The Nuclear Shell Model

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Outline

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- What Shells?
- When does a bound fermionic system show a shell-like behaviour?
- What about the Nucleus? Some basic experimental facts
- The Independent Particle Model, aka, Nuclear Shell Model
- Deformed nuclei and the Nilsson model
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Why Shells?

An atom with Z electrons is described by the following Hamiltonian:

$$H = \sum_i T_i - \sum_i \frac{Ze^2}{r_i} + \sum_{i<j} \frac{e^2}{r_{ij}}$$

If we disregard the repulsion between the electrons (third term), the many body problem is trivial to solve because we can write:

$$H = \sum_i h_i$$

Why Shells?

Thus, to solve the many body Schrödinger equation $H\Psi = E\Psi$, we only need to solve the one body equation $h\psi = \epsilon\psi$

The eigenvalues of this equation are ϵ = $\frac{-Z^2 Ry}{n^2}$

and the eigenvectors ψ = $R_{n'l}(r) \cdot Y_{lm}(\theta, \phi) \cdot \chi_{s,s_z}$

with n=1, 2, ...
n$'$+ l = n
m=-l, -l+1, ... l-1, l
s=1/2, s$_z$=-1/2, +1/2

Why Shells?

The wave function of the ground state (the state of lowest energy) of an atom with Z electrons is the anti-symetrized product of the eigenfunctions of the Z eigenstates of lower energy (Slater determinant). Because, the electrons being fermions, they obey the the Pauli principle and only one electron can occupy a single state characterized by the quantum numbers (n, l, m, s_z). Its energy, the sum of the ϵ's of these states.

Notice that the energy of the states (n, l, m, s_z) only depends on n, therefore for a given n all the states with different values of (l, m, s_z) are degenerated, and it is said that they form a SHELL.

The states with the same values of n and l form an ORBIT

What Shells?

$$d = 2 \cdot \sum_{0}^{n-1} (2l + 1)$$

.

For n=1 d=2, for n=2, d=8, for n=3, d=18, and so on.

At the Z values which correspond to filling completely a number of shells plus one electron there is a large drop in the ionization energies which signal the change from inert noble gases, Z=2, Helium, Z=10, Neon, etc, to the very reactive alkalines. However in our model the next noble gas would have Z=28, Nickel, not a very brilliant prediction!

What Shells?

What happens is that the approximation we have made (hydrogen-like behaviour all around) is no longer valid when we have many electrons. The effect of the electron repulsion can be incorporated in the one body (mean field) potential by means of the Hartree-Fock method. In the HF mean field the grouping of the orbits in shells changes, and the next shell is 3s-3p d=8 giving Z=18, Argon: the next 3d-4s-4p, d=18 leads to Z=36 Krypton, etc: now, indeed, in agreement with the experimental data.

When does a bound fermionic system show a shell-like behaviour?

The atom is a very peculiar system in the sense that it has an external mean field to start with: the attractive potential provided by the nucleus. This is not the case for the system that interest us (the nucleus has no nucleus) neither for others, like the droplets of liquid ^{3}He. Now if we go back to the Hamiltonian we have:

$$H = \sum_i T_i + \sum_{i<j} V(r_{ij})$$

When does a bound fermionic system show a shell-like behaviour?

The key point is whether is it possible to find a mean field U(r), either phenomenologically or via the Hartree-Fock method, such that if we write:

$$H = \sum_i (T_i + U_i) + (\sum_{i<j} V(r_{ij}) - \sum_i U_i) = H_0 + H_R$$

- **Condition I. The eigenvalues of H_0 produce quasi degenerated groups of orbits well separated in energy from one another.**
- **Condition II. H_R can be treated somehow in perturbation theory.**

What do the textbooks tell us about the nucleus?

- It is a system composed of Z protons and N neutrons (A=N+Z)
- Whose low energy behavior can described with non relativistic kinematics
- Bound by the strong nuclear interaction; the restriction of QCD to the space of neutrons and protons
- Which has a complicated form: Strong short range repulsion, spin-spin, spin-orbit and tensor terms, etc

Basic experimental facts

- Which nuclei are stable?
- How much they weight? The mass of a nucleus is the sum of the masses of its constituents minus the energy due to their mutual interactions (binding energy), which is the lowest eigenvalue of its Hamiltonian
- For medium and heavy mass nuclei the binding energy per particle is roughly constant (saturation)
- What are their matter densities and radii? The nuclear radius grows as $A^{1/3}$, therefore the nuclear density is constant (saturation)

The Liquid Drop Model

- **These properties resemble to those of a classical liquid drop, thus the binding energies might be reproduced by a semi empirical mass formula with its volume and surface terms: B= a_v A - a_s $A^{2/3}$**
- **However the drop is charged and the Coulomb repulsion -a_c Z^2/ $A^{1/3}$ favors drops made only of neutrons, therefore an extra term has to be included to reproduce the experimental line of stability: the symmetry term which favors nuclei with N=Z; - a_{sym} $(N\text{-}Z)^2$/ A**

