The Shell Model: An Unified Description of the Structure of the Nucleus

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The Nuclear A-body Problem

- In the Standard Model of Nuclear Structure the elementary components are nucleons (N neutrons and Z protons, N+Z=A). The mesonic and quark degrees of freedom are integrated out
- In most cases non-relativistic kinematics is used
- The bare nucleon-nucleon (or nucleon-nucleon-nucleon) interactions are inspired by meson exchange theories or more recently by chiral perturbation theory, and must reproduce the nucleon-nucleon phase shifts, and the properties of the deuteron and other few body systems

The Nuclear A-body Problem

- **The challenge is to find $\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_A)$ such that**
- **HΨ=EΨ, with**
- **H=** $\sum_{i}^{A} T_i + \sum_{i,j}^{A} V_{2b}(\vec{r}_i, \vec{r}_j) + \sum_{i,j,k}^{A} V_{3b}(\vec{r}_i, \vec{r}_j, \vec{r}_k)$
- **The knowledge of the eigenvectors Ψ and the eigenvalues E make it possible to obtain electromagnetic moments, transition rates, weak decays, cross sections, spectroscopic factors, etc.**

The spherical mean field and the NN interaction

The usual procedure to generate a mean field in a system of N interacting fermions, starting from their free interaction, is the Hartree-Fock approximation, extremely successful in atomic physics. Whatever the origin of the mean field, the eigenstates of the N-body problem are Slater determinants *i.e.* anti-symmetrized products of N single particle wave functions.

The spherical mean field and the NN interaction

In the nucleus, there is a catch, because the very strong short range repulsion and the tensor force make the HF approximation based upon the bare nucleon-nucleon force impracticable.

However, at low energy, the nucleus do manifest itself as a system of independent particles in many cases, and when it does not, it is due to the medium range correlations that produce strong configuration mixing and not to the short range repulsion.

The meaning of the Independent Particle Model

Does the success of the shell model really "prove" that nucleons move independently in a fully occupied Fermi sea as assumed in HF approaches? In fact, the single particle motion can persist at low energies in fermion systems due to the suppression of collisions by Pauli exclusion.

To know more, read the article "Independent particle motion and correlations in fermion systems" V. R. Pandharipande, et al., RMP 69 (1997) 981.

The meaning of the Independent Particle Model

Brueckner theory takes advantage of the Pauli blocking to regularize the bare nucleon- nucleon interaction, in the form of density dependent effective interactions of use in HF calculations or G-matrices for large scale shell model calculations.

The price to pay is that the independent particles are now dressed nucleons which require the use of effective transition operators.

A simple regularized interaction: Brink y Boeker

$$V_c(|\vec{r}_1 - \vec{r}_2|) = \sum_{i=1}^{2} [1 - m_i(1 + P_\sigma P_\tau)]\ v_i\ e^{-|\vec{r}_1 - \vec{r}_2|^2/\mu_i^2}$$

