Toroidal compactification of superstrings

1 Motivation

In this lecture we study toroidal compactification of the (spacetime supersymmetric) superstring theories. The main motivation is to obtain theories which reduce to 4d at low energies. Although the models obtained in this lecture are not interesting to describe the real world (they are non-chiral), they will be useful starting points for further constructions, like orbifolds. Also, toroidal compactification illustrates, just as in bosonic theory, the very striking features of stringy physics. For instance, the phenomenon of T-duality in will reveal that the seemingly different superstring theories are related upon toroidal compactification.

2 Type II superstrings

In this discussion we follow section 13.1 of [1].

2.1 Circle compactification

Let us consider the type IIA, IIB theories compactified to 9d on a circle $\mathbf{S^1}$ of radius R. The 2d fermion sector is completely unchanged by the compactification; the only effects of the compactification are

i) the possibility of boundary conditions with non-zero winding w for the 2d bosonic fields, namely

$$X^9(\sigma+\ell,t) \;=\; X^9(\sigma,t) \;+\; 2\pi R\, w \tag{1}$$

ii) the fact that momentum along x^9 is quantized, $p_9 = k/R$.

In a sector of momentum k and winding w, we have the mode expansion

$$\begin{aligned} X_L(\sigma + t) &= \frac{x_0^9}{2} + \frac{p_{L,9}}{2p^+}(t+\sigma) + \frac{1}{\alpha' p^+} N_B \\ X_R(\sigma - t) &= \frac{x_0^9}{2} + \frac{p_{R,9}}{2p^+}(t-\sigma) + \frac{1}{\alpha' p^+} \tilde{N}_B \end{aligned} \quad (2)$$

with

$$p_L = \frac{k}{R} + \frac{wR}{\alpha'} \quad ; \quad p_R = \frac{k}{R} - \frac{wR}{\alpha'} \quad (3)$$

We have the spacetime mass formulae

$$\begin{aligned} M_L^2 &= \frac{p_L^2}{2} + \frac{2}{\alpha'}(N_B + N_F + E_0) \\ M_R^2 &= \frac{p_R^2}{2} + \frac{2}{\alpha'}(\tilde{N}_B + \tilde{N}_F + \tilde{E}_0) \end{aligned} \quad (4)$$

From these expressions we can obtain the spectrum of 9d states at any radius R. For a generic R, the only massless states are in the sector of $k = 0$, $w = 0$. These states correspond to the zero modes (zero internal momentum) of the KK reduction of the effective field theory of 10d massless modes. Note that these states are present in field theory because they have zero winding.

The proccess of KK reduction to 9d and keeping just the zero mode amounts to simply decomposing the representations with respect to the 10d $SO(8)$ group into representations of the 9d $SO(7)$ group. Working first with e.g. the purely left moving sector, at the massless level we have

Sector	State	$SO(8)$	$SO(7)$
NS	$\psi^i_{-1/2}\|0\rangle$	8_V	$7+1$
R	$(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$	8_S	8
		8_C	8

Notice that the chiral 10d spinors of different chirality reduce to the same spinor representation of $SO(7)$, which does not have chiral representations (there is no chirality in odd dimensions).

In order to glue left and right movers, we may tensor the $SO(8)$ representions for left and right movers to get the 10d fields, and then decompose with respect to $SO(7)$, or decompose the left and right states with respect to $SO(7)$ representations and then tensor them. Both methods give the same result, so we may use any of them at will.

For type IIB theory, the massless 10d fields are the metric, 2-form and dilaton, G, B, ϕ; two gravitinos and two spin 1/2 fields $\psi_{\mu\alpha}$, ψ_α; the scalar axion, a 2-form and a self-dual 4-form, a, $\tilde{B}$, A_4^+. We have the following set of 9d massless states (See table 35 [3] for tensor products in $SO(7)$):

NS-NS									
$8_V,\, 8_V$	$\rightarrow$	$8_V \times 8_V =$	35_V	$+$	28_V	$+$	1		
$\downarrow$			$\downarrow$		$\downarrow$		$\downarrow$		
$7+1,\, 7+1$	$\rightarrow$	$7 \times 7 =$	27	$+$	21	$+$	1		$G_{\mu\nu},\, B_{\mu\nu},\, \phi$
		$7 \times 1 + 1 \times 7 =$	7	$+$	7				$G_{9\mu},\, B_{9\mu}$
		1×1	1						G_{99}
R-NS									
$8_C,\, 8_V$	$\rightarrow$	$8_C \times 8_V =$	56_S	$+$	8_S				
$\downarrow$			$\downarrow$		$\downarrow$				
$8,\, 7+1$	$\rightarrow$	$8 \times 7 =$	$48+8$		$\downarrow$				$\psi_{\mu\alpha},\, \psi_{9\alpha}$
	$\rightarrow$	$8 \times 1 =$			8				ψ_α
NS-R									
$8_V,\, 8_C$	$\rightarrow$	$8_V \times 8_C =$	56_S	$+$	8_S				
$\downarrow$			$\downarrow$		$\downarrow$				
$7+1,\, 8$	$\rightarrow$	$7 \times 8 =$	$48+8$		$\downarrow$				$\psi_{\mu\alpha},\, \psi_{9\alpha}$
	$\rightarrow$	$1 \times 8 =$			8				ψ_α
R-R									
$8_C,\, 8_C$	$\rightarrow$	$8_C \times 8_C =$	1	$+$	28_C	$+$	35_C		
$\downarrow$			$\downarrow$		$\downarrow$		$\downarrow$		
$8,\, 8$	$\rightarrow$	$8 \times 8 =$	1	$+$	$7+21$	$+$	35		$a,\, \tilde{B}_{9\mu},\, \tilde{B}_{\mu\nu},\, A_{9\mu\nu\rho}$

Here $\mu = 2, \ldots, 8$ runs in the seven non-compact directions transverse to the light-cone.

For type IIA theory, the massless 10d fields are the metric, 2-form and dilaton, G, B, ϕ; two gravitinos and two spin 1/2 fields $\psi_{\mu\alpha}$, ψ_α, $\psi_{\mu\dot\alpha}$, $\psi_{\dot\alpha}$; a 1-form and a 3-form A_1 C_3. We have the following set of 9d massless states (See table 35 [3] for tensor products in $SO(7)$):

$$
\begin{array}{ccccccccc}
\text{NS-NS} & & & & & & & & \\
8_V,\, 8_V & \rightarrow & 8_V \times 8_V = & 35_V & + & 28_V & + & 1 & \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & \\
7+1,\, 7+1 & \rightarrow & 7 \times 7 = & 27 & + & 21 & + & 1 & G_{\mu\nu},\, B_{\mu\nu},\, \phi \\
 & & 7\times 1 + 1 \times 7 = & 7 & + & 7 & & & G_{9\mu},\, B_{9\mu} \\
 & & 1 \times 1 & 1 & & & & & G_{99} \\
\text{R-NS} & & & & & & & & \\
8_C,\, 8_V & \rightarrow & 8_C \times 8_V = & 56_S & + & 8_S & & & \\
\downarrow & & & \downarrow & & \downarrow & & & \\
8,\, 7+1 & \rightarrow & 8 \times 7 = & 48+8 & & \downarrow & & & \psi_{\mu\alpha},\, \psi_{9\alpha} \\
 & \rightarrow & 8 \times 1 = & & & 8 & & & \psi_\alpha \\
\text{NS-R} & & & & & & & & \\
8_V,\, 8_S & \rightarrow & 8_V \times 8_S = & 56_C & + & 8_C & & & \\
\downarrow & & & \downarrow & & \downarrow & & & \\
7+1,\, 8 & \rightarrow & 7 \times 8 = & 48+8 & & \downarrow & & & \psi_{\mu\alpha},\, \psi_{9\alpha} \\
 & \rightarrow & 1 \times 8 = & & & 8 & & & \psi_\alpha \\
\text{R-R} & & & & & & & & \\
8_C,\, 8_S & \rightarrow & 8_C \times 8_S = & 8_V & + & 56_V & & & \\
\downarrow & & & \downarrow & & \downarrow & & & \\
8,\, 8 & \rightarrow & 8 \times 8 = & 1+7 & & 21+35 & & & A_9,\, A_\mu,\, C_{9\mu\nu},\, C_{\mu\nu\rho}
\end{array}
$$

Several observations are in order:

• Notice that there is one additional scalar besides G_{99} (which defines the compactification radius), namely A_9. It would be interesting to describe the compactification for an arbitrary background of this field. Unfortunately, it is not known how to couple RR fields to the worldsheet 2d theory, so we will be unable to do this. In later sections, in the compactification of several dimensions, there appear additional scalars arising from the NS-NS sector. For these it is known how to couple the background to the 2d theory, and the latter is exactly solvable (still a free theory), so we will be able to describe the compactification in a general background of these fields, in the complete string theory.

• Notice that both type II theories lead to the same 9d massless spectrum. In particular, notice that chirality of type IIB theory is lost in toroidal compactification, since there is no chirality in 9d. Notice also that the origin of the 9d fields in the RR sector is very different from the 10d viewpoint in the IIA and IIB theories. The low energy effective theory for the massless modes in either case is described by 9d supergravity with 32 supercharges (which is a unique theory).

• The generalization to compactification to lower dimensions (here we refer to square tori, and trivial B-field background, see later for non-triival cases) is very easy. At the massless level, one simply decomposes the representations 8_V, 8_S, 8_C with respect to the surviving Lorentz group, and then tensors them together. In particular it is possible to see that compactification to 4d on $\mathbf{T^6}$ leads to the field content of $\mathcal{N}=8$ 4d supergravity. Notice that again this theory is non-chiral, so it is not useful to describe the real world. The large amound of susy in lower dimensions is related to the fact that compactification on tori does not break any supersymmetry. This will motivate to discuss more involved compactifications in later sections (e.g. Calabi-Yau compactification).

