Appendix: Rudiments of group theory

In this appendix we provide some basic techniques in group theory that we will need to be familiar with. Useful references are [1, 2] and the more formal [3, 4].

1 Groups and representations

1.1 Group

A **group** G is a set on which there exists a multiplication, satisfying

- Closure: For any $g, h \in G$, $g \cdot h \in G$
- Identity element: there exists an element $e \in G$ such that $e \cdot g = g \cdot e = g$ for any $g \in G$
- Inverse: For any $g \in G$ there exists an element g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = e$
- Associativity: $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ for any $g, h, k \in G$

Notice that commutativity $g \cdot h = h \cdot g$ is not required to be a group. If any pair of elements commute, the group is called abelian.

1.2 Representation

A **representation** R of a group is a mapping that, to each element of G associates a linear operator $R(g)$ acting on a vector space V, in a way compatible with the group multiplication, namely

$$R(g)\, R(h) \;=\; R(g \cdot h) \qquad \forall g, h \in G \tag{1}$$

Hence a representation is a homomorphism between G and the set of linear operators on V. If it is an isomorphism (injective and onto), then the representation is called **faithful**.

The vector space V is called the **representation space**, and vectors in V are said to form the representation R of G. The group G is said to acto on V (or on vectors of V) in the representation R.

If the dimension of V is n, and we fix a basis $|e_i\rangle$, any linear operator can be regarded as an $n \times n$ matrix via

$$R(g)_{ij} \;=\; \langle e_i|\, R(g)\, |e_j\rangle \tag{2}$$

So a representation can be defined also as a homomorphism between G and the set of $n \times n$ matrices. We call these matrix representations of G.

Notice that the explicit matrix that represents an element $g \in G$ in a matrix representation, depends on the basis. Hence, it makes sense to define an equivalence relation of matrix representations. Two matrix representations R and R' are **equivalent** if there exist a similarity transformation S ($n \times n$ invertible matrix) such that

$$R'(g) \;=\; S\, R(g)\, S^{-1} \qquad \forall g \in G \tag{3}$$

Namely the matrices $R(g)$ and $R'(g)$ are related by a (g-independent) change of basis in V.

OBS: Often, one find a group acting on a physical system in a particular representation. It is however important to distinguish between the abstract group and its different representations.

1.3 Reducibility

A representation R is **reducible** if it has a matrix version equivalent to a representation with block diagonal matrices

$$R(g) \ = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_2(g) \end{pmatrix} \qquad \forall g \in G \tag{4}$$

Hence V splits into V_1 and V_2, which are acted on, but not mixed, by $R_1(g)$ and $R_2(g)$, respectively.

An **irreducible** representation (**irrep** for short) is one which is not reducible.

1.4 Examples

• The **trivial** representation, which exists for any group G. To every element, it associates the 1×1 matrix 1.

$$R(g) = 1 \quad \forall g \in G \tag{5}$$

It is clearly a homomorphism, but not an isomorphism. It is not a faithful representation

• Irreps of $\mathbf{Z_3}$. The group $\mathbf{Z_3}$ has three elements, 1, g and g^2, with the group multiplication law $g^k \cdot g^l = g^{k+l}$, $g^3 = 1$.

It has three inequivalent irresps, which are all 1-dimensional. One of them is the trivial

$$1 \to 1 \quad ; \quad g \to 1 \quad ; \quad g^2 \to 1 \tag{6}$$

There are two faithful representations

$$\begin{array}{lllll} R_1 & : & 1 \to 1 \; ; & g \to e^{2\pi i/3} \; ; & g^2 \to e^{4\pi i/3} \\ R_2 & : & 1 \to 1 \; ; & g \to e^{4\pi i/3} \; ; & g^2 \to e^{2\pi i/3} \end{array} \tag{7}$$

In fact, it is easy to show that for an abelian group all irreducible representations are necessarily 1-dimensional.

• Group of symmetries of the square. This group is generated by two elements: α, a rotation or 90 degrees around the center of the square, and β, a flip around a vertical axis. Any other element can be obtained by taking products of these. A simple 2-dimensional faithful irrep of this group is

$$\alpha \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad ; \quad \beta \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

and the corresponding product matrix for other elements.

1.5 Operations with representations

It is useful to define them in terms of matrix representation. Let R_1, R_2 be representations of a group G on vector spaces V_1, V_2, on which we specify a basis $|e_i\rangle$, $|f_m\rangle$, of dimensions n_1, n_2 respectively.

• **Sum of representations** We define the sum representation $R_1 \oplus R_2$, acting on $V_1 \oplus V_2$ as

$$R(g) \;=\; \begin{pmatrix} R_1(g) & 0 \\ 0 & R_2(g) \end{pmatrix} \tag{9}$$

It has dimension $n_1 + n_2$, and is clearly reducible.

• **Tensor product of representations**. We define the product represention $R = R_1 \otimes R_2$, acting on $V_1 \times V_2$ (which has basis $|e_i\rangle \otimes |f_m\rangle$) as

$$(R(g))_{im,jn} \;=\; (R_1(g))_{ij}\,(R_2(g))_{mn} \tag{10}$$

It has dimension $n_1 n_2$ and is in general reducible. The decomposition of tensor product representations as sum or irreps is a canonical question in group theory, which can be systematically solved using Clebsch-Gordan techniques.

2 Lie groups and Lie algebras

2.1 Lie groups

A **Lie group** G is a group where the elements are labeled by a set of continuous real parameters, ξ^a, $a = 1, \ldots, N$, with the multiplication law depending smoothly on the latter. Namely

$$g(\xi) \cdot g(\xi') = g(f(\xi, \xi')) \tag{11}$$

with $f^a(\xi, \xi')$ a continuous (usually also C^∞) function of ξ, ξ'.

OBS: The Lie group is a differentiable manifold, and the ξ are coordinates. Usually we define the parameters such that $g(\xi = 0) = e$, the identity element of G. The number of parameters N is called the dimension of the group.

We will be interested in compact Lie groups (which are compact as manifolds), although there exist very important non-compact Lie groups, for instance, the Lorentz group (where the boost parameters correspond to non-compact directions).

Lie groups also have representations. As usual, to each element $g(\xi) \in G$ they associate a linear operator $R(g(\xi))$ on a vector space V, compatibly with the group law. The dimension of V is unrelated to N the dimension of the group. For short we denote $R(g(\xi))$ by $R(\xi)$.

2.2 Lie algebra A(G)

Formally, it is the tangent space to the manifold G at the point corresponding to the identity element, see fig 1. Since the geometry of G is so constrained by the group law, its structure is almost completely encoded just in the tangent space.

