders in α', i.e. including stringy effects (there are proposal for equivalence also to all orders in the spacetime string coupling constant).

There is a lot of evidence in favour of this proposal. For instance, classification of large classes of Calabi-Yau threefols show that they appear in pairs, for each $\mathbf{X}$ there is some $\mathbf{Y}$, with the right relation of Hodge numbers. Obviously, this is necessary but not sufficient for mirror symmetry. Nevertheless it is a compelling piece of evidence.

More convincing is the explicit construction of two different Calabi-Yau geometries starting from a unique 2d interacting conformal field theory, by two different geometric interpretation of the 2d fields. See [3].

The mirror symmetry proposal has very interesting implications. It implies an exact matching of the complex structure moduli space of $\mathbf{X}$ with the Kahler moduli space of $\mathbf{Y}$ (with the Kahler parameters complexified by the addition of scalars arising from B-fields), exactly in α'. This has led to remarkable predictions in *mathematics*, as follows. A non-renormalization theorem of $\mathcal{N}=2$ 4d supersymmetry ensures that the structure (metric) of the vector multiplet moduli space is independent of scalars in hypermultiplets, and vice versa. Recall that α' corrections are controlled by a Kahler parameter, which for type IIB(IIA) is a hypermultiplet (vector multiplet) scalar. This implies that in the compactification of type IIB on $\mathbf{Y}$ the vector multiplet moduli space, i.e. the complex structure moduli space, does not suffer α' corrections, and the result obtained in the supergravity approximation is α' exact. Mirror symmetry proposes that this is exactly the vector multiplet moduli space of type IIA on the mirror $\mathbf{X}$; this is the Kahler moduli space of $\mathbf{X}$, and it suffers from α' corrections. Mirror symmetry is giving us a tool to resum all the α' corrections to the metric in the Kahler moduli space of IIA on $\mathbf{X}$ via its equivalence with the complex structure moduli space of IIB on $\mathbf{Y}$, which is exactly computable from classical geometry in supergravity. The α' corrections on the Kahler moduli space of IIA on $\mathbf{X}$ are interesting, because a non-renormalization theorem ensures that there are no perturbative (in the α' expansion) corrections; on the other hand, there are non-perturbative (in the α' expansion) corrections, due to worldsheet instantons: these are processes mediated by configurations where the closed string wraps around a holomorphic 2-cycle in **X**. Mirror symmetry allows to compute these contributions from the mirror, and to extract from this the number of holomorphic 2-cycles in the Calabi-Yau threefold **X**. These numbers are very difficult to compute from other mathematical means, and easily derived from mirror symmetry. Hence mirror symmetry has attracted the attention of many mathematicians.

3 Compactification of heterotic strings on Calabi-Yau threefolds

In this section we study the more interesting (and difficult) compactification of heterotic theory on Calabi-Yau threefolds. They will lead to models with potential phenomenological application, in the sense that they are similar to the physics of Elementary Particles we observe in Nature.

Notice that since we work in the supergravity approximation, heterotic $SO(32)$ and type I compactifications will be very similar. Also both heterotics require the same tools for this compactification, hence (for historical reasons, and also because they lead to nicer models with the particular ansatz we make (standard embedding)), we center on compactifications of the $E_8 \times E_8$ heterotic.

3.1 General considerations

The original massless 10d fields of the theory are the metric G, the 2-form B, the dilaton ϕ, and the gauge bosons A^a in $E_8 \times E_8$, plus the fermion superparters of all these. We compactify the corresponding supergravity theory on $M_4 \times \mathbf{X_6}$. Clearly, the condition that we get some unbroken 4d supersymmetry, in particular some 4d gravitino, implies that $\mathbf{X_6}$ must be a Calabi-Yau threefold. We see that starting with a single 10d gravitino we will end up with a single 4d gravitino, namely the 4d theory has $\mathcal{N}=1$ supersymmetry. This is very nice, since it is a low enough degree of supersymmetry to allow for chiral fermions. On the other hand, we know that $\mathcal{N}=1$ supersymmetry is considered one of the most promising extensions beyond the Standard Model.

One difference of heterotic compactifications, compared with type II compactifications, are the presence of the 10d nonabelian gauge fields. Hence in the compactification there is the possibility of turning on a non-trivial background for their internal components $A_m(x^4,\ldots,x^9)$. More formally, we need to specify not just a compactification manifold, but also a gauge bundle (a principal G-bundle, with $G\subset E_8\times E_8$) over the internal space $\mathbf{X_6}$. Such bundles are also constrained in order to lead to unbroken 4d susy in the gauge sector of the theory (see below).