• There is no point (besides $R=0$ or $R=\infty$) at which states become light. At $R\to\infty$ we have a tower of states of zero winding and arbitrary momentum which become very light. This corresponds to the decompactification limit of the theory. As $R \to 0$ we instead have a tower of states of zero momentum and arbitrary winding which become light. It is natural to think that this corresponds to the decompactification limit of a dual theory, where momentum is the original winding, etc, just as in the bosonic string theory. We study this in next section

2.2 T-duality for type II theories

Recall from the bosonic theory that the effect of T-duality is to relate a theory compactified on a circle of radius R with a theory compactified on a circle of radius $R' = \alpha'/R$, in such a way that states of momentum, winding (k, w) are mapped to states of momentum, winding $(k', w') = (w, k)$. Equivalently, starting with a 2d field theory of left- and right-moving bosons $X_L(\sigma + t)$, $X_R(\sigma - t)$, with a spacetime geometry spanned by $X(\sigma, t) = X_L + X_R$, T-duality related it to a theory on a spacetime geometry spanned by $X'^9(\sigma, t) = X_L^9 - X_R^9$, $X^\mu(\sigma, t) = X_L^\mu + X_R^\mu$.

In type II theory we also have the 2d fermions. In order to be consistent with 2d susy, we require that the T-dual theory is described also by the fermion field $\psi'^9(\sigma, t) = \psi_L^9(\sigma + t) - \psi_R^9(\sigma - t)$.

Hence, T-duality acts as spacetime parity on the right-movers. It is then intuitive that at the level of the spacetime spectrum, it will flip the chirality of the R groundstate, exchanging $8_C \leftrightarrow 8_S$. Namely, it flips the GSO projection on the right movers. Hence, starting with type IIB theory compactified on radius R the T-dual will describe type IIA theory compactfied on radius $R' = \alpha'/R$. This is T-duality for type II theories. Notice that it implies that the spectrum of massless fields at *generic* radius must be the same for both theories; the full spectrum is the same only for R, R' related by the T-duality relation.

The flip in the GSO projection can be derived more explicitly as follows. Recall that to build the R groundstate one forms the linear combinations of fermion zero modes

$$A_a^{\pm} = \psi_0^{2a} \pm i\psi_0^{2a+1} \tag{5}$$

So T-duality acts as $A_4^{\pm} \leftrightarrow A_4'^{\mp}$. In the original theory, one defines a state $|0\rangle$ satisfying $A_a^-|0\rangle = 0$ and the states surviving the GSO are e.g.

$$|0\rangle \quad , \quad A_{a_1}^+ A_{a_2}^+|0\rangle \quad , \quad A_1^+ A_2^+ A_3^+ A_4^+|0\rangle \tag{6}$$

In the T-dual theory, one would define a state $|0\rangle'$ by ${A_a}'^{-}|0\rangle' = 0$. In terms of the original operators we have $A_a^-|0\rangle' = 0$ for $a = 1,2,3$ and $A_4^+|0\rangle' = 0$. Hence we have

$$|0\rangle' = A_4^+|0\rangle \tag{7}$$

This implies that the $(-1)^F$ eigenvalue of $|0\rangle'$ is opposite to that of $|0\rangle$. This implies the GSO projection is opposite in the T-dual. Indeed, the surviving states (6) read, in the T-dual

$$A_a^+|0\rangle \quad , \quad A_{a_1}^+ A_{a_2}^+ A_{a_3}^+|0\rangle \tag{8}$$

From the viewpoint of the T-dual theory, we are choosing the opposite GSO projection.

It is easy to check the effect that T-duality has on the 10d fields, by comparing the 9d spectra. For instance, for bosonic fields

$$\begin{array}{rcl} IIA & \stackrel{T}{\leftrightarrow} & IIB \\ G_{\mu\nu}, B_{\mu\nu} & \leftrightarrow & B_{\mu\nu}, G_{\mu\nu} \\ A_9, A_\mu & \leftrightarrow & a, \tilde{B}_{9\mu} \\ C_{9\mu\nu}, C_{\mu\nu\rho} & \leftrightarrow & \tilde{B}_{\mu\nu}, A_{9\mu\nu\rho} \end{array} \tag{9}$$

The beautiful conclusion of T-duality is that IIA and IIB theories are much more intimately related than expected. In fact, they can be regarded as different limits of a unique theory, namely type II compactificaion on $\mathbf{S^1}$ in the limits of $R \to 0$ and $R \to \infty$.

2.3 Compactification of several dimensions

In this section we study compactification on a d-dimentional torus $\mathbf{T^d}$. These compactified theories contain more additional scalar fields, which correspond to 10d fields with some internal indices. Hence the vaccuum expectation value of these scalars correspond to specifying the backgrounds for the metric and other fields in the internal manifold.

We are interested in studying the set of possible toroidal compactifications, that is, the set of vevs that these scalar fields can acquire. This is called the moduli space of (toroidal) compactification. Unfortunately, it is not known how to quantize the 2d theory exactly if backgrounds for RR fields are turned on. So we will restrict to turning on backgrounds for the NS-NS fields, namely the metric and 2-form [1]

We describe $\mathbf{T^d}$ by periodic coordinates $x^i \simeq x^i + 2\pi R$, and define its geometry by a constant metric tensor G_{ij}. We also introduce a background for the 2-form, B_{ij}, which must be constant so as not to induce cost in energy (for constant B, its field strength vanishes).

The light-cone gauge-fixed action for an arbitrary metric background reads (see equation after (27) in lecture on quantization of closed bosonic

[1]The moduli spaces including RR backgrounds can be studied in the supergravity approximation; we postpone this discussion to coming lectures, since the analysis is most useful to study non-perturbative properties of string theory.

string)

$$L_G \;=\; -p^+\partial_t x^-(t) \;+\; \frac{1}{4\pi\alpha'}\int_0^\infty d\sigma\, G_{ij}\,(\partial_t X^i\partial_t X^j \;-\; \partial_\sigma X^i\partial_\sigma X^j\,) \qquad (10)$$

where we have used $p^+ = \frac{\ell}{2\pi\alpha'}g_{\sigma\sigma}$, and set $\ell = 2\pi\alpha' p^+$, so $g_{\sigma\sigma} = 1$.

To this we must add the term that describes the interaction of the string with the B-field, which reads

$$L_B \;=\; \frac{1}{4\pi\alpha'}\int_0^\infty d\sigma\,\epsilon^{ab}\,B_{ij}\,\partial_a X^i\partial_b X^j \;=\; \frac{1}{2\pi\alpha'}\int_0^\infty d\sigma\,B_{ij}\,\partial_t X^i\partial_\sigma X^j \quad (11)$$

In total we have

$$L \;=\; \frac{1}{2\pi}\int_0^\infty d\sigma\;\left[\frac{1}{2\alpha'}G_{ij}\,(\partial_t X^i\partial_t X^j \;-\; \partial_\sigma X^i\partial_\sigma X^j\,) \;+\; \frac{1}{\alpha'}B_{ij}\,\partial_t X^i\partial_\sigma X^j\right] (12)$$

The presence of the backgrounds and the periodicity of the coordinates x^i do not modify the oscillator piece for the 2d bosons. We are already familiar with this fact for the metric background, from our experience with circle compactifications. For backgrounds of the B-field, this follows because the lagrangian term in L_B is a total derivative

$$\epsilon^{ab}\partial_a X^i\partial_b X^j B_{ij} = \partial_a(\epsilon^{ab}X^i\partial_b X^j B_{ij}) \qquad (13)$$

so it is insensitive to the 2d local dynamics, and feels only the topology of the 2d field configuration (namely, the winding number).

Thus it is enough to work with the zero oscillator number piece in the mode expansion of the 2d bosons. In a sector of momenta and winding $k_i, w^j \in \mathbf{Z}$ this reads

$$X^i(\sigma,t) \;=\; x_0^i \;+\dot{x}^i\,t \;+\frac{2\pi R}{\ell}\,w^i\,\sigma \qquad (14)$$

where $\dot{x}^i$ will be related to k_i below. Plugging this ansatz into the lagrangian, we get

$$L \;=\; \frac{\ell}{2\pi}\,\left[\,\frac{1}{2\alpha'}\,G_{ij}\,(\dot{x}^i\dot{x}^j \;-\; (\frac{2\pi R}{\ell})^2\,w^i\,w^j\,) \;+\; \frac{1}{\alpha'}\,B_{ij}\,\dot{x}^i\,\frac{2\pi R}{\ell}\,w^j\,\right] \tag{15}$$

The canonical momentum conjugate to x^i is

$$p_i \;=\; \frac{\partial L}{\partial \dot{x}^i} \;=\; \frac{\ell}{2\pi R}\,(G_{ij}\dot{x}^j + B_{ij}w^j\,\frac{2\pi R}{\ell}) \tag{16}$$

It is quantized in units of $1/R$, namely $p_i = k_i/R$. This leads to

$$\dot{x}^i \;=\; \frac{G^{ij}}{p^+}\,(\,\frac{k_j}{R} \;-\; \frac{R}{\alpha'}B_{jl}w^l\,) \tag{17}$$

and

$$X^i(\sigma,t) \;=\; x^i_0 \;+\; \frac{G^{ij}}{p^+}\,(\,\frac{k_j}{R} \;-\; \frac{R}{\alpha'}B_{jl}w^l\,)\,t + \frac{R}{\alpha' p^+}\,w^i\sigma \tag{18}$$