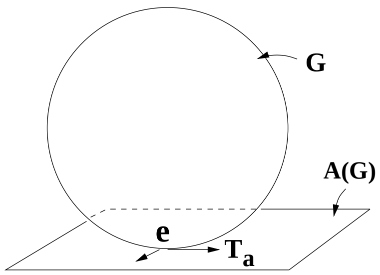

Figure 1: The Lie algebra is in a very precise sense the tangent space to the Lie group at the point corresponding to the identity element.

Recall the differential geometry definition of tangent space of a manifold M at a point P. It is the vector space generated by the objects ∂_a, $a = 1, \ldots, \dim M$; the latter are vectors, formally defined as mappings from the space of functions on M, $\mathcal{F}(\mathcal{M})$ to the real numbers

$$\begin{aligned} \partial_a : \quad \mathcal{F}(\mathcal{M}) \to & \quad \mathbf{R} \\ f(x) \to & \quad \partial_a f(x)|_P \end{aligned} \tag{12}$$

In Lie groups, the natural functions of G are matrix valued functions compatible with the group law, namely representations. Hence we define the vectors T_a as mappings from the space of representations of G, $\mathcal{R}(G)$ to the space of matrices Mat

$$\begin{aligned} T_a : \quad \mathcal{R}(\mathcal{M}) \to & \quad \text{Mat} \\ R(g(\xi)) \to & \quad -i\partial_a R(g(\xi))|_{\xi=0} \end{aligned} \tag{13}$$

This formal definition is used to emphasize that the properties of the T_a are properties of the group and not of any particular representation. In this sense, this can be formally written as '$T_a = -i\partial_a g|_e$'. However, it is often useful to discuss properties etc in terms of representations.

For a fixed representation R, we call $-i\partial_a R(\xi)|\xi = 0$ the representation of T_a in the representation R, and call it t_a^R. It is interesting to note that changes of coordinates in G induce linear transformations on the T_a's, as follows

$$T'_a = \frac{\partial \xi^b}{\partial \xi'^a} T_b \tag{14}$$

We can form linear combinations and multiply the T_a's, as induced from sum and product of matrices. Roughly speaking the Lie algebra is the algebra generated by the T_a's with this sum and product. The linear combinations $\sum_a \lambda_a T_a$ are called generators of the group/algebra (often, just the T_a are called generators of the algebra).

2.3 Exponential map

Generators provide infinitesimal transformations

$$g(0,\ldots,\delta\xi^a,\ldots,0) = e + \partial_a g\, \delta\xi^a = e + i\, T_a\, \delta\xi^a \tag{15}$$

In fact, they are associated to whole one-parameter subgroups of G (which are said to be generated by T_a). In any representation R

$$\begin{aligned} R(0,\ldots,\xi^a+\delta\xi^a,\ldots,0) &= R(0,\ldots,\delta\xi^a,\ldots,0)\, R(0,\ldots,\xi^a,\ldots,0) = \\ &= (1+\partial_a R|_{\xi=0}\delta\xi^a)\, R(0,\ldots,\xi^a,\ldots,0) \end{aligned} \tag{16}$$

On the other hand

$$R(0,\ldots,\xi^a+\delta\xi^a,\ldots,0) = R(0,\ldots,\xi^a,\ldots,0) + \partial_a R|_{\xi=0}\delta\xi^a \tag{17}$$

So we get

$$\partial_a R(0,\ldots,\xi^a,\ldots,0) = i\, t_a^R\, R(0,\ldots,\xi^a,\ldots,0) \tag{18}$$

Hence

$$R(0,\ldots,\xi^a,\ldots,0) \;=\; e^{it_a^R\xi^a} \quad (\text{no sum}) \tag{19}$$

In the abstract group/algebra

$$g(0,\ldots,\xi^a,\ldots,0) \;=\; e^{iT_a\,\xi^a} \quad (\text{no sum}) \tag{20}$$

In fact, any element of the group $g(\xi)$ continously connected to the identity can be written as

$$g(\xi) = e^{i\sum_a T_a\,\xi^a} \tag{21}$$

for a suitable generator $\sum_a \xi^a T_a$ in the algebra, see figure 2. So the whole group can be recovered from the structure of the algebra [1]

2.4 Commutation relations

The generators T_a satisfy simple commutation relations

$$[T_a, T_b] = i f_{abc} T_c \tag{22}$$

where f_{abc} are called the structure constants of the group/algebra.

i) They are determined by the group multiplication law. To see this, consider the group element $g(\lambda)$ defined by

$$g_{ab}(\lambda) \;=\; e^{i\lambda T_b}\, e^{i\lambda T_a}\, e^{-i\lambda T_b}\, e^{-i\lambda T_a} \tag{23}$$

[1] In fact, some global information on the group may not be recovered from the algebra. There are groups which are globally different yet have the same Lie algebra. They are typically quotients of each other, so they differ in their homotopy groups. The group recovered from the algebra is the so-called universal cover group, which is the only simply connected group with that algebra. This subtle issue is what makes $SU(2)$ and $SO(3)$ have the same Lie algebra although $SU(2)$ is simply connected and $SO(3) = SU(2)/\mathbf{Z_2}$.

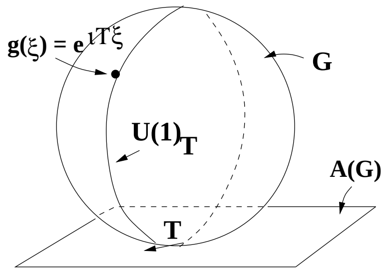

Figure 2: Any element in the group (in the component continuously connected to the identity) can be obtained from a generator in the Lie algebra by the exponential map.

Expanding around $\lambda = 0$, we have

$$g_{ab}(\lambda) \;=\; 1 \;+\; \lambda^2\,[T_a, T_b] \;+\ldots \tag{24}$$

Since $g(\lambda)$ is a group element, infinitesimally close to the identity, it also has the expansion as identity plus some element in the algebra

$$g_{ab}(\lambda) \;=\; 1 \;+\; \lambda^2 \sum_c f_{abc} T_c \tag{25}$$

By comparing, we get the commutation relations (22)

ii) They determine the group multiplication law, at least for elements connected to the identity. To see that, consider two group elements $e^{i\lambda^a T_a}$ and $e^{i\sigma^a T_a}$, their product is some element $e^{i\rho^a T_a}$. The Lie algebra information is enough to find the ρ^a in terms of the λ^b, σ^c. By expansion of the relation

$$e^{i\lambda^a T_a}\, e^{i\sigma^a T_a} \;=\; e^{i\rho^a T_a} \tag{26}$$

we get

$$\rho^a \;=\; \lambda^a \;+\; \sigma^a \;-\; \frac{1}{2}\, f_{abc}\, \lambda^b\, \sigma^c \;+\; \ldots \tag{27}$$

this verifies our claim.