More on Experimental Data

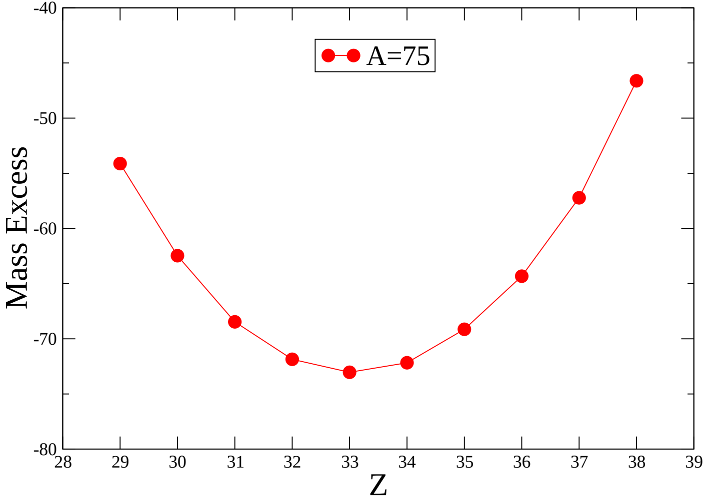
- Nuclei are quantal objets which have discrete energy levels characterized by their total angular momentum J and their parity.
- Each state has a well defined excitation energy and magnetic and electric moments. It may also have a size or density distribution different from that of the ground state
- Excited states may decay by coupling to the electromagnetic field, emitting photons of different multipolarities, hence they have an associated half life and different branching ratios to different final states

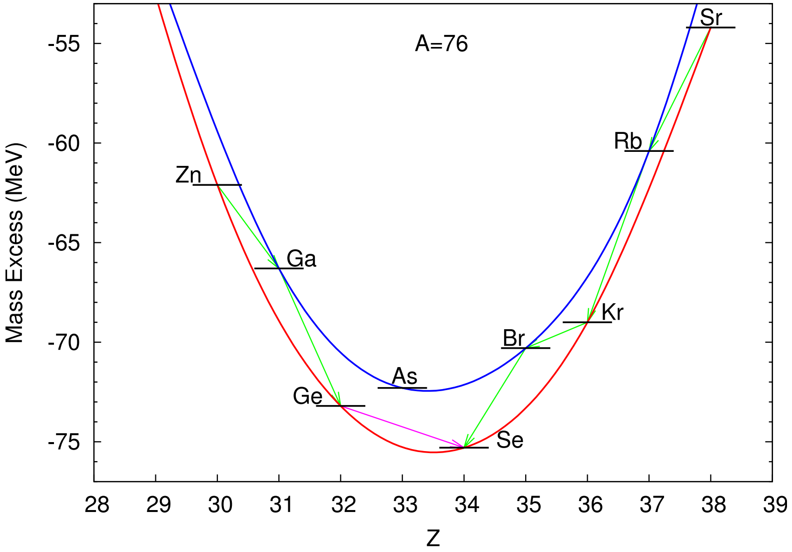
More on Experimental Data

- The nuclear states couple also to the weak field and may β-decay to a more bound isobar with one more/less unit of charge. This is the most frequent decay mechanism for nuclei in their ground states, albeit they may also decay by α or proton emission. All these decays are characterized by their half-lives and branching ratios. Excited states can have even more decay modes as for instance one and two neutron emission.
- At the end of the day, the validity of a nuclear model must be gauged by its success in predicting or explaining all this body of experimental data

Beyond the LDM

- Even with this addition the LDM cannot explain the fact that there is an anomalously large fraction of even-N even-Z nuclei among the stable ones and only a few odd-odd. Indeed, it cannot produce the splitting of the mass parabolas for even A. This requires a new ad hoc addition; the pairing term which is clearly beyond the liquid drop picture


A=75
Mass Excess
Z
-40
-50
-60
-70
-80
28
29
30
31
32
33
34
35
36
37
38
39


A=76
Sr
Rb
Zn
Ga
Kr
Br
As
Ge
Se
-55
-60
-65
-70
-75
Mass Excess (MeV)
28
29
30
31
32
33
34
35
36
37
38
39
Z

Magic Numbers

- In addition, when the neutron and proton separation energies were examined, it was realised that they showed peaks at very precise (and the same) numbers of neutrons and protons, reminiscent of the ones found in the ionisation potentials of the noble gases. This big surprise gained to these numbers the label ”magic numbers”, not a very scientific one indeed!
- In order to explain the magic numbers, (2, 8, 20, 28, 50, 82, 126) the IPM (or naive shell model) of the nucleus was postulated.