Splitting between the left and right movers, we have

$$\begin{aligned} X^i_L(\sigma+t) &\;=\; \frac{x^i_0}{2} \;+\; \frac{p_L}{2p^+}\,(t+\sigma) \\ X^i_R(\sigma-t) &\;=\; \frac{x^i_0}{2} \;+\; \frac{p_R}{2p^+}\,(t-\sigma) \end{aligned} \tag{19}$$

with

$$\begin{aligned} p_{L,i} &\;=\; \frac{k_i}{R} \;+\; \frac{R}{\alpha'}\,(G_{ij} - B_{ij})\,w^j \\ p_{R,i} &\;=\; \frac{k_i}{R} \;+\; \frac{R}{\alpha'}\,(-G_{ij} - B_{ij})\,w^j \end{aligned} \tag{20}$$

and mass formulae read

$$\begin{aligned} M^2_L &\;=\; \frac{2}{\alpha'}(N_B + N_F + E_0) \;+\; \frac{p^2_L}{2} \\ M^2_R &\;=\; \frac{2}{\alpha'}(\tilde{N}_B + \tilde{N}_F + \tilde{E}_0) \;+\; \frac{p^2_R}{2} \end{aligned} \tag{21}$$

Narain lattice

The 2d-dimensional lattice of momenta (p_L, p_R) has two very special properties. It is even with respect to the Lorentzian (d,d) signature scalar product

$$(p_L, p_R) \cdot (p'_L, p'_R) \; = \; \alpha' \sum_i (p^i_L p_{L,i} - p^i_R p_{R,i}) \; = \; 2 \sum_i (k^i w'_i + w^i k'_i) \; \in \; \mathbf{Z} \quad (22)$$

and it is self-dual. These two properties ensure that the 1-loop partition function for the theory is modular invariant. Namely, the partition function has roughly speaking the structure

$$Z(\tau) \; = \; \ldots \sum_{(k,w)} q^{\alpha' p_L^2/2} \, \overline{q}^{\alpha' p_R^2/2} \; = \; \ldots \sum_{(p_L,p_R)} q^{\alpha' p_L^2/2} \, \overline{q}^{\alpha' p_R^2/2} \quad (23)$$

It is easy to see that the even and self-duality properties ensure that this is invariant under $\tau \to \tau + 1$ and $\tau \to -1/\tau$, resp. So each choice of background fields determines a (lorentzian) even and self-dual lattice of momenta (p_L, p_R). This is the so-called Narain lattice.

Conversely, any choice of ((d,d) lorentzian) even and self-dual lattice $\Gamma_{d,d}$ can be used to *define* a consisten modular invariant toroidal compactification of type II theory, by simply using the vectors in the lattice to provide the sectors of momenta (p_L, p_R) in the theory.

This description, first introduced by Narain [4] in the heterotic context, is useful to provide a complete classification of all possible toroidal compactifications (which correspond to free worldsheet theories). Hence they allow to compute the moduli space of such compactifications, as follows.

A general theorem in mathematics states that all possible (d,d) lorentzian even self-dual lattices are isomorphic, namely any two such lattices differ by an $SO(d,d)$ rotation. This does not mean that there is a unique physical compactification, because the physics is not invariant under arbitrary $SO(d,d)$ transformations. In particular, the spacetime mass of a state with momenta (p_L, p_R) depends on $p_L^2 + p_R^2$, which is only $SO(d) \times SO(d)$ invariant. This is illustrated in figure 1. Hence, physically different theories are classified by

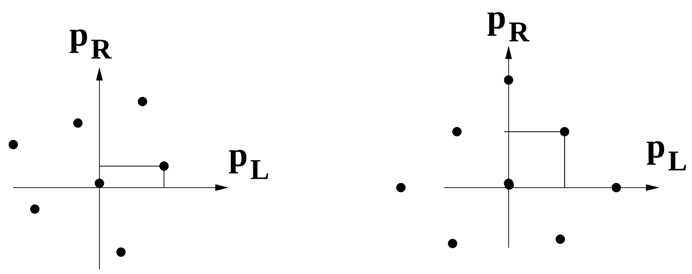

Figure 1: Heuristic picture of the relation between lattices and physical theories. Although the two lattices are related by a rotation in 2d spcace, the physics is sensitive to the independent values of p_L and p_R, and therefore not invariant under the rotation. Physics is not invariant under the mathematical isomorphism that relates the two lattices. The rotation parameters encode the background fields.

elements in the coset $SO(d,d)/[SO(d)\times SO(d)]$. This is (almost, see below) the moduli space of compactifications. Note that it has dimension d^2.

It would be interesting to be able to provide an interpretation of a compactification defined by these abstract lattices, in terms of background fields as those introduced above. In fact, the number of background fields is also $d(d+1)/2$ (for G_{ij}) plus $d(d-1)/2$ (for B_{ij}), namely a total of d^2. This suggests that any abstract lattice corresponds to a particular choice of background fields.

In fact we can be even more specific: The background fields themselves are the rotation parameters in $SO(d,d)/[SO(d)\times SO(d)]$. For instance, it is easy to show that the lattice of momenta for generic B_{ij}

$$(p_{L,i},p_{R,i}) \;=\; (\,\frac{k_i}{R}\,+\,\frac{R}{\alpha'}\,(G_{ij}-B_{ij})\,w^j\,;\,\frac{k_i}{R}\,+\,\frac{R}{\alpha'}\,(-G_{ij}-B_{ij})\,w^j\,) \qquad (24)$$

are related to the lattice of momenta for $B_{ij}=0$

$$(p_{L,i},p_{R,i}) \;=\; (\,\frac{k_i}{R}\,+\,\frac{R}{\alpha'}\,G_{ij}\,w^j\,;\,\frac{k_i}{R}\,-\,\frac{R}{\alpha'}\,G_{ij}\,w^j\,) \qquad (25)$$

by the rotation matrix

$$M_B = \begin{pmatrix} \delta_i^j - \frac{1}{2}B_i^j & \frac{1}{2}B_i^j \\ -\frac{1}{2}B_i^j & \delta_i^j + \frac{1}{2}B_i^j \end{pmatrix} \tag{26}$$

which is in $SO(d,d)$ because $M_B = \exp\frac{1}{2}\begin{pmatrix} -B & B \\ -B & B \end{pmatrix}$. Similarly, the momenta for generic G_{ij} can be related to the momenta for cubic metric $G_{ij} = \delta_{ij}$ via an $SO(d,d)$ rotation

$$M_G = \begin{pmatrix} \cosh S & \sinh S \\ \sinh S & \cosh S \end{pmatrix} = \exp\frac{1}{2}\begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \tag{27}$$

where S_{ij} is a symmetric matrix.

From either viewpoint we reach the conclusion that the moduli space of compactifications with these backgrounds is $SO(d,d)/[SO(d) \times SO(d)]$. In fact, this statement needs some refinement. In the description in terms of abstract lattices, it is clear that there may exist some finite $SO(d,d)$ transformations, not in $SO(d) \times SO(d)$, which leave the lattice Γ invariant as a whole, although acting non-trivially on the individual points (p_L, p_R). Since the lattice defines the physics, we should mod out by those transformations. They correspond to rotation matrices with integer entries, and generate a group denoted $SO(d,d;\mathbf{Z})$. Therefore the complete moduli space is

$$SO(d,d)/[SO(d) \times SO(d) \times SO(d,d;\mathbf{Z})] \tag{28}$$

These latter transformations act nontrivially on the winding and momentum quantum numbers, and also relate theories with different backgrounds. They include large diffeomeorphisms of $\mathbf{T^d}$, large gauge transformations of the B_{ij}, and also T-dualities (sign flips of right-moving momenta). For this reason, $SO(d,d;\mathbf{Z})$ is often called the T-duality group.

Some observations are in order

• States in the theory must form representations of the T-duality group: Since it leaves the theory invariant, there must be sets of states which are shuffled among themselved by the action of the symmetry. They thus lie in representations of the group. Representations of $SO(d,d;\mathbf{Z})$ are easy to construct from representations of $SO(d,d)$ by restriction. To give one example of this discussion, the d states $k_i = 1$, $w^j = 0$ and the d states $k_i = 0$, $w^j = 1$ form a $2d$-dimensional representaion of $SO(d,d;\mathbf{Z})$, which is the representation obtained from restriction of the vector representation of $SO(d,d)$.

• Again, we recall that toroidal compactifications contain more moduli than those discussed here. The inclusion of the additional backgrounds leads to large moduli spaces. They cannot be computed in full-fledged string theory, but can be computed in the supergravity approximation (which is reliable since the large amount of supersymmetry protects the structure of moduli space to a large extent).

• Finally, there will be enlarged duality groups, which act nontrivially on the states and on the backgrounds. A novelty, to be studied in later lectures, is that these enlarged duality groups act nontrivially on the string coupling, and therefore relate weakly coupled and strongly coupled regimes of string theory. The corresponding duality multiplets therefore contains perturbative string states (such as strings with momentum and winding) and non-perturbative states (the so-called branes) Hence dualities provide an extremely useful tool to study non-perturbative phenomena in string theory.