The commutation relations satisfy the Jacobi identities

$$[T_a, [T_b, T_c]] \;+\; [T_c, [T_a, T_b]] \;+\; [T_b, [T_c, T_a]] \;=\; 0 \tag{28}$$

(as in any representation they are simply matrices which obviously satisfy this relation). This can be easily translated into a relation among the structure constants.

A representation R of the Lie algebra is a mapping that to each T_a it associates a linear operator t_a^R (acting on a space V of some dimension n, independent of the dimension N of the group), consistently with linear combinations and with the commutation relations, namely

$$[t_a^R, t_b^R] = i f_{abc} t_c^R \tag{29}$$

Clearly the structure constants are a property of the group/algebra and not of the representation.

Clearly, given a representation of the group we can build a representation of the algebra (by taking representations of group elements close to the identity $t_a^R = -i\partial_a R(\xi)$), and viceversa (by the exponential mapping $R(\xi) = e^{it_a^R \xi^a}$).

The structure constants depend on the choice of basis in the Lie algebra, so it is convenient to fix a canonical choice. To fix it, consider the quantity $\mathrm{tr}\,(t_a^R t_b^R)$ in any representation R; it is a real and symmetric matrix, which can be diagonalized by a change of basis in the Lie algebra. Once we are in such basis $\mathrm{tr}\,(t_a^R t_b^R) = k_R \delta_{ab}$ and we obtain the structure constants as

$$f_{abc} = -\frac{i}{k_R}\,\mathrm{tr}\,([t_a^R, t_b^R] t_c^R) \tag{30}$$

and are completely antisymmetric.

Since this can be played for any representation R, it shows that there exists a basis in the abstract Lie algebra where (22) hold with completely antisymmetric structure constants.

In the remaining of this lecture we will center on compact Lie groups, for which any representation is equivalent to a unitary representation. In such representation all generators are hermitian and the structure constants are real.

2.5 Some useful representations

There is a very useful representation which is canonically built in the structure of the Lie algebra. It is the **adjoint** representation, which is N-dimensional (same dimension as the group). Consider an N-dimensional vector space, with a set of basis vectors labeled by the generators of the algebra $|T_a\rangle$, $a = 1, \ldots, N$. And represent T_a by the linear operator $t_a^{\rm Adj}$ defined by

$$t_a^{\rm Adj}|T_b\rangle \;=\; |[T_a, T_b]\rangle \;=\; i\, f_{abc}\, |T_c\rangle \tag{31}$$

Namely we have the matrix elements $(t_a^{\rm Adj})_{bc} = -i f_{abc}$.

Given any representation R, with generators represented by t_a^R, we can build another representation R^*, called the **conjugate representation**, with generators represented by $-(t_a^R)^T$. It is a simple exercise to check that it also provides a representation of the algebra.

3 SU(2)

To warm up before the study of more general Lie algebras, we study the construction of representations for $SU(2)$, the simplest non-abelian group. The Lie algebra is given by

$$[J_a, j_b] = i\epsilon_{abc} J_c \tag{32}$$

A familiar representation is provided by the Pauli matrices $J_a = \sigma_a/2$, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{33}$$

In this representation, elements of the group correspond to 2×2 unitary matrices. This particular representation arises as the action of the 3d rotation group on spin 1/2 particles. We will be interested in constructing more general representations in a more systematic way.

3.1 Roots

We first put the Lie algebra in **Cartan-Weyl form**. To do that, the first step is to choose a maximal set of mutually commuting generators (this is the so-called **Cartan subalgebra**, whose dimension is called the **rank** of the group/algebra). For $SU(2)$ any pair of generators is non-commuting, there is at most one such generator, say J_3.

Next, se take the remaining generators are form linear combinations

$$J^{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \tag{34}$$

such that they have simple commutation relations with the Cartan generator J_3

$$[J_3, J^+] = J^+ \quad ; \quad [J_3, J^+] = -J^- \tag{35}$$

In intuitive terms, this tells us the charges of $J^{\pm}$ with respect to the $U(1)$ subgroup generated by the Cartan J_3. In the adjoint representation, we have the relation $J_3|J^{\pm}\rangle = \pm|J^{\pm}\rangle$; upon exponentiation, $g(\xi)|J^{\pm} = e^{\pm i\xi J_3}|J^{\pm}\rangle$, namely $|J^{\pm}$ tranform with charges $\pm$ under the $U(1)$ generated by J_3. By abuse of language we use the same language for $J^{\pm}$ themselves.

Figure 3: Fig. a) shows the root diagram for the $SU(2)$ Lie algebra; Fig b) shows the general structure of the weights for irreducible representations of this algebra.

We also have

$$[J_3, J_3] = 0 \quad ; \quad [J^+, J^-] = J_3 \tag{36}$$

This are the commutation relations for the algebra written in the Cartan-Weyl form. The charges of the different generators with respect to the $U(1)$ generated by the Cartan J_3 are called the **roots** of the algebra. In our case we have the roots -1, 0, $+1$ for J^-, J_3, J^+ respectively.

The roots of an algebra are drawn in a **root diagram**, as in figure 3a). Such picture encodes all the information about the algebra.

3.2 Weights

Let us now discuss the construction of irreps. The representation space is a vector space spanned by a set of basis vectors. It is natural to take a basis where the representative of J_3 is diagonal, and then it is natural to label each vector in the basis by its J_3 eigenvalue, $|\mu\rangle$. Hence we have by construction

$$J_3|\mu\rangle = \mu|\mu\rangle \tag{37}$$

The eigenvalues μ are in principle real numbers, which give us the charge of the corresponding eigenstate with respect to the $U(1)$ generated by J_3. Such charges are called **weights** of the representation. The irrep is essentially defined by giving the set of weights for all basis vector in the representation space, and it is usual to draw the weights in a **weight diagram** (see below) that encodes all information about the representation.

We define the **highest weight** as the highest of all eigenvalues, and call it j. Soon we will see that the complete irrep is defined just in terms of its highest weight.

An important fact is that **weights** in an irrep differ by roots. Starting with a state of weight $|\mu\rangle$, we can build the states $J^{\pm}|\mu\rangle$, which are eigenstates of J_3 with eigenvalues $\mu \pm 1$

$$J_3 J^{\pm}|\mu\rangle \;=\; ([J_3, J^{\pm}] + J^{\pm}J_3)|\mu\rangle \;=\; (\pm J^{\pm} + \mu J^{\pm})|\mu\rangle \;=\; (\mu \pm 1)J^{\pm}|\mu\rangle \quad (38)$$

So the states $J^{\pm}|\mu\rangle$ must be either zero or they are part of our basis vectors. Hence there should exist weigths which are equal to $\mu \pm 1$, namely weights differ by roots.