The Independent Particle Model

The basic idea of the IPM is to assume that, at zeroth order, the result of the complicated two body interactions among the nucleons is to produce an average self-binding potential. Mayer and Jensen (1949) proposed a mean field consisting in an isotropic harmonic oscillator plus a strongly attractive spin-orbit potential and an orbit-orbit term.

$$H = \sum_i h(\vec{r}_i)$$

$$h(r) = -V_0 + t + \frac{1}{2} m\omega^2 r^2 - V_{so}\vec{l} \cdot \vec{s} - V_B l^2$$

The Independent Particle Model

Later, other functional forms , which follow better the form of the nuclear density and have a more realistic asymptotic behavior, *e.g.* the Woods-Saxon well, were adopted

$$V(r) = V_0 \left(1 + e^{\frac{r-R}{a}}\right)^{-1}$$

with

$$V_0 = \left(-51 + 33\frac{N-Z}{A}\right) MeV$$

and

$$V_{ls}(r) = \frac{V_0^{ls}}{V_0}(\vec{l} \cdot \vec{s})\frac{r_0^2}{r}\frac{dV(r)}{dr} \; ; \; V_0^{ls} = -0.44\, V_0$$

The Independent Particle Model

The eigenvectors of the IPM ($h\phi_{nljm} = \epsilon_{nlj}\phi_{nljm}$) are characterized by the radial quantum number n, the orbital angular momentum l, the total angular momentum j and its Z projection m.

$$\phi_{nljm} = R^{ho}_{n,l}(r) \cdot (Y_{l,l_z}(\theta,\phi) \times \chi_{s,s_z})^j_m$$

with the choice of the harmonic oscillator. The eigenvalues are:

$$\epsilon_{nlj} = -V_0 + \hbar\omega(2n + l + 3/2)$$

$$-V_{so}\frac{\hbar^2}{2}(j(j+1) - l(l+1) - 3/4) - V_B\hbar^2 l(l+1)$$

In order to reproduce the nuclear radii, $\hbar\omega = 41A^{-1/3}$ p=(2n+l) is the principal quantum number of the oscillator.

The IPM, degeneracies and magic numbers

An orbit of total angular momentum has d=2j+1

An harmonic oscillator shell of principal quantum number p has d=(p+1)(p+2)

Therefore the magic numbers of the HO are (N or Z) = $\sum_0^p$ (p+1)(p+2); 2, 8, 20, 40, 70, 112, etc,

The addition of the spin-orbit term regroups the orbits into new shells which produce the correct magic numbers:

- p=0 d=2 (2)
- p=1 d=6 (8)
- p=2 d=12 (20)
- $0f_{7/2}$ d=8 (28)
- p=3 – $0f_{7/2}$ + $0g_{9/2}$ d=22 (50)
- p=4 – $0g_{9/2}$ + $0h_{11/2}$ d=32 (82)
- p=5 – $0h_{11/2}$ + $0i_{13/2}$ d=44 (126)

Vocabulary

- STATE: a solution of the Schrödinger equation with a one body potential; e.g. the H.O. or the W.S. It is characterized by the quantum numbers *nljm* and the projection of the isospin t_z
- ORBIT: the ensemble of states with the same *nlj*, e.g. the 0d5/2 orbit. Its degeneracy is (2j+1)
- SHELL: an ensemble of orbits quasi-degenerated in energy, e.g. the *pf* shell
- MAGIC NUMBERS: the numbers of protons or neutrons that fill orderly a certain number of shells
- GAP: the energy difference between two shells
- SPE, single particle energies, the eigenvalues of the IPM hamiltonian

Caveat

- Condition II does only apply in the immediate neighbourhood of nuclei with magic proton and neutron numbers. Hence, in most nuclei, H_R, mainly composed of pairing and quadrupole-quadrupole interactions, plays a dominant role
- Even if this is so, the IPM solutions provide a very adequate basis to solve non-pertubatively the full nuclear many body problem

The wave function of the nucleus in the IPM

- The WF of the ground state of a nucleus (N, Z) is the product of one Slater determinant for the protons and another for the neutrons, built with the N/Z states ϕ_{nljm} of lower energy
- Except if N and Z are such that they correspond to the complete filling of a set of orbits, the solution is not unique. If we have one particle in excess or in defect, this is not a problem because of the magnetic degeneracy. In all the remaining cases the many body solutions of the IPM do not have a well defined total angular momentum J, as they should due to the rotation invariance of the Hamiltonian.