3 Heterotic superstrings

In the discussion we follow section 11.6 of [2]

3.1 Circle compactification without Wilson lines

This is the simplest compactification, although not the most generic one (additional background fields, Wilson lines, are turned on in later sections). We simply take spacetime to be $M_9 \times \mathbf{S^1}$ (so we make one coordinate periodic, $x^9 \simeq x^9 + 2\pi R$) and turn on no background for the 10d gauge fields. As usual, the compactification only modifies the theory by the inclusion of winding sectors, and the restriction to quantized momenta in the compact direction. Therefore, different sector of the theory will be labelled by left and right moving momenta

$$p_{L,R} = \frac{k}{R} \pm \frac{R}{\alpha'}\, w \tag{29}$$

as well as the internal 16d lattice left moving momenta P^I in the $E_8 \times E_8$ or $Spin(32)/\mathbf{Z_2}$ lattices. Defining the internal left moving 16d dimensionful momenta $P_L = \sqrt{2/\alpha'}\, P$, the mass formulae are given by

$$\begin{aligned} M_L^2 &= \frac{P_L^2}{2} + \frac{p_L^2}{2} + \frac{2}{\alpha'}\,(N_B - 1) \\ M_R^2 &= \frac{p_R^2}{2} + \frac{2}{\alpha'}\,(\tilde{N}_B + \tilde{N}_F + \tilde{E}_0) \end{aligned} \tag{30}$$

The spectrum of massless states at a generic radius (in particular at large radius) is the $p_L = p_R = 0$ sector. This corresponds to $k = w = 0$, hence we recover the zero modes of the (field theory) KK reduction from 10d to 9d. States are just the group theory decomposition of the massless states in 10d. We have

$$
\begin{array}{ccccccccc}
\text{NS} & & & & & & & & \\
8_V,\ \alpha^i_{-1}|0\rangle & \rightarrow & 8_V\times 8_V = & 35_V & + & 28_V & + & 1 & \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & \\
7+1,\ 7+1 & \rightarrow & 7\times 7 = & 27 & + & 21 & + & 1 & G_{\mu\nu},\ B_{\mu\nu},\ \phi \\
 & & 7\times 1+1\times 7 = & 7 & + & 7 & & & G_{9\mu},\ B_{9\mu} \\
 & & 1\times 1 & 1 & & & & & G_{99} \\
\text{R} & & & & & & & & \\
8_C,\ \alpha^i_{-1}|0\rangle & \rightarrow & 8_C\times 8_V = & 56_S & + & 8_S & & & \\
\downarrow & & & \downarrow & & \downarrow & & & \\
8,\ 7+1 & \rightarrow & 8\times 7 = & 48+8 & & \downarrow & & & \psi_{\mu\alpha},\ \psi_{9\alpha} \\
 & \rightarrow & 8\times 1 = & & & 8 & & & \psi_\alpha \\
\text{NS} & & & & & & & & \\
8_V,\ \alpha^I_{-1}|0\rangle & \rightarrow & 8_V\times 1 = & 8_V & & & & & \\
\downarrow & & & \downarrow & & & & & \\
7+1,\ 1 & \rightarrow & & 7+1 & & & & & A^I_\mu,\ A^I_9 \\
8_V,\ |P_I\rangle_{P^2=2} & \rightarrow & 8_V\times 1 = & 8_V & & & & & \\
\downarrow & & & \downarrow & & & & & \\
7+1,\ 1 & \rightarrow & & 7+1 & & & & & A_{P,\mu},\ A_{P,9} \\
\text{R} & & & & & & & & \\
8_C,\ \alpha^I_{-1}|0\rangle & \rightarrow & 8_C\times 1 = & 8_C & & & & & \\
\downarrow & & & \downarrow & & & & & \\
8,\ 1 & \rightarrow & & 8 & & & & & \psi^I \\
8_C,\ |P_I\rangle_{P^2=2} & \rightarrow & 8_C\times 1 = & 8_C & & & & & \\
\downarrow & & & \downarrow & & & & & \\
8,\ 1 & \rightarrow & & 8 & & & & & \psi_P
\end{array}
$$

The first set of states is the gravity multiplet of 9d supergravity with 16 supersymmetries. The second set of states are 9d vector supermultiplets with respect to 16 supersymmetries, namely 9d gauge bosons, gauginos and real scalars in the adjoint of the gauge group, which is $E8 \times E_8$ or $SO(32)$. Hence the 10d gauge group from the internal lattice is unbroken. In addition, there is the usual $U(1)^2$ gauge group arising from the familiar KK mechanism from the 10d graviton and B-field.

The generalization to lower dimensions is very easy, one simply needs to decompose the fields with respect to representations of the corresponding Lorentz group. Notice that in any of these compactifications chirality is lost. In particular, compactifications to 4d lead to theories with 4d $\mathcal{N}=4$ supersymmetry, which are automatically non-chiral.

Notice that in the above construction (i.e. without gauge field backgrounds) the pattern of enhance gauge symmetries at special values of R is exactly like in bosonic string theory. That is, the generic $U(1)^2$ gauge symmetry from the graviton and B-field enhance to $SU(2)^2$ at $R=\sqrt{\alpha'}$. Notice that there are no values of R for which the enhancement of the group involves both the $U(1)^2$ and the original 10d group. This will be different when we include Wilson lines.

Finally, we would like to mention that the $E_8 \times E_8$ and $SO(32)$ heterotic theories are self-T-dual. The $E_8 \times E_8$ heterotic theory on a circle of radius R is equivalent (up to relabeling of k and w) to the $E_8 \times E_8$ heterotic theory on a circle of radius $R' = \alpha'/R$ (and similarly for the $SO(32)$ heterotic theory). This would suggest that the two heterotics are not as intimately related as type IIA and IIB theories. We will see that they are: if one considers the more general case of compactifications with Wilson lines, there are T-dualities relating compactifications of the two heterotic theories.

3.2 Compactification with Wilson lines

The compactifications discussed above are not the most general circle compactifications. Note that the resulting 9d theory had additional scalars besides G_{99}, namely the scalar fields A_9^a in the adjoint of the gauge group. A vev for these scalars corresponds to turning on backgrounds for the internal components of the gauge fields, the so-called Wilson lines. In this section we discuss Wilson lines, first in the context of field theory, then in the context of heterotic string theory.

3.3 Field theory description of Wilson lines

Consider the following toy model of compactification from 5d to 4d. Consider a gauge theory with gauge group G in a spacetime $\mathbf{M_4} \times \mathbf{S^1}$, with $\mathbf{S^1}$ parametrized by the periodic coordinate $x^4 \simeq x^4 + 2\pi R$.

We also turn on a constant backbround for the internal component of the gauge bosons A_4^a. Locally, this is pure gauge, namely it could be gauge away, but the gauge parameter would not be a single-valued in $\mathbf{S^1}$ and thus would not define a global function. For instance, for $G = U(1)$, the gauge background can be locally gauge away with a gauge transformation

$$A_\mu \to A_\mu \partial_\mu \lambda \;\text{ with }\; \lambda = -\langle A_4 \rangle x^4 \tag{31}$$

and λ is not globally well defined on $\mathbf{S^1}$.

The gauge non-triviality of the gauge background can be encoded in the gauge-invariant object, called the Wilson line, defined by

$$W^a = \exp i \int_{\mathbf{S^1}} A^a = \exp(2i\pi R A_4^a) \tag{32}$$

Notice that A_4^a is periodic with period $1/R$. It is convenient to define $\tilde{A}_4^a = 2\pi R A_4^a$ which has period 1.

From the 4d viewpoint, the Wilson lines or gauge backgrounds of this kind are interpreted as giving a vacuum expectation value to the 4d fields A_4^a, which are 4d scalars transforming in the adjoint of the gauge group.

This makes it clear that, using global transformations in the gauge group, one can always diagonalize the hermitian matrix of vevs. This means that one can always rotate within the gauge group to a configuration where the gauge backgrounds are non-zero only for Cartan generators. We will denote the gauge background in this basis by A_4^I, with $I = 1, \ldots, \mathrm{rank}\, G$. This is a vector of Wilson line vevs.

We are interested in obtaining the spectrum of light 4d fields. To obtain them we should expand the 5d action around the background defining the compactification (namely, the circle geometry and the gauge background). The 5d action for the gauge fields roughly reads

$$S_{5d} \; = \; \int_{M_4 \times \mathbf{S^1}} \mathrm{tr}\, \mathcal{F}_{MN} \mathcal{F}^{MN} \tag{33}$$

with

$$\mathcal{F}_{MN} \; = \; \partial_{[M} \mathcal{A}_{N} + [\mathcal{A}_M, \mathcal{A}_N] \quad ; \quad \mathcal{A}_M = \sum_a A_M^a \, t^a \tag{34}$$

The terms $|[\mathcal{A}_M, \mathcal{A}_N]|^2$ in the compactification lead to 4d mass terms for gauge bosons $|[\mathcal{A}_\mu, \mathcal{A}_4]| \simeq tr(A_\mu^a A_\mu^a) W^2$ unless the generators associated with the gauge bosons commute with the generators associated with the gauge background. This is called the commutant of the subgroup where the gauge background was turned ont. To understand better which gauge bosons survive, we describe their generators in the Cartan-Weyl basis.

Gauge bosons of Cartan generators always have zero mass terms in 4d (since they always commute with the background, because it is embedded in Cartan generators as well). The rank of the 4d gauge group is the same as for the 5d group.

For non-Cartan generators, associated with some non-zero root α, the corresponding gauge boson survives in the massless sector if the commutator vanishes

$$[\langle H_I \rangle, E_\alpha] = \alpha_I A_4^I = 0 \tag{35}$$

Namely we obtain massless 4d gauge bosons for $\alpha \cdot A_4 = 0$. Recalling the periodicity in A_4^I, careful analysis leads to the slightly more relaxed $\alpha \cdot \tilde{A}_4 \in \mathbf{Z}$.

Recalling that the α_I are integer, and the A_4^I are continuous parameters, it is clear that generically the only surviving massless gauge bosons are the Cartan generators, generically the 4d group is broken to $U(1)^r$, with $r =$ rank G. For special choices of Wilson line (i.e. at particular points in Wilson line moduli space) we will obtain enhanced non-abelian gauge symmetries. For instance, for zero Wilson lines the 4d group equal to G. Turning on small wilson lines starting from a point of enhanced symmetry, breaks the gauge group. From the viewpoint of the 4d theory this is understood as a Higgs effect due to the scalars in the adjoint of the enhanced gauge group.

To give a simple example, consider $G = U(n)$, and consider that the Wilson line along x^4 corresponds to $(\mathcal{A}_4^I) = (0, \ldots, 0, a)$. For generic a, the only elements of $SU(n)$ that preserve the background (commute with the Cartan with Wilson line) are the $U(n-1)$ rotations in the first $n-1$ entries, times the total trace $U(1)$. The unbroken group is $U(n-1) \times U(1)$.

There is an alternative description of what fields remain massless in the 4d theory in the presence of Wilson lines, which is valid not just for gauge bosons but for any 5d field ψ charged under the 5d gauge group. Recalling that in a gauge theory all derivatives must be promoted to covariant derivatives, involving the gauge field, and that derivatives are related to momenta, it is clear that the natural momentum in the fifth direction x^4 is not associated to ∂_4, but to

$$D_4\psi = \partial\psi + q_I A_4^I \psi$$
$$P_4 = (k + q_I \tilde{A}_4^I)/R \quad (36)$$

with $k \in \mathbf{Z}$.