Since by definition $\mu = j$ was the highest weight, the structure of the basis vectors is

$$|j\rangle\,,\; |j-1\rangle\,,\; |j-2\rangle\;\ldots \quad (39)$$

On the other hand, the representations we are interested in are finite dimensional, so the representation should end. To compute when, we must realize that $J^{-}|\mu\rangle \simeq |\mu - 1\rangle$ up to a normalization factor. Namely, one has

$$\begin{aligned} J^{-}|\mu\rangle \;&=\; N_\mu\,|\mu - 1\rangle \\ J^{+}|\mu\rangle \;&=\; N_\mu\,|\mu + 1\rangle \end{aligned} \quad (40)$$

and the coefficent can be computed to be

$$N_\mu = \frac{1}{\sqrt{2}}\sqrt{(j+\mu)(j-\mu+1)} \quad (41)$$

which means that the representation is finite-dimensional if some $\mu = -j$

$$J^{-}|-j\rangle = 0 \quad (42)$$

Since μ's differ by integers, j and $-j$ must differ by an integers, which implies the constraint that j must be integer or half odd.

Hence irreps of $SU(2)$ are characterized by a highest weight, which must be an integer of half-odd number. The representation space is spanned by the basis vectors

$$|j\rangle\,,\ |j-1\rangle\,,\ |j-2\rangle\ \ldots\ |-j\rangle \tag{43}$$

which is $(2j+1)$-dimensional. The matrices representing generators in this space are easy to obtain from the actions of $J^{\pm}$, J_3 on the basis vectors. All the information of the irrep with highest weight j is encoded in a weight diagram as in figure 3b.

4 Roots and weights for general Lie algebras

The idea is to generalize to any Lie algebra the procedure introduced for $SU(2)$.

4.1 Roots

First we put the Lie algebra in the **Cartan-Weyl form**. The first step is to pick a maximal set of mutually commuting hermitian [2] generators, which we call H_i, $i = 1\ldots, r$. The number of such generators is called the **rank** r of the group; they generate the **Cartan subalgebra** of the Lie algebra. Upon exponentiation, they generate a $U(1)^r$ subgroup of the Lie group.

The second step is to take linear combinations of the remaining operators so that they have easy commutators with the H_i. To do that, we go to the

[2] By abuse of language we talk about a hermitian generator in the abstract algebra, as a generator which is represented by a hermitian operator/matrix in any unitary representation.

adjoint representation, with basis vectors $|T_a\rangle$, and construct the matrix

$$M_{ab}^{(i)} = \langle T_a|H_i|T_b\rangle \tag{44}$$

Diagonalizing simultaneously the matrices $M^{(i)}$ (they commute since they represent the Cartan generators, which commute in the abstract algebra), we get a new basis of vectors $|E_\alpha\rangle$, which are eigenstates of the H_i (better, of their representatives in the adjoint representation). We label each such state by its r eigenvalues α_i with respect to H_i.

$$H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle \tag{45}$$

At the level of the abstract algebra, this induces some linear combinations of the original generators T_a into some generators E_α with commutation relations

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{46}$$

These are not hermitian, rather $E_\alpha^\dagger = E_{-\alpha}$

Using the Jacobi identity it is also possible to show that

$$\begin{aligned} [E_\alpha, E_{-\alpha}] &= \sum_i \alpha_i H_i \\ [E_\alpha, E_\beta] &= E_{\alpha+\beta} \text{ if } \alpha+\beta \text{ is root} \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{47}$$

The r-dimensional vectors α are called the **roots of the Lie algebra**, and they provide the charges of the E_α with respect to the $U(1)^r$ generated by the Cartan subalgebra.

4.2 Weights

To describe irreps, we choose a basis of the representation space where all matrices representing the Cartan generators are diagonal, and we label the vectors in the basis (eigenstates of the matrix representing H_i) by the corresponding eigenvalues. By abuse of language, we denote H_i the matrix representing the abstract H_i in the representation. We have

$$H_i|\mu\rangle = \mu_i|\mu\rangle \qquad i = 1, \ldots, r \tag{48}$$

The r-dimensional vectors μ are called **weights of the representation**. The set of weights of a representation characterize the representation.

OBS: Notice that the weights are a property of the representation, while the roots are a property of the algebra. Notice also that the weights of the adjoint representation are the roots of the Lie algebra (this is beacuse the adjoint is a very canonical representation, built into the structure of the algebra itself).

OBS: Notice that in an irreducible representation there may be different states with the same weight vectors. One (special) example is the states $|H_i\rangle$ in the adjoint representation, which all have weight equal to zero. One must be careful in dealing with situations where different vectors have same weights.

In a given representation, weights are not arbitrary. Rather, as in $SU(2)$, **weights differ by roots**. Namely, starting with an state $|\mu\rangle$ we can construct $E_{\pm\alpha}|\mu\rangle$ which is an eigenstate of the H_i, with eigenvalue $\mu_i \pm \alpha_i$, as follows

$$H_i E_{\pm\alpha}|\mu\rangle \;=\; (\alpha_i E_{\pm\alpha} + E_{\pm\alpha} H_i)\,|\mu\rangle \;=\; (\mu_i \pm \alpha_i) E_{\pm\alpha}|\mu\rangle \tag{49}$$

So there must in principle exist a weight in the representation given by the vectors $\mu + \alpha$, and a corresponding state $|\mu \pm \alpha\rangle$. In fact, as in $SU(2)$ we have a relation modulo a coefficient

$$E_{\pm\alpha}|\mu\rangle \;=\; N_{\mu,\pm\alpha}|\mu \pm \alpha\rangle \tag{50}$$

and for some μ we will have $N_{\mu,\pm\alpha} = 0$, which ensures that representations are finite-dimensional, and impose some additional constraints on the possible values of the weights μ. The sets of allowed irreps and the corresponding weights is difficult to analyze in general, and we leave their discussion for specific examples, see sections 6.

It is worth pointing out that the analogy with $SU(2)$ is quite precise. In fact, for any non-zero root α, the generators $E_{\pm\alpha}$, $\sum_i \alpha_i H_i$ form an $SU(2)$ subalgebra of the Lie algebra. Defining $E^{\pm} = \frac{1}{|\alpha|} E_{\pm\alpha}$, $E_3 = \frac{1}{|\alpha|^2} \sum_i \alpha_i H_i$ we have the commutators

$$[E_3, E^{\pm}] = \pm E^{\pm} \quad ; \quad [E^+, E^-] = E_3 \tag{51}$$

which is an $SU(2)$ algebra in the Cartan-Weyl form. This means that for any μ the states $|\mu + k\alpha\rangle$ form an irrep of this $SU(2)$.

For future convenience, we use this a bit further. This irrep will contain some highest and lowest $SU(2)$ weight states $|j\rangle$ and $|-j\rangle$, namely there exist integers p, q such that

$$\begin{aligned} E_\alpha |\mu + p\alpha\rangle = 0 \quad &; \qquad j = \frac{\alpha \cdot \mu}{|\alpha|^2} + p \\ E_{-\alpha} |\mu - q\alpha\rangle = 0 \quad &; \qquad -j = \frac{\alpha \cdot \mu}{|\alpha|^2} - q \end{aligned} \tag{52}$$

so we get $\frac{\alpha\cdot\mu}{|\alpha|^2} = -\frac{1}{2}(p-q)$. This is the master formula extensively used in the classification of Lie algebras, see section 5.