$$+ V_{ls} = \frac{-12\ \mathrm{MeV}}{\hbar^2 \sqrt{A}}\ \vec{l} \cdot \vec{s}$$

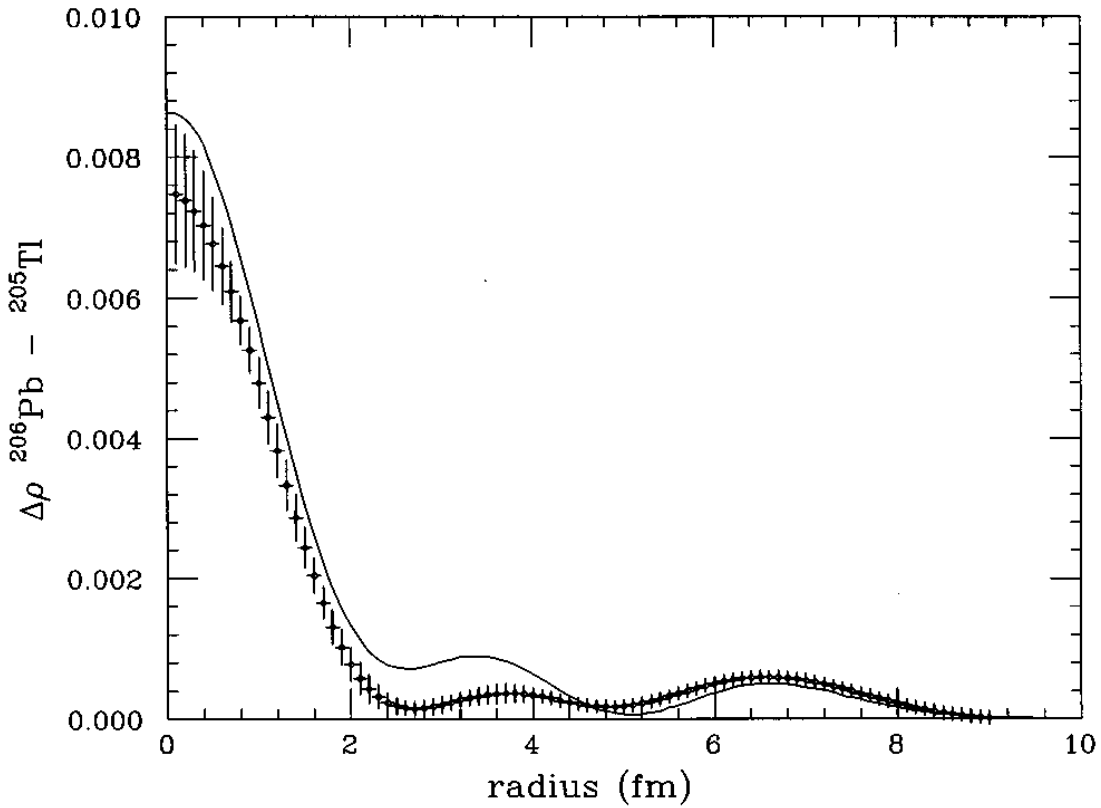
with

i	μ_i(fm)	v_i (MeV)	m$_i$
1	0.7	471.1	-0.43
2	1.4	-163.8	0.51

The meaning of the Independent Particle Model

In spite of that, the nuclear quasi-particles resemble extraordinarily to the mean field solutions of the IPM. This was demonstrated by the beautiful electron scattering experiment of Cavedon *et al.* (1982) in which they extracted the charge density difference between ^{206}Pb and ^{205}Tl, that, in the IPM is just the square of the $2s_{1/2}$ orbit wave function.

The meaning of the Independent Particle Model



The shape of the $2s_{1/2}$ orbit is very well given by the mean field calculation. To make the agreement quantitative the calculated density has to be scaled down by an effective occupation number

Beyond the IPM; The mean field way

- HF-based approaches rely on the use of density dependent interactions of different sort; Skyrme, Gogny, or Relativistic Mean Field parametrizations
- The correlations are taken into account breaking symmetries at the mean field. Particle number for the pairing interaction and rotational invariance for the quadrupole-quadrupole interaction
- Projections before (VAP) or after (PAV) variation are enforced to restore the conserved quantum numbers
- Ideally, configuration mixing is also implemented through the GCM

Beyond the IPM; The Interacting Shell Model (ISM)

Is an approximation to the exact solution of the nuclear A-body problem using effective interactions in restricted spaces

The single particle states (i,j, k, ...), which are the solutions of the IPM, provide as well a basis in the space of the occupation numbers (Fock space). The many body wave functions are Slater determinants:

$$\Phi = a^{\dagger}_{i_1}, a^{\dagger}_{i_2}, a^{\dagger}_{i_3}, \ldots a^{\dagger}_{i_A}|0\rangle$$

We can distribute the A particles in all the possible ways in the available single particle states, getting a complete basis in the Fock space. The number of Slater determinants will be huge but not infinite because the theory is no longer valid beyond a certain cutt-off.

A formal solution to the A-body problem

Therefore, the "exact" solution can be expressed as a linear combination of the basis states:

$$\Psi = \sum_{\alpha} \Phi_{\alpha}$$

and the solution of the many body Schödinger equation

$$H\Psi = E\Psi$$

is transformed in the diagonalization of the matrix:

$$\langle \Phi_{\alpha} | H | \Phi_{\beta} \rangle$$

whose eigenvalues and eigenvectors provide the "physical" energies and wave functions

The Interacting Shell Model (ISM) or (SM-CI)

- **A Shell Model calculation amounts to diagonalizing the effective nuclear hamiltonian in the basis of all the Slater determinants that can be built distributing the valence particles in a set of orbits which is called valence space. The orbits that are always full form the core.**
- **If we could include all the orbits in the valence space (a full No Core calculation) we should get the ”exact” solution.**

Effective interactions for SM-CI calculations

- The effective interactions are obtained from the bare nucleon-nucleon interaction by means of a regularization procedure aimed to soften the short range repulsion. In other words, using effective interactions we can treat the A-nucleon system in a basis of independent quasi-particles. As we reduce the valence space, the interaction has to be renormalized again in a perturbative way. Up to this point these calculations are in fact "ab initio"
- Moreover, the realistic NN interactions seem to be correct except for its simplest part, the monopole hamiltonian responsible for the evolution of the spherical mean field. Therefore, we surmise that the three body forces will only contribute to the monopole hamiltonian.