The 4d mass of the KK modes of this field is given by $m^2 = P_4^2$, for varying k. Clearly we obtain 4d massless fields only if $q \cdot \tilde{A}_4 \in \mathbf{Z}$. This generalizes the condition on gauge bosons, which is recovered by recalling that the roots α^I are simply the charges of the gauge bosons under the corresponding Cartan generator.

Before concluding, we would like to mention how this generalizes to compactification of several dimensions, i.e. $\mathbf{T^d}$ compactifications. In this case, we can turn on gauge backgrounds along any of the internal directions, A_i^a. Now in order to turn on this background without any cost in vacuum energy (so that we are still describing a vacuum of the theory) we have to avoid that backgrounds in different directions contribute to the energy via the commutators $[\mathcal{A}_i, \mathcal{A}_j]$ in the higher dimensional gauge kinetic term. This implies that backgrounds in the different direction commute among themselves. (From the viewpoint of the 4d theory, it implies a conditions on the corresponding scalar vev, which is condition of minimization of the scalar potential). On the other hand, it means that the corresponding matrices (in the gauge indices) can be simultaneously diagonlized, i.e. the complete background can be rotated to the Cartan generators. Therefore, the most general configuration of Wilson lines corresponds to backgrouns A_i^I for the Cartan generators. Clearly, the basic rule is that we obtain massless fields for states with charge vector q^I satisfying $q \cdot \tilde{A}_i \in \mathbf{Z}$, for any $i = 1, \ldots d$. Namely, each Wilson line acts independently.

In later sections we will see how this effective field theory description arises in string theory, at least in the limit of large radii.

3.4 String theory description

Narain lattice

In order to discuss compactification with Wilson lines in string theory, is to couple the gauge background to the 2d worldsheet theory. Happily, in the presence of constant gauge backgrounds the 2d theory is still free, and so exactly solvable. The gauge backgrounds A_i^I in a T^d compactification can be seen to couple e.g. to the 2d bosons through a term

$$S_A = \int d^2\xi\, \epsilon^{ab}\, \partial_a X^i\, \partial_b X^I A_i^I \tag{37}$$

The complete action is quadratic, a free theory.

The canonical quantization of the complete lagragian in the presence of backgrounds G_{ij}, B_{ij} and A_i^I is discussed in [5]. This is analogous to our study of type II compactification on $\mathbf{T^d}$, but there is a subtlety in that the 2d fields X^I are constrained to be purely left moving. The use of Dirac method of quantization of constrained systems implies a subtle additional piece in the canonical momenta. Skipping the details, the result for the left and right moving momenta in this compactifications are given by

$$\begin{aligned} P_L^I &= \sqrt{\frac{2}{\alpha'}}\,(\,P^I \,+\, RA_i^I\, w^i\,) \\ p_{L,i} &= \frac{k_i}{R} + \frac{R}{\alpha'}\,(G_{ij} - B_{ij})\, w^j \,-\, P^I A_i^I \,-\, \frac{R}{2}\, A_i^I A_j^I\, w^j \\ p_{R,i} &= \frac{k_i}{R} + \frac{R}{\alpha'}\,(-G_{ij} - B_{ij})\, w^j \,-\, P^I A_i^I \,-\, \frac{R}{2}\, A_i^I A_j^I\, w^j \end{aligned} \tag{38}$$

The formulae are given by

$$\begin{aligned} M_L^2 &= \frac{P_L^2}{2} + \frac{p_L^2}{2} + \frac{2}{\alpha'}\,(N_B - 1) \\ M_R^2 &= \frac{p_R^2}{2} + \frac{2}{\alpha'}\,(\tilde{N}_B + \tilde{N}_F + \tilde{E}_0) \end{aligned} \tag{39}$$

The lattice of momenta (38) is even with respect to the Lorentzian scalar product $P_L^I P_L^{I\prime} + p_L^i p'_{L,i} - p_R^i p'_{R,i}$, and self-dual. This ensures that the partition function for these theories is modular invariant for any choice of background fields, so they define consistent vacua of the theory.

As in type II compactifications, we are interested in the structure of the set of vacua of these theories, namely the moduli space for the scalars in the compactified theory. Following Narain, any $\mathbf{T^d}$ compactification can be defined in terms of an abstract $(16+d,d)$ lorentzian even and self-dual lattice $\Gamma_{16+d,d}$ of momenta. Mathematical theorems ensure that (p,q) lorentzian even self-dual lattices exist iff $p-q$ is a multiple of 8, which is fortunately satisfied in our case. Also, for $d>1$ all $(16+d,d)$ even self-dual lattices are isomorphic, up to a rotation in $SO(16+d,d)$. Again, this does not mean that all physical compactifications are equivalent, because the physics (e.g. the mass formulae) is invariant only under $SO(16+d)\times SO(d)$. Therefore, the set of inequivalent $\mathbf{T^d}$ compactifications of the theory is the coset $SO(16+d,d)/[SO(16+d)\times SO(d)]$.

This space has dimension $(16+d)d$, so a vacuum of the compactified theory is defined by $(16+d)d$ parameters. In fact, this is the number of parameters that define a background configurations, namely d^2 from G_{ij}, B_{ij} and $16d$ from the Wilson lines A_i^I. In fact, it is possible to see that these background fields are indeed the $SO(16+d,d)$ rotation parameters. Namely, the momenta (38) for generic values of B_{ij}, A_i^I are related to those for $B_{ij=0}$, $A_i^I=0$

$$\begin{aligned} P_L^I &= \sqrt{\frac{2}{\alpha'}}\,P^I \\ p_{L,i} &= \frac{k_i}{R} + \frac{R}{\alpha'}\,G_{ij}\,w^j \\ p_{R,i} &= \frac{k_i}{R} - \frac{R}{\alpha'}\,G_{ij}w^j \end{aligned} \tag{40}$$

by the matrix

$$M_{B,A} \;=\; \begin{pmatrix} \delta^i_J & \sqrt{\frac{2}{\alpha'}}A^i_J & -\sqrt{\frac{2}{\alpha'}}A^i_J \\ -\sqrt{\frac{2}{\alpha'}}A^I_j & \delta^i_j - \frac{1}{2}B^i_j - \frac{\alpha'}{4}A^I_jA^{I,i} & \frac{1}{2}B^i_j = \frac{\alpha'}{4}A^I_jA^{I,i} \\ -\sqrt{\frac{2}{\alpha'}}A^I_j & -\frac{1}{2}B^i_j - \frac{\alpha'}{4}A^I_jA^{I,i} & \delta^i_j + \frac{1}{2}B^i_j = \frac{\alpha'}{4}A^I_jA^{I,i} \end{pmatrix} \qquad (41)$$

which is an $SO(16+d,d)$ rotation since

$$M_{B,A} \;=\; \begin{pmatrix} 0 & \sqrt{\frac{2}{\alpha'}}A^i_J & -\sqrt{\frac{2}{\alpha'}}A^i_J \\ -\sqrt{\frac{2}{\alpha'}}A^I_j & -B^i_j & B^i_j \\ -\sqrt{\frac{2}{\alpha'}}A^I_j & -B^i_j & B^i_j \end{pmatrix} \qquad (42)$$

As in type II, the momenta for generic G_{ij} are related to those for cubic metric by a rotation

$$M_G \;=\; \begin{pmatrix} \cosh S & \sinh S \\ \sinh S & \cosh S \end{pmatrix} = \exp\frac{1}{2}\begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \qquad (43)$$

As in type II, we should be careful in constructing the moduli space, since there may exist finite $SO(16+d,d)$ transformations which leave a lattice of momenta invariant, although acting non-trivially on individual states. These transformations form the group $SO(16+d,d;\mathbf{Z})$ and corresponds to large diffeomorphisms of $\mathbf{T^d}$, shifts on B_{ij}, A^I_i by whole periods, and T-dualities. Since theories related by these rotations are physically equivalent, the moduli space has really the structure

$$SO(16+d,d)/[SO(16+d)\times SO(d)\times SO(16+d,d;\mathbf{Z})] \qquad (44)$$

This result will be useful in the discussion of non-perturbative dualities in compactifications of heterotic theories, etc, in later lectures.

Spectrum

At generic R the spectrum of light states is easily computed. For instance we obtain massless states from the decomposition of $8_V \times \alpha^i_{-1}|0\rangle$ and $8_C \times \alpha^i_{-1}|0\rangle$, which lead to the 4d $\mathcal{N}=4$ supergravity multiplet. Notice that it includes gauge bosons arising from the 10d metric and 2-form with one internal index.

We also get massless states from the decomposition of $(8_v + 8_C) \times \alpha^I_{-1}|0\rangle$, they correspond to 4d $\mathcal{N} = 4$ $U(1)^{16}$ vector multiplets. Finally, states with nonzero 16d momentum lead to massless states if $p_L = p_R = 0$, $P_L^2 = 4/\alpha'$. This can only be achieved in the $w^i = 0$ sector where

$$\begin{aligned} P_L^I &= \sqrt{\frac{2}{\alpha'}}\, P^I \\ p_{L,i} &= \frac{(k_i - P \cdot \tilde{A}_i}{R} \\ p_{R,i} &= \frac{(k_i - P \cdot \tilde{A}_i}{R} \end{aligned} \qquad (45)$$

$$(46)$$

So massless states correspond to $P^2 = 2$, $P \cdot \tilde{A}_i \in \mathbf{Z}$. This result, valid for generic R (and thus also for large R) reproduces the field theory analysis, as should be the case. These modes correspond to the KK reduction of the 10d $\mathcal{N} = 1$ vector multiplets in the presence of Wilson lines. For generic Wilson lines the non-abelian gauge bosons do not survive and the 4d gauge symmetry is simply $U(1)^{16}$.