The basic strategy to build irreps is therefore as follows. We need to introduce the concept of a highest weight. To do so, we define a positive vector in the r-dimensional space of roots/weights/charges, $v > 0$ if $v_1 > 0$; if $v_1 = 0$ we say that $v > 0$ if $v_2 > 0$; etc. We say that one vector v is higher than other vector w, $v > w$, if $v - w > 0$. This allows to define the **highest weight** μ_0 of a representation the weight such that $\mu_0 > \mu$ for any other weight μ.

The concept of positivity allows to split the set of non-zero roots into the set of positive roots and of negative roots. For $\alpha > 0$ the E_α are raising operators and the $E_{-\alpha}$ are lowering operators. The highest weight vector is characterized by the fact that it is annihilated by the raising operators (if not, we would get states $|\mu_0 + \alpha\rangle$ with weight higher than $|\mu_0\rangle$, which was defined as the highest!).

The representation is build by applying lowering operators to the highest weight state, in all possible inequivalent ways, until we exhaust the representation (namely, until we start finding zeroes upon application of lowering operators). That this happens is guaranteed because states form representations of the $SU(2)$'s associated to each α, and such representations are finite-dimensional from our experience with $SU(2)$

4.3 SU(3) and some pictures

Instead of giving the commutation relations of the $SU(3)$ algebra, all the relevant information is provided by the root diagram of the algebra, shown in figure 4. Namely, the rank is two; the Cartan subalgebra is spanned by two generators H_1, H_2, which are mutually commuting. The remaining eight generators are labelled E_α, $E_{-\alpha}$ for $\alpha = (1,0)$, $(1/2, \sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$, and have commutation relations

$$[H_i, E_{\pm\alpha}] = \pm\alpha_i E_{\pm\alpha} \tag{53}$$

Notice the $SU(2)$ subalgebras along the different α's, which graphically correspond to lines along which the roots reproduce the root diagram of $SU(2)$.

Some representations

Instead of writing the explicit matrices providing a particular representation of the $SU(3)$ algebra, we can instead provide the weight diagram of the

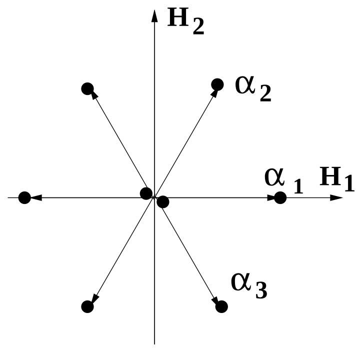

Figure 4: The root system of the $SU(3)$ Lie algebra. The positive roots are $\alpha_1 = (1,0)$, $\alpha_2 = (1/2, 1/(2\sqrt{3})$, $\alpha_3 = (1/2, -1/(2\sqrt{3})$. The two roots at $(0,0)$ correspond to the Cartan generators.

corresponding representation.

A familiar representation is the fundamental representation, which is 3-dimensional, and on which the generators are represented as 3×3 hermitian matrices (the Gell-Mann matrices). Upon exponentiation, the group elements are represented as 3×3 unitary matrices.

This representation can be equivalentely described by the weights in picture 5a. The action of the Cartans on the states $|\mu = (\pm 1/2, 1/(2\sqrt{3}))$, $(0, -1/\sqrt{3})$ is

$$H_i|\mu\rangle = \mu_i|\mu\rangle \tag{54}$$

The action of non-zero root generators E_α is

$$E_\alpha|\mu\rangle = N_{\mu,\alpha}|\mu + \alpha\rangle \tag{55}$$

Notice that the states form representations under the $SU(2)$ subalgebras of the non-zero roots. That is, weights along lines parallel to the root diagram of the corresponding $SU(2)$ subgroup differ by the corresponding root.

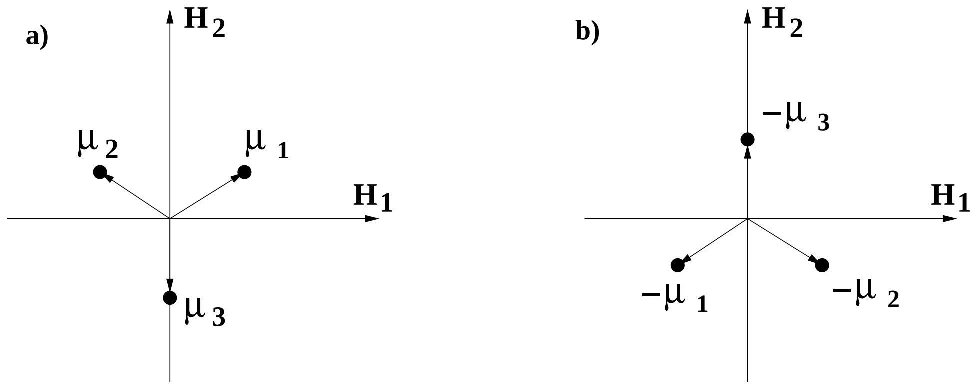

Figure 5: The weight diagram for the fundamental (a) and antifundamental representations of $SU(3)$.

The construction of the irrep is as follows. The highest weight is $|(1/2, 1/(2\sqrt{3}))\rangle$, so this is annihilated by the positive roots $\alpha_1 = (1,0)$, $\alpha_2 = (1/2, 1/(2\sqrt{3})$, $\alpha_3 = (1/2, -1/(2\sqrt{3})$. The remaining states are obtained as

$$
\begin{aligned}
E_{-\alpha_1}|(1/2, 1/(2\sqrt{3}))\rangle &\simeq |(-1/2, 1/(2\sqrt{3}))\rangle \\
E_{-\alpha_2}|(1/2, 1/(2\sqrt{3}))\rangle &\simeq |(0, -1/\sqrt{3}))\rangle
\end{aligned}
\tag{56}
$$

The conjugate representation, the antifundamental, which is obtained by minus the transposed GellMann matrices, has weights opposite to those of the fundamental. Namely, conjugation of the representation flips the charges of objects. The weights are shown in figure 5b

5 Dynkin diagrams and classification of simple groups

The discussion in this section will be very sketchy. For more information, see chapter VIII of [1]. However, the discussion is not too relevant, one can jump to the results directly.