Before discussing the bundles in more detail, let us wonder whether we really need non-trivial bundles, or else compactifications with trivial gauge bundle are consistent. The answer is that such compactifications are inconsistent if the Calabi-Yau is non-trivial (i.e. is not a six-torus). To see this, recall the Green-Schwarz terms in the 10d action, that we mentioned in the discussion of heterotic (or type I) 10d anomalies. In particular, there is a term of the form

$$\int_{10d} B_6\wedge(\mathrm{tr}\,F^2-\mathrm{tr}\,R^2) \qquad (13)$$

where F and R are the curvatures of the gauge and tangent bundle, and B_6 is the dual to the NSNS 2-form, $dB_6=*dB_2$. This leads to an action for B_6 which can be written

$$\int_{10d} H_3\wedge dB_6+\int_{10d} B_6(\mathrm{tr}\,F^2-\mathrm{tr}\,R^2) \qquad (14)$$

where H_3 is the field strength for B_2. This leads to the equations of motion

$$dH_3 = \mathrm{tr}\, F^2 - \mathrm{tr}\, R^2 \tag{15}$$

Taking this equation in cohomology (both sides are closed), the left hand side is exact so corresponds to the zero class. We get

$$[\mathrm{tr}\, F^2] = [\mathrm{tr}\, R^2] \quad \text{namely} \quad c_2(E) = c_2(R) \tag{16}$$

the second Chern class of the gauge bundle must equal that of the tangent bundle. The latter is trivial only for the six-torus, so consistency of the equations of motion requires the internal gauge bundle to be non-trivial.

Thus we need to specify a connection in a non-trivial principal G-bundle to have a consistent compactification. The requirements on this connection in order to have unbroken 4d supersymmetry is that the curvatures obey the conditions

$$F_{ij} = 0 \quad ; \quad F_{\overline{i}\overline{j}} = 0 \quad ; \quad g^{i\overline{j}} F_{i\overline{j}} = 0 \tag{17}$$

Again, explicit solutions to these equations are difficult to find. However, there is a theorem (by Donaldson, Uhlenbeck and Yau) which guarantees the existence of a solution for gauge bundles satisfying the (simpler to check, since they are almost topological) conditions

i) The complexified vector bundle (with fiber given by the vector space of *complex* linear combinations of the basis vectors) is holomorphic (i.e. transition functions are holomorphic).

ii) The bundle is stable. This is a complicated to state condition, which in physics terms ensures that the gauge field configuration is stable against decay into product of bundles.

The classification or even the construction of stable holomorphic bundles over a Calabi-Yau is a difficult task even for mathematicians, so we will not say much about this.

Happily, there is a very natural gauge bundle that satisfies the above conditions, and can be used for any Calabi-Yau manifold, therefore leading to a 4d $\mathcal{N} = 1$ supersymmetric compactification. It amounts to taking the gauge bundle to be isomorphic to the tangent bundle, and the gauge connection to be the same, at each point, to the spin connection. This is called the standard embedding, or embedding the spin connection on the gauge degress of freedom.

Note that since $F = R$ it automatically satisfies the condition $c_2(F) = c_2(R)$. Also note that due to the Calabi-Yau property, the tangent bundle has holonomy $SU(3)$, so the non-trivial part of the gauge bundle is embedded in an $SU(3)$ subgroup of one of the E_8, i.e. $H = SU(3)$.

We emphasize that the standard embedding is just a possible choice of consistent gauge background in the heterotic compactification. Any other choice of bundle, with different structure group, etc, would lead to equally consistent models. In this lecture we however center on standard embedding models for simplicity.

3.2 Spectrum

Before entering the construction of the final 4d spectrum, recall the basic 4d $\mathcal{N} = 1$ supermultiplets.

i) the gravity multiplet, containing the 4d metric and one gravitino

ii) the vector multiplet, containing the gauge bosons and the gauginos (Majorana fermions in the adjoint)

iii) the chiral multiplet, containing a complex scalar and a Weyl fermion, both in some representation of the gauge group.

With this information it will be enough to determine just the spectrum of bosons or of fermions.

The reduction of the 10d $\mathcal{N}=1$ sugra multiplet leads to the following bosonic fields in 4d

Het		Gravity	$h_{1,1}$ Chiral	$h_{2,1}$ Chiral	Chiral
G	$\to$	$g_{\mu\nu}$	$h_{1,1}$	$2h_{2,1}$	
B	$\to$		$\int_{\Sigma_a} B$		c
ϕ	$\to$				ϕ

Thus we get $h_{1,1}+h_{2,1}+1$ chiral multiplets, neutral under the gauge group.