On the other hand, by tunning some backgrounds, it is possible to achieve situations where some vector in the lattice of momenta satisfies

$$P_L^2 + p_L^2 = 4/\alpha' \qquad (47)$$

leading to some enhancement of the gauge symmetry breaking due to states $(8_V + 8_C) \times |P_L, p_L\rangle$. One simple particular case is tunning the Wilson lines to zero.

Notice that in general the new massless states at enhances symmetry points involve non-zero spacetime winding and momentum. This means that they are charged under the $U(1)^{2d}$ gauge bosons arising from the 10d metric and 2-form, in addition to being charged under the $U(1)^{16}$ from the internal 16d 'space'. The complete non-abelian group gathers Cartan generators of very different origin in 10d language!. The general recipe is that any non-abelian (simply laced[2]) group of rank $\leq 16 + 2d$ can appear as the gauge group in a corner of moduli space of $\mathbf{T^d}$ compactifications.

As a final comment, let us mention that moving away from such points (of enhanced gauge symmetry) in moduli space corresponds to a Higgs effect from the viewpoint of the lower dimensional effective field theory. This is similar to what we saw for the bosonic theory.

T-duality of $E_8 \times E_8$ and $SO(32)$ circle compactification

The fact that the moduli space of e.g. $\mathbf{S^1}$ compactifications of heterotic string theory is connected implies that a single theory in 9d can receive two interpretations, as compactification of $E_8 \times E_8$ heterotic on a radius R with Wilson lines A_i^I, and as compactification of $SO(32)$ heterotic on a different radius R' with different Wilson lines $A_i^I{}'$. Both compactifications are physically equivalent, although look different in 10d language. They are hence related by T-duality transformation. In this section we study the simplest example of these T-dualities (we follow section 11.6 of [2]).

Consider compactification of $E_8 \times E_8$ and $SO(32)$ heterotic theories on $\mathbf{S^1}$'s of radii R and R' repectively, with $G_{99} = 1$, $G'_{99} = 1$. The momenta lattice read

$$
\begin{aligned}
P_L^I &= \sqrt{\frac{2}{\alpha'}}\,(\,P^I + RA^I\,w^i\,) \\
p_{L,R} &= \frac{k}{R} \pm \frac{R}{\alpha'}\,w - P\cdot A - \frac{R}{2}\,A\cdot A\,w \qquad (48)
\end{aligned}
$$

[2] A group is simply laced if all its roots have length square equal to 2).

and similarly for primed parameters. Consider the choice of Wilson lines

$$(\tilde{A}^I) = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};0,0,0,0,0,0,0,0)$$
$$(\tilde{A}^{I\prime}) = (1,0,0,0,0,0,0,0;1,0,0,0,0,0,0,0) \tag{49}$$

for $E_8 \times E_8$ and $SO(32)$, resp.

The T-duality is the statement that these two theories are equivalent if $R = \alpha'/(2R)$. To show this one would have to see that the two Narain lattices are exactly the same. This can be done [6], but is a bit involved, so we will be happy by just showing the matching of some subsets of states.

For instance, it is easy to see that in either case the gauge group is defined by the surviving non-zero root vectors

$$(\underline{\pm,\pm,0,0,0,0,0,0};0,0,0,0,0,0,0,0) \quad ; \quad (0,0,0,0,0,0,0,0;\underline{\pm,\pm,0,0,0,0,0,0})$$

satisfying $P \cdot A \in \mathbf{Z}$. They correspond to a group $SO(16) \times SO(16)$ in both cases.

We can also match other states. Let us consider states uncharged under the 10d gauge group, i.e. neutral under $SO(16) \times SO(16)$, which have $P_L = 0$ and so $P^I = -\tilde{R}A^I_j w^j$. Using the particular form of the vectors P^I for the lattices, it can be seen that this condition requires that w is even, $w = 2m$. Hence, the spacetime left and right moving momenta are

$$p_{L,R} = \frac{k}{R} \pm \frac{wR}{\alpha'} + \frac{wR}{2}\frac{A\cdot A}{R^2} = \frac{k+2m}{R} \pm \frac{2mR}{\alpha'} \tag{50}$$

and similarly for primed quantities. Defining $\tilde{k} = k + 2m$, we get

$$p_{L,R} = \frac{\tilde{k}}{R} \pm \frac{2mR}{\alpha'}$$
$$p'_{L,R} = \frac{\tilde{k}'}{R'} \pm \frac{2m'R'}{\alpha'} \tag{51}$$

We see that the two theories are equivalent for $R' = \alpha'/(2R)$, $\tilde{k}' = m$, $m' = \tilde{k}$. $E_8 \times E_8$ heterotic theory and $SO(32)$ heterotic theory can be considered different (decompactification) limits of this 9d theory. We then have a picture similar to that of type II theories.

4 Toroidal compactification of type I superstring

In this section we study type I superstring compactified on a circle. Generalization to $\mathbf{T^d}$ is analogous and will be mentioned only briefly.

Recall thet type I theory is a theory of unoriented closed and open strings. We have the 10d massless fields G, B, ϕ, and $SO(32)$ gauge bosons (and superpartners). This field content is the same as for the $SO(32)$ heterotic, which means that in the large R regime the results (which are well described by field theory in this regime) will agree with those in heterotic theory. The string theory description, however, will be very different, and the stringy features, like gauge enhancement or T-duality will be very different.

Before entering the detailed discussion, let us point out that in a general toroidal compactification it is possible to turn on background for the RR 2-form B; however, it is not known how to couple such backgrounds to the 2d worldsheet theory. Hence, the only backgrounds we will be able to describe exactly in the string theory are metric and Wilson line backgrounds.

4.1 Circle compactification without Wilson lines

We start discussing the simplest case of compactification on a circle of radius R, with zero gauge background. We have to describe the closed and open string sector independently.

Closed string sector

The toroidal compactification of the closed sector of type I is simply the

Ω projection of the toroidal compactification of type IIB theory. In type IIB theory on a circle, different sectors of the theory are characterized by the momentum and winding, k and w, which define the mode expansion of the compactified direction (for clarity we omit the index in X^9)

$$\begin{aligned} X_L(\sigma+t) &= \frac{x_0}{2}+\frac{p_L}{2p^+}+\frac{1}{\alpha' p^+}N_B \\ X_R(\sigma-t) &= \frac{x_0}{2}+\frac{p_R}{2p^+}+\frac{1}{\alpha' p^+}\tilde{N}_B \end{aligned} \tag{52}$$

The effect of Ω on k, w is easy to find out, by recalling that it maps X to X^Ω such that

$$X^\Omega(\sigma,t)=X(-\sigma,t) \tag{53}$$

This implies that *Omega* acts by $x_0\to x_0$, $k\to k$, $w\to -w$.

Hence Ω-invariant states are linear combinations of states in opposite winding sectors, schematically $|w\rangle+|-w\rangle$. This implies that winding number is not a well defined quantum number for states in this theory. This will be a relevant point in understanding some features of the T-dual version.

In the $w=0$ sector, Ω relates states within this sector. This implies that we get the usual projection on the operator piece of the states; namely in the NSNS sector the states of the form

$$\psi^i_{-1/2}|w=0\rangle\otimes\tilde{\psi}^j_{-1/2}|w=0\rangle \tag{54}$$

survive only by taking the symmetrized product, exactly as in the original 10d theory. Indeed it is easy to check that the $w=0$ sector gives the KK reduction of the massless fields in the original 10d theory.

In sectors of $w\neq 0$ (these are massive states, but we are interested in discussing them at this point), there exist Ω-invariant combinations of winding excitations of these states both in symmetrized and antisymmetrized prodcucts. For instance, in the NSNS sector the state

$$\psi^{[i}_{-1/2}|w\rangle \otimes \tilde{\psi}^{j]}_{-1/2}|w\rangle + \psi^{[i}_{-1/2}|-w\rangle \otimes \tilde{\psi}^{j]}_{-1/2}|-w\rangle \quad (55)$$

survives. It can be considered as a winding excitation of the field B_{ij} since it is in a sense left-right antisymmetric. Nevertheless it is invariant under Ω due to the additional action on winding number. The observation that winding excitations of Ω-odd 10d massless fields are Ω invariant will be relevant in the discussion of the T-dual picture.

In any event, the spectrum of states massless at generic R is obtained by the Ω-invariant states in the $k=0$, $w=0$ sector of the IIB theory. As expected, this is simply the zero modes of the KK reduction of the 10d $\mathcal{N}=1$ supergravity multiplet.

Notice that since the parent IIB theory did not have any enhanced symmetries at special values of R, neither does the closed sector of type I theory.

Open string sector

(We start the discussion in compactfications without Wilson lines; inclusion of the latter will be discussed in later sections.)

A key difference between the compactification of open string sectors and closed string sectors is the absence of winding. As shown in figure 2, open strings can always unwind in a compact dimension. This agrees with the fact that winding was defined using the periodicity in σ for closed strings, and this does not exist in open strings.

Hence, the only effect of the circle compactification in the open string sector is that the internal momentum is now quantized and equal to k/R. Since there is no winding, compactification of open strings is very much like KK compactification in field theory.

We have the mode expansion for 2d bososn

$$X(\sigma,t) \;=\; x_0 + \frac{k}{Rp^+} + \text{oscillators} \quad (56)$$

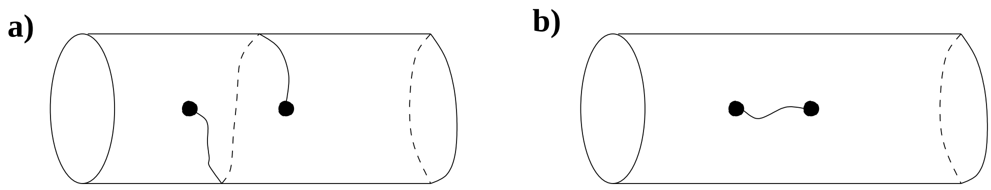

Figure 2: Open strings with NN boundary conditions in a compact direction cannot wind around it. String seemingly wrapped on the internal circle are in the same topological sector as strings with no winding.

leading to the mass formula

$$m^2 \,=\, \frac{1}{\alpha'}(N_B + N_F + E_0) \,+\, \frac{k^2}{R^2} \tag{57}$$

Thus massless states correspond to $k = 0$ and reproduce the zero modes of the KK reduction of the 10d massless fields. Namely $SO(32)$ gauge bosons, one real scalar in the adjoint representation, and fermion superpartners. States with non-zero k are the KK replicas of these zero modes. Again, there are no special values of R at which new states become massless.