The information we have obtained is also useful in yielding information that can be used to classify all possible Lie algebras. In fact in the study of representations we obtained some interesting constraints. For instance, recall the master formula that for any representation, the fact that $|\mu + k\alpha$ for a representation of $SU(2)_\alpha$ implied that the weights satisfy

$$\frac{\alpha \cdot \mu}{|\alpha|^2} = -\frac{1}{2}(p - q) \tag{57}$$

In particular we may apply this to the adjoint representation, where the weight μ is a root. Requiring that the states $|\beta + k\alpha\rangle$ form a representation of $SU(2)_\alpha$, and that the states $|\alpha + k\beta\rangle$ form a representation of $SU(2)_\beta$, we get

$$\frac{\alpha \cdot \beta}{|\alpha|^2} = -\frac{1}{2}m \quad ; \quad \frac{\beta \cdot \alpha}{|\beta|^2} = -\frac{1}{2}m \quad ; m.m' \in \mathbf{Z} \tag{58}$$

We obtain a constraint on the relative angle of the roots

$$\cos^2 \theta_{\alpha,\beta} \;=\; \frac{(\alpha \cdot \beta)^2}{|\alpha|^2|\beta|^2} \;=\; \frac{mm'}{4} \tag{59}$$

The angle is constrained to be 0, 30, 45, 60, 90, 120, 135, 150 or 180 degrees.

5.1 Simple roots

We now define a **simple root** as a positive root which cannot be written as a sum of positive roots with positive coefficients. Simple roots have nice properties, in particular the set of simple roots of an algebra is linearly independent, and there are r simple roots; so simple roots provide a basis of root space.

Moreover, the angles between simple roots are more constrained. To see this, notice that if α and β are simple roots, then $\alpha - \beta$ is *not* a root [3]. Now going to the adjoint representation, $E_{-\alpha}$ must annihilate E_β (since otherwise it would create a state $|E_{\beta-\alpha}\rangle$, but $\beta - \alpha$ is not a root!), so $|E_\beta\rangle$ is the lower weight state $|-j\rangle$ for the subalgebra $SU(2)_\beta$, and we get

$$2\frac{\alpha \cdot \beta}{|\alpha|^2} = -p \quad ; p \in \mathbf{Z}^+ \tag{60}$$

Hence the quantities $2\frac{\alpha\cdot\beta}{|\alpha|^2}$ are non-positive integers for simple roots. Using

$$2\frac{\alpha \cdot \beta}{|\alpha|^2} = -p \quad ; \quad 2\frac{\alpha \cdot \beta}{|\beta|^2} = -p' \tag{61}$$

we get $\cos\theta_{\alpha,\beta} = -\frac{1}{2}\sqrt{pp'}$, and this forces the angles between simple roots to be 90, 120, 135 or 150 degrees.

5.2 Cartan classification

The only invariants of the set of simple roots are the relative lengths and angles of the simple roots. Use of this information is enough to recover the complete system of roots, since simple roots provide a basis. Hence the problem of classification of Lie algebras is the problem of classifying sets of r linearly independent vectors in r-dimensional space with non-positive integer values of $2\alpha \cdot \beta/|\alpha|^2$.

In the classification it is important to note the following. Two r_1- resp r_2-dimensional systems of simple roots, satisfying the above properties, can

[3] If it were, it would be positive or negative; if it is positive then $\alpha = \beta + (\alpha - \beta)$ contradicts the fact that α is simple; if it is negative, then $\beta = \alpha + (\beta - \alpha)$ contradicts that β is simple

always be combined into a new $(r_1 + r_2)$-dimensional simple root system, by simply joining orthogonally the two initial systems. Clearly we are interested in root systems which cannot be split into orthogonal subsystems.

This is related to the concept of invariant subalgebra. Given and algebra A, an invariant subalgebra B is a subalgebra such that the commutator of any element in B with any element in A is still in A. Upon exponentiation, Lie algebras with invariant subalgebras lead to non-simple groups, namely groups which split as product of groups, $G = G_1 \times G_2$.

So one is in principle interested in classifying simple groups (as any other is obtained by taking products) and Lie algebras without invariant subalgebras (simple Lie algebras). Lie algebras with invariant subalgebras manifest as root systems which split into two orthgonal subsystems. Hence we are interested in classifying simple root systems without such subsystems. Any other can be obtained by simple adjunction.

The problem of classifying simple root systems of this kind has been solved. The result, called the Cartan classification can be recast is giving the relative lengths and angles between the simple roots. This is conveniently codified in a picture called the Dynkin diagram. The classification of **Dynkin diagrams** for simple Lie algebras is given in figure 6. The rules to obtain the simple root system from the diagram are as follows.

• Each node corresponds to a simple root (hence the number of nodes is the rank of the Lie algebra/group)

• The number of lines joining two nodes gives us the angle between the two simple roots: no line means 90^o, one line means 120^o, two lines means 135^o, three lines means 150^o.

• Dark nodes correspond to shorter roots (the relative lengths can be found from (59)

Clearly, Dynking diagrams corresponding to non-simple algebras are ob-

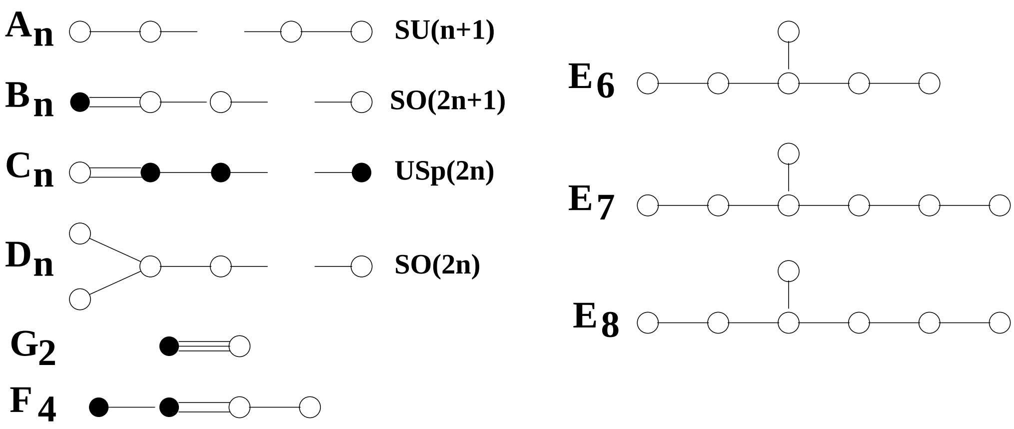

Figure 6: Dynkin diagrams for simple Lie algebras. There are four infinite series (labeled by a positive integer r giving the number of nodes), and some exceptional algebras. Notice that for small rank some algebras are isomorphic and have the same Dynking diagram (e.g. $A_3 = D_3$, namely $SU(4) \simeq SO(6)$. The groups arising from the A, B, C, and D series were known in classical mathematics before Cartan and are known as classical Lie groups, they are listed to the right of the corresponding diagram.

tained by adjoining in a disconnected way Dynkin diagrams for simple algebras (so that we adjoin orthogonally the two subsystems of simple roots).