In the compactification of the 10d $\mathcal{N}=1$ $E_8\times E_8$ vector multiplet, it is easy to identify the resulting 4d vector multiplets. This can be done by realizing that the gauge symmetries surviving in 4d are those gauge transformations in $E_8\times E_8$ which leave the background invariant. Thus the 4d gauge group is the commutant of the subgroup H with non-trivial gauge background turned on.

For the standard embedding $H=SU(3)$, embedded within one of the two E_8's. Thus, the other E_8 is untouched and survives in the 4d gauge group. About the E_8 on which we embed the $SU(3)$, the unbroken 4d gauge group by realizing that E_8 has a maximal rank subgroup $E_6\times SU(3)$ and we embed the gauge connection on the last factor. The adjoint representation of E_8 decomposes as (see below)

$$\begin{aligned} E_8 \quad & \to E_6\times SU(3) \\ 248 \quad & \to (78,1)+(1,8)+(27,3)+(\overline{27},\overline{3}) \end{aligned} \tag{18}$$

The generators commuting with $SU(3)$ must be singlets under it, so the unbroken 4d group is E_6 (times E_8).

To verify the above decomposition, recall that the generators of E_8 are 8 Cartans H^I and the non-zero roots

$$(\underline{\pm,\pm,0,0,0,0,0,0}) \quad ; \quad \frac{1}{2}(\pm,\pm,\pm,\pm,\pm,\pm,\pm,pm) \tag{19}$$

(with an even number of minus signs in the second set).

The decomposition (18) is as follows

$$\begin{array}{rcl} SU(3) & \to & H_1 - H_2 \,,\, H_1 + H_2 - 2H_3 \\ & & (\underline{+,-,0}) \\ E_6 & & H_1 + H_2 + H_3 \,,\, H_4 \,,\, H_5 \,,\, H_6 \,,\, H_7 \,,\, H_8 \\ & & (0,0,0,\underline{\pm,\pm,0,0,0}) \\ & & \frac{1}{2}(+,+,+,\pm,\pm,\pm,\pm,\pm) \\ & & \frac{1}{2}(-,-,-,\pm,\pm,\pm,\pm,\pm) \\ (27,3) & & (\underline{+,0,0},\pm,0,0,0,0,0) \\ & & (\underline{-,-,0},0,0,0,0,0) \\ & & \frac{1}{2}(\underline{+,-,-},\pm,\pm,\pm,\pm,\pm) \\ (\overline{27},\overline{3}) & & (\underline{-,0,0},\pm,0,0,0,0,0) \\ & & (\underline{+,+,0},0,0,0,0,0) \\ & & \frac{1}{2}(\underline{-,+,+},\pm,\pm,\pm,\pm,\pm) \end{array} \quad (20)$$

Thus we get 4d $\mathcal{N}=1$ vector multiplets of $E_6 \times E_8$. This is very interesting, since the group E_6 has been considered as a candidate group for grand unification models. So in a sense, it is relatively close to the Standard Model (we simply point out that slightly more complicated models, with other structure group on the gauge bundle, can lead to gauge groups even closer to that of the Standard Model).

Finally, we need to discuss the spectrum of chiral multiplets. To obtain these it is more convenient to obtain the fermionic components that arise in the KK reduction of the 10d gaugino. Let us discuss the general idea of how to do this, before going to the particular case of the standard embedding.

For simplicity, we center on the E_8 factor broken by the compactification, the corresponding gaugino transforms in the adjoint of the original gauge group E_8. In the breaking of the gauge group $E_8 \to H \times G_{4d}$, the adjoint of E_8 suffers a general decomposition

$$\begin{array}{ccc} E_8 & \to & H \times G_{4d} \\ 248 & & \sum_i (R_{H,i}, R_{G,i}) \end{array} \tag{21}$$

The ansatz for the profile of the 10d gaugino in the KK reduction is of the form

$$\begin{aligned} \lambda_\alpha(x^0,\ldots,x^9) &= \textstyle\sum_i \Big(\xi_4^{R_{H,i}}(x^4,\ldots,x^9)\psi_{-1/2}^{R_{G,i}}(x^0,\ldots,x^3) + \\ & \qquad \xi_{\overline{4}}^{R_{H,i}}(x^4,\ldots,x^9)\psi_{1/2}^{R_{G,i}}(x^0,\ldots,x^3)\Big) \end{aligned} \tag{22}$$

where ξ_4, $\xi_{\overline{4}}$ are spinors of opposite chiralities in the internal 6d and $\psi_{\pm 1/2}$ as spinors of opposite chiralities in 4d. The singlet component of ξ gives rise to the 4d gauginos.