4.2 T-duality

In this section we study the T-dual of the type I theory, also called type I' theory.

Closed string sector

Again, the closed string sector presents an infinite tower of states (with $k = 0$ and arbitrary w) which become light as $R \to 0$. This suggests the existence of a T-dual theory, which becomes decompactified in this limit. In this section we find out the structure of this T-dual theory, which is related to the original one by

Original	T-dual
R	$R' = \alpha'/R$
$k,\ w$	$k' = w,\ w' = k$
$X_L,\ \psi_L$	$X_L,\ \psi_L$
$X_R,\ \psi_R$	$X'_R = -X_R,\ \psi'_R = \psi_R$

(the action is only on the coordinate along the compact direction 9, on which we are T-dualizing).

In the **closed string sector**, the dual theory described by (X', ψ') corresponds to type IIA theory (since T-duality flips the right moving GSO projection) compactified on a circle of radius $R' = \alpha'/R$, and modded out by an orientifold projection. The orientifold action on X' can be obtained by reading the Ω action on left and right movers

$$X_L^\Omega(\sigma + t) = X_R(-\sigma - t) \quad ; \quad X_R^\Omega(\sigma - t) = X_L(-\sigma + t) \quad ; \tag{58}$$

and constructing $X^{\Omega\prime} = X_L^\Omega - X_R^\Omega$ and $X' = X_L - X_R$. We obtain

$$X^{\Omega\prime}(\sigma,t) = X_L^\Omega(\sigma + t) - X_R^\Omega(\sigma - t) = X_R(-\sigma,t) - X_L(-\sigma + t) = -X'(-\sigma,t) \tag{59}$$

Hence the T-dual is type IIA theory on a circle modde out by an orientifold action $\Omega\mathcal{R}$, where $\mathcal{R}$ is a geometric action $x^9 \to -x^9$. It is easy to verify that the action (59) on the mode expansion is to flip the momentum and leave winding invariant, as should be the case for the T-dual of Ω.

Recalling our lecture on unoriented strings, recall that we claimed that one can mod out a theory by Ω only if it is left-right symmetry (i.e. IIB theory). Here we are modding by $\Omega\mathcal{R}$ and this can be done only if the theory is left-right symmetric up to a GSO shift (i.e. IIA theory).

Notice that $\mathcal{R}$ has fixed points at two diametrically opposite point in the dual circle, see figure 3. These are regions where the orientation of a string can flip. They are 9-dimensional subpaces of 10d space, and are called

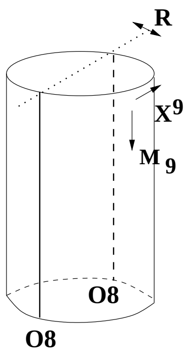

Figure 3: O8-planes in type I' theory.

orientifold 8-planes, O8-planes for short (they involve 8 spatial plus one time direction).

The existence of these special points implies that the compactification violates translation invariance. This is not strange, since staes in the original model did not have winding as a good quantum number; hence in the T-dual, momentum is not a good quantum number, so there are violations of translation invariance in the internal coordinate.

Finally, let us mention that states are in general linear combinations of states of the original theory in sectors of opposite internal momentum. In the $k=0$ sector this implies the usual projection, and that only ΩR even states arise. However, in sectors of $k\neq 0$ there exist momentum excitations of fields which are ΩR odd in the 10d theory. This has the interesting consequence that such 10d fields are not identically vanishing in the model, but rather propagate in the 'bulk', away form the orientifold planes. The orientifold projection impose the boundary condition that 10d ΩR odd fields vanish at the O8-plane location, and so lead to no zero modes. Hence, in the bulk the theory is still locally type IIA theory, and it is the O8-planes that project out part of the zero modes (although KK excitations survive).

Open string sector

We now study the open string sector in the T-dual version. The local 2d dynamics of the T-dual open string sector should be that of an (orientifold version of) type IIA theory. In particular, it implies that the interior of open string worldsheets propagates in 10d. However, since the original open string sector does not have winding number in x^9, the T-dual open string sector has no momentum in x^9. This implies that such fields propagate only in 9d.

The resolution to this seeming paradox can be understood by finding out the boundary conditions for the open strings in the T-dual. In the original theory we have Neumann boundary conditions at the open string endpoints

$$\begin{aligned} &\partial_\sigma X(\sigma,t)|_{\sigma=0,\ell}=0 \\ &\partial_\sigma X_L(\sigma+t)|_{\sigma=0,\ell}+\partial_\sigma X_R(\sigma-t)|_{\sigma=0,\ell}=0 \end{aligned} \tag{60}$$

This can be written as

$$\partial_t X_L(\sigma+t)|_{\sigma=0,\ell}-\partial_t X_R(\sigma-t)|_{\sigma=0,\ell}=0 \tag{61}$$

Namely, in terms of the T-dual coordinate $X=X_L-X_R$

$$\partial_t X'(\sigma,t)|_{\sigma=0\ell}=0 \tag{62}$$

These are Dirichlet boundary conditions (the corresponding open strings are said to have DD boundary conditions in x^9). They imply that the open string endpoints cannot move from a fixed value of the coordinate x^9, so the open string states are forced to move in 9d only. However the inside of the open string can still move in 10d. See figure 4.

One may question whether this is consistent. For instance, the open string sector is not translational invariance in x^9, but neither is the underlying closed string sector, so this is not worrisome. Another issue is that we

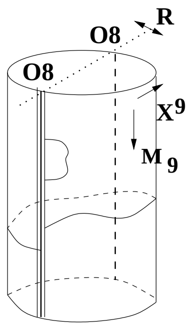

Figure 4: Open strings in type I' have endpoints at a fixed position in x^9, although their 'inside' can still move in 10d.

obtained Neumann boundary conditions as some correct boundary conditions to recover the familiar equations of motion for the 2d theory in the inside of the open string worldsheet. In fact, we can check that Dirichlet boundary conditions do the job as well. Recall that the variation of the Polyakov action is

$$\begin{aligned} \delta S_P &= -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\xi\, g^{ab}\partial_a X^\mu \partial_b \delta X^\mu = \\ &= -\frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} de\,(g^{ab}\delta X^\mu \partial_b X_\mu)|_{\sigma=0}^{\sigma=\ell} + \frac{1}{2\pi\alpha'}\int_\Sigma d^2\xi\, \delta X_\mu\, g^{ab}\partial_a\partial_b X^\mu \end{aligned} \quad (63)$$

Dirichlet boundary conditions in x^9 imply that $\delta X^9 = 0$ at $\sigma = 0, \ell$. Hence DD boundary conditions on x^9 and NN boundary conditions on the remaining coordinates ensure that the first term in the second line vanishes and we recover the correct 2d dynamics in the interior of the open strings.

It is interesting to notice that the mode expansion for the T-dual coordinate $X'(\sigma,t)$ contains a winding term and no momentum

$$X'(\sigma,t) = \frac{2\pi R'}{\ell} w'\sigma + \text{osc.} \quad (64)$$

which is indeed allowed by DD boundary conditions. Pictorially, existence of winding for open strings with endpoints stuck at points in x^9 is manifest in figure 4. Notice that the endpoints of all open strings are necessarily located at the same point in $\mathbf{S^1}$. This can be seen directly from the above

$$X'(\sigma = \ell, t) - X'(\sigma = 0, t) = 2\pi R w' \tag{65}$$

so the open strings stretch whole periods of x^9, such that endpoints always lie at $x^9 = 0$. This is true regardless of the Chan-Paton indices carried by the string. The presence of wilson lines in the original picture will modify this last fact, as we show later on.

A very intuitive picture, which becomes even more useful in more complicated situations (like with non-trivial Wilson lines in the original picture), is to consider that the model contains some objects, spanning the 9d hyperplane at $x^9 = 0$, called D8-branes, and on which open strings are forced to end. In fact, the precise picture is that there exist one such D8-brane for each possible value of the Chan-Paton index (32 D8-branes for the T-dual of type I). An open string endpoint with Chan-Paton index a must end on the a^{th} D8-brane. In the present situation, all 32 D8-branes are sitting at the same location in x^9.

The open string spectrum is easy to recover in this language. In the massless sector, we have open strings with all possible combinations of Chan-Paton factors (i.e. ending on the 32 D8-branes in all possible ways). This would lead to a 9d $U(32)$ vector multiplet with respect to the 16 unbroken supersymmetries. Since the open strings are sitting on top of an orientifold plane, we have to keep ΩR invariant states, leading to a 9d $SO(32)$ vector multiplet with respect to the 16 unbroken supersymmetries.

Notice that this gauge sector propagates in a 9d subspace of spacetime, while gravity and other fields still propagate in 10d. The possibility of constructing models of this kind has led to the brane-world idea, the proposal that perhaps the Standard Model that we observe is embedded in a brane which spans a subpace in a full higher dimensional spacetime. This would lead to the existence of extra dimensions which are detectable only using gravitational experiments. We will learn more about branes, and model building with them in later lectures.

4.3 Toroidal compactification and T-duality in type I with Wilson lines

As in heterotic theories, upon compactification there exist 9d scalars tranforming in the adjoint representations of the gauge group. Their vevs parametrize the possibility of turning on constant backgrounds for the internal components of the gauge fields. In this section we study the modifications they introduce for type I.