6 Some examples of useful roots and weights

There are some systems of roots and weights that we will encounter in our study of string theory. In this section we list some of them. A more complete reference, which includes a systematic discussion of tensor products or irreps, and decomposition of representations under subgroups, is the appendices of [5].

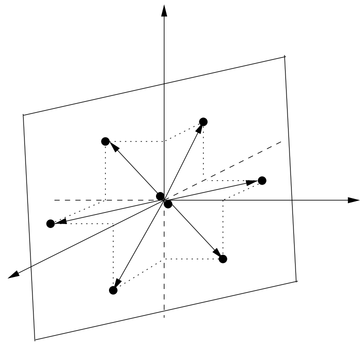

Figure 7: The root system of $SU(3)$ described as a set of vectors lying in a 2-plane in 3-dimensional space.

6.1 Comments on $SU(k)$

Roots

Although $SU(k)$ (or its algebra A_{k-1}) has rank $k-1$, it is convenient and easier to describe its roots as k-dimensional vectors, which lie on an $(k-1)$-plane. Besides the $k-1$ zero roots associated to the Cartan generators, the non-zero roots are given by the k-dimensional vectors

$$(\underline{+,-,0,\ldots,0}) \tag{62}$$

where $+$, $-$ denote $+1$, -1, and where underlining means permutation, namely the $+$ and $-$ can be located in any (non-coincident) positions. Note that all roots satisfy one relation $\sum_{i=1}^{n} v_i = 0$, so they live in a $(k-1)$-plane Π in $\mathbf{R^n}$. There are a total of k^2-1 roots, which is the number of generators of $SU(k)$.

Fixing a basis within the $(k-1)$-plane it is straightforward to read out the roots as $(k-1)$-dimensional vectors. The picture of the root system of $SU(3)$ in this language is given in figure 7.

The extra direction in the diagram can be regarded as associated to the extra $U(1)$ generator in $U(k) = SU(k)\times U(1)$. Hence, $SU(k)$ weight diagrams embedded in $(k-1)$-planes parallel to Π but not passing through the origin are associated to states which, in addition to being in a representation of $SU(k)$, also carry some charge under the additional $U(1)$.

Weights

A familiar representation is the fundamental representation. The corresponding weights, given as k-dimensional vectors but inside the $(k-1)$-plane Π are,

$$\frac{1}{n}\,(\underline{n-1,-1,\ldots,-1}) \tag{63}$$

Notice that weights differ by roots, so application of generators associated to non-zero roots relate states with different weights (or give zero if they take us out of the representation).

In situations where the gauge group is $U(k)$ so there is an additional $U(1)$ generator, the fundamentals of $SU(k)$ may carry some charge, so the weights satisfy the relation $\sum_{i=1}^{n} v_i = q$ for some non-zero constant q giving (up to normalization) the charge under the additional $U(1)$. Very often one finds fundamentals from weights of the form

$$(\underline{+,0,\ldots,0}) \tag{64}$$

or

$$\frac{1}{2}(\underline{+,-,\ldots,-}) \tag{65}$$

Notice that the weights (63) can be written as

$$(\underline{+,0,\ldots,0})\;-\;(1/n,\ldots,-1/n) \tag{66}$$

where the second term removes the piece corresponding to the additional $U(1)$ charge. By abuse of language, we will often use things expressions like

(64) or (65) to denote the fundamental even in situations where there is no additional $U(1)$, removing implicitly the piece corresponding to this charge.

The weights for the antifundamental representation are the opposite to those for the fundamental, namely

$$(\underline{-, 0, \ldots, 0}) \tag{67}$$

By this, we mean

$$\frac{1}{n} (\underline{-(n-1), 1, \ldots, 1}) \tag{68}$$

or any other shifted version, with the understanding that the additional $U(1)$ charge should be removed.

Other representations can be obtained by taking tensor products of the fundamental (using the techniques of Yound tableaux, not discussed in this lecture, see [1] for discussion). The corresponding weights are obtained by adding the weights of the fundamental representation.

For instance, the two-index antisymmetric representation has $k(k-1)/2$ weights

$$(\underline{+, +, 0, \ldots, 0}) \tag{69}$$

while the two-index symmetric representation has $k(k+1)/2$ weights

$$(\underline{+, +, 0, \ldots, 0}) \quad ; \quad (\underline{\pm 2, 0, \ldots, 0}) \tag{70}$$

They are obtained by adding two times weights of the fundamental representation in a way consisten with antisymmetry or symmetry of the representation.

It is straightforward to derive familiar facts like the equivalence of the antifundamental representation and the $(k-1)$-index antisymmetric representation. They have the same weights.

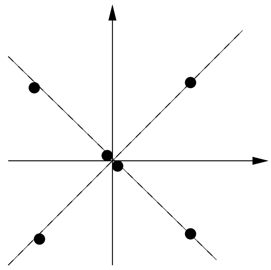

Figure 8: Root diagram for $SO(4)$. In fact it splits as two orthogonal $SU(2)$ root systems.

6.2 Comments on $SO(2r)$

Roots

Besides the n zero roots, the non-zero roots for the D_r Lie algebra are given by the r-dimensional vectors

$$(\underline{\pm, \pm, 0, \ldots, 0}) \tag{71}$$

Meaning that the + and - can be choses arbitrarily in any non-coincident position. The total number of roots is $2r(2r-1)/2$.

The root system of $SO(4)$ is shown in figure 8. The fact that there are two subsets of orthogonal roots means that there are invariant subalgebras. In fact, $SO(4) \simeq SU(2) \times SU(2)'$, with non-zero roots of the latter being given by

$$SU(2) \,:\, (++), (--) \quad ; \quad SU(2)' \,:\, (+-), (-+) \tag{72}$$

Notice also that the Dynkin diagram for D_2 are two disconnected nodes, so is the same as two A_1 Dynkin diagrams.

It is important to notice that the root system of $SO(2r)$ contains the roots of $SU(r)$, so by exponentiation the group $SO(2r)$ contains a subgroup

$SU(r)$.

Weights

An important representation is the **vector representation**, which is $2r$-dimensional and has weights

$$(\underline{\pm, 0, \ldots, 0}) \tag{73}$$

Notice that it is a real representation, since its conjugate has opposite weights, but the representation (as a whole) is invariant under such change.

When the group is regarded as the group of rotational isometries of a $2r$ dimensional euclidean space, the vector representation in which vectors of this space transform.

More representations can be obtained by taking tensor products of the vector representation. These are the respresentations under which tensors in the euclidean space transform under rotations.

There are some additional representations which cannot be obtained from tensor products of the vector representation. These are the spinor representations. For D_r there are two inequivalent irreducible spinor representations, both with dimension 2^{r-1}, and weights

$$\begin{aligned} \text{spinor} &: \quad (\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}) \quad , \ \#- = \text{even} \\ \text{spinor}' &: \quad (\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}) \quad , \ \#- = \text{odd} \end{aligned} \tag{74}$$

These spinor representations are said to have different chirality [4].