As usual, the 4d mass of a chiral left handed 4d fermion $\psi_{-1/2}^{R_{H,i}}$ in the representation $R_{G,i}$ of G_{4d} is given by the eigenvalue of the kinetic operator on the corresponding internal wavefuncion $\xi_4^{R_{H,i}}$. This is the 6d Dirac operator for fermions in the 4, coupled to an H-bundle in the $R_{H,i}$ representation. Hence we obtain a left handed chiral 4d fermion in the $R_{G,i}$ for each solution of the equation

$$\not{D}_{R_{H,i}} \xi_4^{R_{H,i}} = 0 \tag{23}$$

The number of fermions $n_{R_{G,i}}^-$ is hence the dimension of $\ker \not{D}_{R_{H,i}}$.

In general, the number of zero modes of $\not{D}_{R_{H,i}}$ is *not* given by a topological quantity of $\mathbf{X_6}$ or the bundle. The reason is that the KK reduction of the 10d gaugino can also lead to 4d right-handed chiral fermions in the representation $R_{H,i}$. The number $n_{R_{G,i}}^+$ of such zero modes is given by the dimension of

$\ker \not{D}^{\dagger}_{R_{H,i}}$. Since two 4d chiral fermions of opposite 4d chiralities and in the same representation of the gauge group can couple to get a Dirac mass, they can disappear from the massless spectrum. This can be triggered by a small change of the geometry of the manifold or the gauge bundle, while staying in the same topological sector. Therefore, the individual numbers of massless chiral fermions $n^{\pm}_{R_{G,i}}$ are *not* topological. However, in all these processes of Dirac mass generation, the difference between the two two numbers is conserved. Indeed, it was known to mathematicians that the difference,

$$\mathrm{ind} \not{D}_{R_{H,i}} = \dim \ker \not{D}_{R_{H,i}} - \dim \ker \not{D}^{\dagger}_{R_{H,i}} \tag{24}$$

called the index of the Dirac operator (coupled to a suitable bundle) can be expressed in terms of characteristic classes of the tangent and gauge bundles

$$\mathrm{ind}\ \not{D}_{R_{H,i}} = \int_{\mathbf{X_6}} ch(F)\sqrt{\hat{A}(R)} \tag{25}$$

where it is understood that one must expand the Chern character (computed in the representation $R_{H,i}$) and A-roof genus, and pick the degree 6 piece to integrate it.

This is satisfactory enough, since we expect that generically vector-like pairs of fermions pick up large masses, of the order of the cutoff scale (the string scale or compactfication scale) since there is no symmetry or principle that forbids it. Hence, the only fields that we see in Nature would be the unpaired chiral fermions (this is a version of Georgi's 'survival hypothesis')

Returning to the case of the standard embedding, we are interested in obtaining the (net) number or fermions in the 27 of E_6. Since the gauge connection is determined by the spin connection, the index theorem gives the number of such 4d fermions in terms of just the topology of $\mathbf{X_6}$. It can be shown that the index theorem gives

$$n^{-}_{27} - n^{+}_{27} = \xi(\mathbf{X}_6)/2 = h_{1,1} - h_{2,1} \tag{26}$$

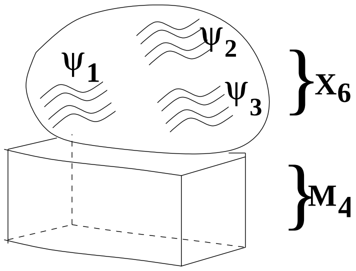

Figure 2: The replication of chiral families has a geometric origin in heterotic compactifications on Calabi-Yau spaces.

where $\xi(\mathbf{X_6})$ is the so-called Euler characteristic of $\mathbf{X_6}$. Therefore we get this number of chiral multiplets in the 27 of E_6. This is very remarkable because in E_6 grand unification the Standard Models families arise from representaions 27, hence $\xi/2$ is the number of fermion families in this kind of compactification. As we discussed above, this is a beautiful geometric origin for the number of families, as they arise from different zero energy resonances of a 10d field in the internal space! (see figure 2).

This number can be quite large in simple examples. For instance, for the quintic Calabi-Yau we get a model with 100 families, far more than we would like. In any event, there exist Calabi-Yaus where this number is small, and can be even three.