The three pillars of the shell model

- The Effective Interaction
- Valence Spaces
- Algorithms and Codes

Dimensions

- **If the number of states in the valence space for the protons is D_π and for the neutrons is D_ν**
- **The dimension of the basis for n_π valence protons and n_ν valence neutrons is:**

$$\left(\begin{array}{c} D_\pi \\ n_\pi \end{array} \right) \times \left(\begin{array}{c} D_\nu \\ n_\nu \end{array} \right)$$

- **For instance for ^{48}Cr in the *pf*-shell, D=23 474 025. In reality we only need the M=0 Slater Determinants and this gives D_0=1 963 461**
- **The maximum dimension in the *pf*-shell corresponds to ^{60}Zn, D_0= 2 292 604 744**

The Hamiltonian

- As our basis is provided by the IPM, it is natural to express the many body states and the Hamiltonian in terms of creation and annihilation of particles in IPM states
- In addition, this approach makes it possible to distinguish the components of the Hamiltonian which only contribute to the spherical mean field from those which are responsible for the many body correlations

The Hamiltonian

$$\mathcal{H} = \sum_r \epsilon_r \, \hat{n}_r + \sum_{r \le s, t \le u, \Gamma} W^{\Gamma}_{rstu} Z^{+}_{rs\Gamma} \cdot Z_{tu\Gamma},$$

where Z^{+}_{Γ} (Z_{Γ}) is the coupled product of two creation (annihilation) operators

$$Z^{+}_{rs\Gamma} = [a^{\dagger}_r a^{\dagger}_s]^{\Gamma}$$

where Γ is a shorthand for (J,T); r, s ..... run over the orbits of the valence space; ϵ_r are the single particle energies and W^{Γ}_{rstu} the two body matrix elements:

$$W^{\Gamma}_{rstu} = \langle j_r j_s (JT) | V | j_t j_u (JT) \rangle$$

How to compute the two-body matrix elements

In the occupation number representation (Fock space) all the information about the interaction is contained in its two body matrix elements. The many body problem then reduces to the manipulation of the creation and annihilation operators using the Wick theorem and techniques alike.

The most general method to compute the two body matrix elements is due to Slater and carries its name. When the independent particle wave functions are those of the harmonic oscillator or if they can be represented by linear combination of a few harmonic oscillator states, the method of choice is that of Brody and Moshinsky

The two-body matrix elements: Slater

For a general central interaction, the two body matrix elements involve integrals in the coordinates of the two particles, $\vec{r}_1$ and $\vec{r}_2$ of a product of functions of $\vec{r}_1$ and $\vec{r}_2$ (the IPM wave functions) and a function of $|\vec{r}_1 - \vec{r}_2|$ (the potential) which is not at all straightforward. Slater's method consists in expanding the interaction in a complete set of functions, for instance, the Legendre polynomials:

$$V(|\vec{r}_1 - \vec{r}_2|) = \sum_{k=0}^{\infty} v_k(r_1, r_2) P_k(cos\ \theta_{12})$$

with

$$P_k(cos\ \theta_{12}) = \sum_{\kappa} \frac{4\pi}{(2k+1)} Y_k^{\kappa^*}(\Omega_1) Y_k^{\kappa}(\Omega_2)$$

Two body matrix elements; Moshinsky's transformation

The hamiltonian of two nucleons in an harmonic oscillator potential can be written as:

$$H = \frac{1}{2m}(\vec{p}_1^2 + \vec{p}_2^2) + \frac{1}{2}m\omega^2(\vec{r}_1^2 + \vec{r}_2^2)$$

Using the transformation $\vec{r}=\vec{r}_1-\vec{r}_2$; $\vec{R}=\frac{1}{2}(\vec{r}_1+\vec{r}_2)$, it becomes:

$$H = \frac{1}{m}\vec{p}^2 + \frac{1}{4}m\omega^2\vec{r}^2 + \frac{1}{4m}\vec{P}^2 + m\omega^2\vec{R}^2$$

Two body matrix elements; Moshinsky's transformation

Therefore the wave function of the two nucleons can be written in two ways:

$$|n_1 l_1 m_1\rangle|n_2 l_2 m_2\rangle \quad \text{and} \quad |nlm\rangle|N\Lambda M_\Lambda\rangle$$

Both provide equivalent basis for the problem, and therefore it is possible to express the states of one basis as a linear combinations of the states of the other. This is the Moshinsky transformation.

$$|n_1 l_1, n_2 l_2; LM\rangle = \sum_{n,l,N,\Lambda} \langle nl, N\Lambda; L|n_1 l_1, n_2 l_2; L\rangle |nl, N\Lambda; LM\rangle$$

The coefficients of Brody and Moshinsky

The coefficients are named after Brody and Moshinsky which tabulated them in a book published by the UNAM in 1960. The second edition published by Gordon and Breach in 1967 is probably the only physics book which is bilingual Spanish English

The coefficients of Brody and Moshinsky

Now, the two body matrix elements of a central interaction $V(r)$ are easily obtained as:

$$\langle n_1 l_1, n_2 l_2; LM | V(r) | n_1' l_1', n_2' l_2'; LM \rangle =$$

$$\sum_{n,n',l,N,\Lambda} \langle nl, N\Lambda; L | n_1 l_1, n_2 l_2; L \rangle \langle n' l, N\Lambda; L | n_1' l_1', n_2' l_2'; L \rangle \langle nl | V(r) | n' l \rangle$$

More general interactions can be easily incorporated with an extra work of angular momentum algebra.