Spinor representations and Clifford algebras

[4] Clearly there discussion of spinors under the Lorentz group in even dimensional space can be recovered from the group theory of spinor representations of $SO(2r)$ (with a few subtleties arising from the non-compactness of the Lorentz group). A nice discussion of Lorentz spinors can be found in the appendices of [6].

There is a canonical and very useful way to describe the spinor representations of $SO(2r)$, related to representations of **Clifford algebras**. We briefly review this here, since it will appear in our construction of string spectra.

Consider the algebra of objects Γ^i, $i = 1, \ldots, 2r$, satisfying

$$\{\Gamma^i, \Gamma^j\} = 2\delta_{ij} \tag{75}$$

It is called a Clifford algebra. It is important to remark that this is not a Lie algebra! In particular it is not defined in terms or commutators.

The important point is that this algebra is invariant under the group of transformations

$$\Gamma'^i = R^i_j \Gamma^j \tag{76}$$

where R is a $2r \times 2r$ orthogonal matrix. This group is precisely $SO(2r)$, and we have found it acting on the set of Γ^i in the fundamental representation.

The fact that the Clifford algebra (75) has an $SO(2r)$ invariance menas that any representation of the Clifford algebra must also form a representation of $SO(2r)$. In fact, given a hermitian matrix representation for the Γ^i, the hermitian matrices $J^{ij} = \frac{-i}{4}[\Gamma^i, \Gamma^i]$ can be seen to form a (possibly reducible) hermitian matrix representation of the $SO(2r)$ algebra, which is

$$[J^{ij}, J^{kl}] = i\,(\delta^{ik} J^{jl} + \delta^{jl} J^{ik} - \delta^{il} J^{jk} - \delta^{jk} J^{il}) \tag{77}$$

So our purpose is to build a representation of the Clifford algebra, and the resulting representations of $SO(2r)$. The standard technique to build a representation of the Clifford algebra is to form linear combinations of the Γ^i which can act as raising and lowering operators. We define

$$A_a = \frac{1}{\sqrt{2}}(\Gamma_{2a} + i\Gamma_{2a-1}) \quad ; \quad A_a^\dagger = \frac{1}{\sqrt{2}}(\Gamma_{2a} - i\Gamma_{2a-1}) \quad , \quad a = 1, \ldots, r \tag{78}$$

They satisfy the relations

$$\{A_a^\dagger, A_b^\dagger\} = \{A_a, A_b\} = 0 \quad ; \quad \{A_a^\dagger, A_b\} = \delta_{ab} \tag{79}$$

So they behave as fermionic oscillator ladder operators. Notice that in this language only an $SU(r)$ invariance is manifest, with the $A_a^\dagger$, A_a transforming in the fundamental resp. antifundamental representations.

To build a representation of the Clifford algebra, we introduce a 'groundstate' for the harmonic oscillator

$$A_a|0\rangle = 0 \tag{80}$$

The representation is built by applying raising operators to this 'groundstate' in all possible inequivalent ways. We have

$$\begin{array}{ll} \text{states} & \text{number} \\ |0\rangle & 1 \\ A_a^\dagger|0\rangle & r \\ A_a^\dagger A_b^\dagger|0\rangle & r(r-1)/2 \\ \ldots & \ldots \\ A_{a_1}^\dagger \ldots A_{a_k}^\dagger|0\rangle & \begin{pmatrix} r \\ k \end{pmatrix} \\ \ldots & \ldots \\ A_1^\dagger \ldots A_r^\dagger|0\rangle & 1 \end{array} \tag{81}$$

The bunch of $\begin{pmatrix} r \\ k \end{pmatrix}$ states arising from applying k operators to the groundstate clearly forms a k-index completely antisymmetric tensor representation of the $SU(r)$ invariance group.

The total number of states is 2^r. Constructing the Lorentz generators, it is possible to check that the weights are of the form

$$(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}) \tag{82}$$

Moreover, it is easy to realize that the weights among the above with has k $+1/2$'s correspond to the weights of a k-index completely antisymmetric tensor representation of $SU(r)$, in agreement with our above statement.

The above weights therefore define a representation of the $SO(2r)$ group (although only $SU(r)$ invariance was manifest in intermediate steps). Now this representation is reducible. REcalling that the $SO(2r)$ generators are constructed with products of two Γ^i's, it is clear that they are unable to relate states (81) with even number of Γ's to states with odd number of Γ's. More formally, one can introduce the chirality operator $\Gamma = \Gamma^1 \ldots \Gamma^{2r}$ which commutes with all $SO(2r)$ generators (and anticommutes with the Γ^i), and can be used to distinguish the two subsets of states.

This means that the 2^r-dimensional representation is reducible into two 2^{r-1}-dimensional irreducible representations, with weights given in (74), called the chiral spinor representations.

6.3 Comments on $SO(2r+1)$

We will not say much about $SO(2r+1)$, since most of the relevant facts about its representations can be obtained by noticing that it is a subroup of $SO(2r+1)$ and that it contains an $SO(2r)$ subgroup.

Let us simply say that it has an $(2r+1)$-dimensional vector representation, out of which other tensor representations can be obtained by tensor produce. It also has a unique spinor representation, of dimension 2^r which is irreducible [5].

The tensor product of representations and decomposition under subgroups can be found in standard tables, like the appendices in [5].

[5] This underlies the fact that there are no chiral spinors in euclidean spaces of odd dimension.

6.4 Comments on $USp(2n)$

We will not say much about these, since these groups rarely appear in particle physics or in string theory. Moreover, most of its properties can be derived from the trick that it can be constructed from $U(2n)$ by keeping the subset of roots invariant under an involution. We will see more of this as we need it.

6.5 Comments on exceptional groups

The most interesting one is E_8, since it appears automatically in the construction of the heterotic superstring. Moreover, properties of E_6, E_7 etc are easy to derive since they are subgroups of E_8. For details we refer to the properties listed in tables like the appendices in [5].

For the moment, the only data we need is the root system of E_8. This has rank 8 and dimension 248, and the 240 non-zero roots are of the form

$$\begin{array}{ll} (\underline{\pm, \pm, 0, 0, 0, 0, 0, 0}) & \\ (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}) & , \quad \#- = \text{even} \end{array} \tag{83}$$

Notice that there is a nice subset of $SO(16)$ roots, given by the first line of non-zero roots (along with the 8 Cartan generators). With respect to this $SO(16)$ subalgebra, the states associated with the vectors in the second line are transforming in a 2^{8-1}-dimensional chiral spinor representation of $SO(16)$.

We will find good application of these facts for instance in the identification of the spectrum of the heterotic theory.

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