Note that the KK reduction would also lead to other fields, like singlets of E_6 (arising from internal wavefunctions in the 8 of $SU(3)$). These can be obtained from the index theorem, although the topological invariants are much more difficult to compute, so we skip their discussion.

3.3 Phenomenological features of these models

Let us start by mentioning that far more realistic models have been constructed explicitly. In particular one can achieve smaller gauge groups, closer to the Standard Model one, by adding Wilson lines breaking E_6. All examples of heterotic compactification show some generic features, which can be considered as predicitions of this setup (although there exist other ways in which string theory can lead to something similar to the Standard Model, with different phenomenological features).

• The string scale must be around the 4d Planck scale. The argument is as follows. The 10d gravitational and gauge interactions have the structure

$$\int d^{10}x\, \frac{M_s^8}{g_s^2}\, R_{10d} \quad ; \quad \int d^{10}x\, \frac{M_s^6}{g_s^2}\, F_{10d}^{\,2} \tag{27}$$

where M_s, g_s are the string scale and coupling constant, and R_{10d}, F_{10d} are the 10d Einstein and Yang-Mills terms. Upon Kaluza-Klein compactification on $\mathbf{X_6}$, these interactions reduce to 4d and pick up a factor of the volume V_6 of $\mathbf{X_6}$

$$\int d^4x\, \frac{M_s^8\, V_6}{g_s^2}\, R_{10d} \quad ; \quad \int d^4x\, \frac{M_s^6\, V_6}{g_s^2}\, F_{10d}^{\,2} \tag{28}$$

From this we may express the experimental 4d Planck scale and gauge coupling in terms of the microscopic parameters of the string theory configuration

$$M_P^2 \,=\, \frac{M_s^8\, V_6}{g_s^2} \,\simeq\, 10^{19}\,\mathrm{GeV} \quad ; \quad \frac{1}{g_{YM}^2} \,=\, \frac{M_s^6\, V_6}{g_s^2} \,\simeq\, \mathcal{O}(.1) \tag{29}$$

From these we obtain the relation

$$M_s \,=\, g_{YM}\, M_P \,\simeq\, 10^{18}\,\mathrm{GeV} \tag{30}$$

This large string scale makes string theory very difficult to test, since it reduces to an effective field theory at basically any experimentally accessible energy.

• This large cutoff scale makes the proton very stable, since in principle baryon number violating operators are suppressed by such large scale.

• Gauge and gravitational interactions have a similar coupling constant at the string scale, since they are controlled by the vev of the dilaton, which is universal. This is in reasonable good agreement with the renormalization group extrapolation of low energy couplings up in energy (assuming no exotic physics beyond supersymmetry in the intermediate energy region).

• The compactification scale cannot be too small. In order to avoid unobserved Kk replicas of Standard Model gauge bosons, the typical radius of the internal space should be much smaller than an inverse TeV. Other arguments about how the volume moduli modify the gauge couplings of string theory at one loop suggest that the compactification scale should be quite large to get weak gauge couplings. Usually one takes the compactification scale close to the string scale.

• The Yukawa couplings are given by the overlap integral of the internal wavefunctions of zero modes of the Dirac operator in $\mathbf{X_6}$. These are difficult to compute, in particular for the more realistic models which do not have standard embedding. So it is difficult to analyze the generic patterns of fermion masses at the string scale.

Finally, let us mention that this construction is very remarkable. We have succeeded in relating string theory with something very close to the observed properties of Elementary Particles. However, the setup has several serious problems, which are being addressed although no satisfactory solution exists for the moment

• How to break supersymmetry without generating a large cosmological constant?

• The models contain plenty of massless or very light fields, in particular the moduli that parametrize the background configuration. How to get rid of these?

• The vacuum selection problem. There is no criterion in the theory that tells us that a background is preferred over any other. Is the string theory that corresponds to our world special in any sense? Or is it a matter of chance or of anthropic issues that we see the world as it is?

Despite these open questions, we emphasize again the great achievement that we have reviewed today. We have provided a class of theories unifying gauge and gravitational interactions, and leading to 4d physics similar to the physics observed in Nature!

References

[1] P. Candelas, G. T. Horowitz, A. Strominger, E. Witten, 'Vacuum configurations for superstrings', Nucl. Phys. B258 (1985) 46.

[2] E. Witten, 'Phases of N=2 theories in two-dimensions', Nucl. Phys. B403 (1993) 159.

[3] B. R. Greene, M.R. Plesser, 'Duality in Calabi-Yau moduli space', Nucl.Phys. B338 (1990) 15.