Clearly the closed string sector is insensitive to the presence of Wilson lines, since it contains states neutral under the gauge symmetry. The only modifications occur in the open string sector. To describe them, we need to couple the gauge background to the 2d theory. This is easily done by recalling the rule that an open string endpoint with Chan-Paton a has charge ± 1 under the $U(1)$ gauge boson arising in the sector of aa open strings. This implies that the worlsheet action must be modified by a boundary term

$$\Delta S = \int dt \; - iq_a A_i^a \, \partial_t X^\mu \tag{66}$$

Before the orientifold projection, there are 32 $U(1)$ gauge bosons, which are paired by the orientifold action. In terms of this parent $U(32)$ original theory, the most general wilson line consistent with the Ω action is

$$(A_i^a) = \frac{1}{2\pi R}(\theta_1, \theta_2, \ldots, \theta_{16}; -\theta_1, -\theta_2, - \ldots, \theta_{16}) \tag{67}$$

After the orientifold action, the surviving Cartans are linear combinations of the above; in terms of the $U(1)^{16}$ Cartan subalgebra of $SO(32)$ the wilson line is described by

$$(A_i^I) = \frac{1}{2\pi R}(\theta_1, \theta_2, \ldots, \theta_{16}) \tag{68}$$

Although the latter expression is more correct, it is sometimes more intuitive to use (67) to display the Chan-Patons and their orientifold images explicitly.

The Wilson line has the only effect of shifting the internal momentum, as discussed above in field theory terms. Namely, for an open string in the ab Chan-Paton sector (and so, with charges $(+1,-1)$ under $U(1)_a \times U(1)_b$, we have

$$p = \frac{k}{R} + \frac{\theta_a - \theta_b}{2\pi R} \tag{69}$$

here we are using the notation (67), so $\theta_{a+16} = -\theta_a$. The spacetime mass formula for these states is

$$m^2 \; = (\frac{k}{R} + \frac{\theta_a - \theta_b}{2\pi R})^2 + \frac{1}{\alpha'}(N_B + N_F - 1) \tag{70}$$

For generic R and θ_a, the gauge group is broken to $U(1)^{16}$, since only aa states are able to lead to massles modes. When several, say N eigenvalues θ_a coincide and are not zero or π, then there are additional massless fields, leading to $U(N)$ gauge bosons and superpartners. Finally, when n eigenvalues vanish or are equal to π, the gauge symmetry is $SO(2n)$.

The moduli space of compactifications is difficult to obtain, and there is no analog of the Narain lattice. Hence, without further ado, we turn to the discussion of T-duality.

T-duality

The T-dual closed string sector in still given by type IIA theory on a circle, modded out by $\Omega\mathcal{R}$. The T-dual of the open strings is slightly modified by the Wilson lines. By simply mapping the mode expansion of the original into the mode expansion of the T-dual, we find that the dual coordinate has shifted winding

$$X'(\sigma,t) \;=\; \text{const.} \;+\; \frac{2\pi R'}{\ell} w'\sigma + \frac{2\pi R'}{\ell}\,\frac{(\theta_a - \theta_b)}{2\pi}\sigma \;+\; \text{osc.} \tag{71}$$

This implies that the open string endpoints of ab strings are at different locations in x^9

$$X'(\sigma=\ell,t) - X'(\sigma=0,t) \;=\; 2\pi R'\, w' \;+\; \theta_a R' \;-\; \theta_b R' \tag{72}$$

The mass formula for ab strings is

$$m^2 = \frac{R}{\alpha'}\,(w + \frac{\theta_a}{2\pi} - \frac{\theta_b}{2\pi})^2 \;+\; \frac{1}{\alpha'}(N_B + N_F - E_0) \tag{73}$$

In more intuitive terms, recall our description of an endpoint with Chan-Paton a as ending on the a^{th} D8-brane. What we have found is that $\theta_a R'$ is the location in x^9 of the a^{th} D8-brane. The ab open strings start on the a^{th} and end on the b^{th} D8-brane, so their length is $\theta_a R - \theta_b R$, modulo the period $2\pi R$. This stretching contributes to the mass of the corresponding state. See figure 5.

The D8-brane picture makes the gauge symmetry enhancements clear. Generically the D8-branes are located at different positions, so the generic gauge symmetry is $U(1)^{16}$ (since only aa strings have zero stretching). When several, say N θ_a's coincide, several D8-branes overlap, and the corresponding ab strings are massless, leading to $U(N)$ gauge symmetries. Finally, if N θ_a are zero or π, D8-branes and their orientifold images coincide on top of an O8-plane, leading to $SO(2N)$ gauge symmetry.

It is interesting to re-interpret the RR tadpole cancellation conditions in the T-dual language. In this case, the crosscap diagrams are located on top of the O8-planes, and in a sense compute the RR charge of these objects (the

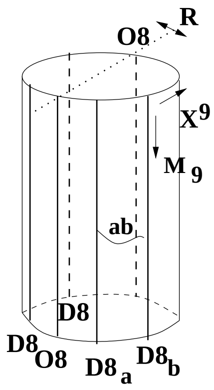

Figure 5: Open string endpoints in the T-dual of type I with Wilson lines are fixed on D8-branes at different positions in the circle. Their stretching is determined by the location of the D8-branes.

strength of their coupling to the RR 9-form (dual to the original 10-form). The disk diagrams are located on top of the D8-branes, and compute the RR charge of these objects. RR tadpole cancellation condition corresponds to the requirement that the fluxlines of the RR 9-form have nowhere to go in the internal space, which is compact, so the total charge must vanish (see fig6). This is Gauss law in a compact space [3] . It is possible to compute these tadpoles as we did for type I, and obtain that each O8-plane has -16 times the charge of a D8-brane. Hence we have $2 \times (-16) + 32 \times 1 = 0$.

We conclude with some relevant observations

• The generalization of this idea to further T-dualities is clear. In the closed string sector the orientifold action acquires an additional geometric

[3]Equivalentely, one can check that the KK reduction of the 9-form has a zero mode, which corresponds to a 9-form in 9d, which has no kinetic term. RR tadpole cancellation can be recovered as the consistency condition for its equations of motion. This description is more analogous (T-dual) to the one used in type I.

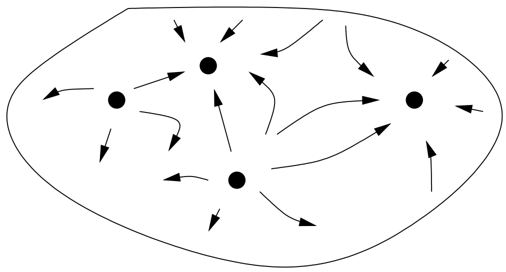

Figure 6: Schematic picture of the interpretation of RR tadpole cancellation as charge cancellation in a compact space.

piece inverting the T-dualized coordinate. Hence, in general we will find theories obtained from toroidal compactification of type IIA/B modded out by Ωg, where g is a geometric action flipping r coordinates, with r even/odd for IIB/IIA. This introduces 2^r O$(9-r)$-planes, which can be seen to carry $32/2^r$ units of RR charge. In order to cancel the RR tadpoles, we introduce 32 D$(9-r)$-branes, which can be at arbitrary locations, but respecting the $\mathbf{Z_2}$ symmetry imposed by g.

• The original type I theory also admits a description in terms of O-planes and D-branes. The Ω projection can be said to introduce an O9-plane (which fills spacetime completely), and the open strings (which can end anywhere in 10d space) can be said to end on D9-branes (which fill spacetime completely).

We should not worry too much about understanding all the details of D-branes at this point. Such objects will reappear in a different way in subsequent section. In fact they correspond to new non-perturbative states in type II string theory. This can be understood already in our picture: recalling that the bulk of spacetime is described by type IIA theory, if one takes the decompactification limit in which the O8-planes go off to infinity, keeping the D8-branes in the middle of the interval, we are roughly left with

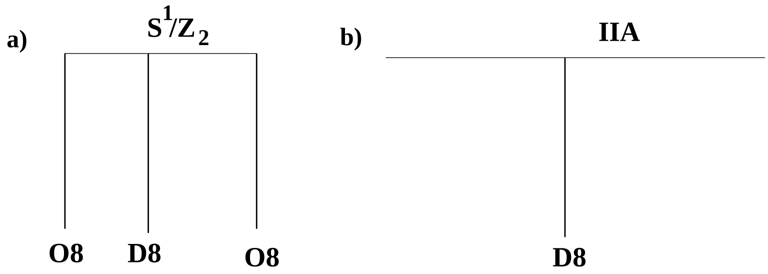

Figure 7: The decompactification limit of type I' keeping the D8-branes at finite distance produces type IIA theory with a topological defect (domain wall) given by the D-brane.

non-compact type IIA theory in the presence of D8-branes, see figure 7. This shows that there exist states in type IIA string theory which are not obtained as perturbative excitations of the type IIA string. Rather, this states should be regarded as a non-perturbative state, analogous in many respects to a soliton. We will come back to these states in later lectures.

5 Final comments

Let us summarize this lecture by emphasizing that we have shown an extremely intimate relation between the different string theories, once we start compactifying them. See figure 8.

This is all very nice, but we should recall that we started out studying string theory as a theory with the potential to unify the interactions we observe in Nature. The theories we have obtained have too much supersymmetry to allow for chirality, so they are quite hopeless as theories of our world. Therefore, we will turn to the study of other compactifications in subsequent lectures.

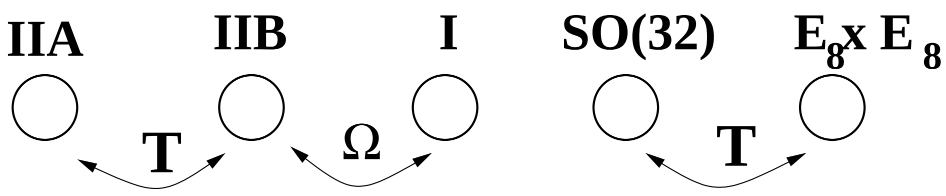

Figure 8: .

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