The wave function of the nucleus in the IPM plus schematic pairing

- **Thus, already at this stage, it is necessary to incorporate dynamical effects that go beyond the spherical mean field obtain physically sound solutions. The minimal choice is to assume that pairs of identical particles on top of a filled orbit are always coupled to total angular momentum zero, due to the strong residual two body pairing interaction**
- **Lets work out the case of the Calcium isotopes as a textbook example**

The IPM description of the Calcium isotopes

- ^{40}Ca is doubly magic. All the orbits of the p=1, 2, and 3 HO shells are filled for neutrons and protons. Therefore the WF of its ground state is a single Slater determinant and "a fortiori" has $J^\pi=0^+$ a fact borne out by experiment. A nice, if trivial, triumph of the IPM.
- The next IPM orbit is the $0f_{7/2}$ followed by $1p_{3/2}$: if we add a neutron, we have several candidates for the GS, (j=7/2, m), but all of them are degenerate in energy, what makes the choice of m irrelevant. Definitely the IPM prediction for the GS of ^{41}Ca is $J^\pi=7/2^-$, and, trivially its first excited state has $J^\pi=3/2^-$. A new success of the IPM.

The IPM description of the Calcium isotopes

- Let's move to ^{42}Ca. Now we have more choices; (j=7/2, m), (j=7/2, m'). This gives 28 combinations which correspond to the values of J allowed by the Pauli principle $J^{\pi}=0^{+}$, 2^{+}, 4^{+} and 6^{+} with M degeneracies 2J+1, 1+5+9+13=28. Within the pure IPM all have the same energy.
- What one should do now is to compute the expectation value of the residual interaction in these states, to break the degeneracy. And indeed, the effective -pairing like- residual neutron neutron interaction privileges the 0^{+} over the other couplings. Again this is what the experiments tell us.
- If we disregard the other possible couplings, the GS of ^{43}Ca would be $J^{\pi}=7/2^{-}$, as it is. We can continue applying the same recipe as far as we want in neutron number. What will be your the prediction for ^{57}Ca?

The IPM description of other observables

- Within the IPM some properties of the nucleus stem just from those of the odd nucleon alone, for instance their ground state magnetic moments
- It is also useful to define the single particle limit of the γ and β decay transition probabilities. In the former case these are called Weisskopft units. Transitions which carry many WU's indicate the onset of collectivity.

	λ=1	λ=2	λ=3	λ=4
E	$1. \times 10^{14} A^{2/3} E^3$	$7.3 \times 10^7 A^{4/3} E^5$	$34. \times A^2 E^7$	$1.1 \times 10^{-5} A^{8/3} E^9$
M	$5.6 \times 10^{13} E^3$	$3.5 \times 10^7 A^{2/3} E^5$	$16 \times A^{4/3} E^7$	$4.5 \times 10^{-6} A^2 E^9$

(energies in MeV)

- Allowed and super allowed β decays have reduced transition probabilities O(1) corresponding to log ft values 3-5

The IPM assets

- The IPM explains the magic numbers, the spins and parities of the ground states and some excited states of doubly magic nuclei plus or minus one nucleon, their magnetic moments, etc. As we have just seen, with the addition of an schematic pairing term it can go a bit further in semi-magic nuclei (Schmidt lines).
- What is less well known is that in the large A limit, the IPM can reproduce the volume, the surface and the symmetry terms of the semi-empirical mass formula as well.

The IPM and the semi-empirical mass formula

- **Let's take the IPM with an HO potential and neglect the spin orbit term. Then:**

$$H = \sum_i t_i - V_0 + \frac{1}{2} m\omega^2 r_i^2$$

- **the single particle energies are:** $\epsilon_i = -V_0 + \hbar\omega(p_i + 3/2)$
- **and** $< r_i^2 >= b^2(p_i + 3/2)$ **with** $b^2 = \dfrac{\hbar}{m\omega}$
- **The degeneracy of each shell is d=(p+1)(p+2) for protons and for neutrons**

The IPM and the semi-empirical mass formula

- **Assume N=Z. To accommodate A/2 identical particles we need to fill the shells up to p=p_F**
- **Experimentally, the radius of the nucleus is given by** $< r^2 >= \frac{3}{5}R^2 = \frac{3}{5}(1.2A^{1/3})^2$
- **And in the IPM by:**

$$< r^2 >= \frac{2}{A}\sum_{i}^{A/2} < r_i^2 >= \frac{2}{A}\sum_{p=0}^{p_F} b^2(p+3/2)(p+1)(p+2)$$

From

$$\frac{A}{2} = \sum^{p_F}(p+1)(p+2)$$

it obtains at leading order, $p_F = (\frac{3}{2}A)^{1/3}$

The IPM and the semi-empirical mass formula

- **Putting everything together we find, at leading order in p_F,**
  $b^2 = A^{1/3}$ **and** $\hbar\omega = 41 \cdot A^{-1/3}$
- **We can now compute the total binding energy as:**

$$B = \sum_{i=1}^{A} (-V_0 + \hbar\omega(p_i + 3/2))$$

- **that gives at leading order**

$$\frac{B}{A} + V_0 = \hbar\omega \cdot \frac{p_F^4}{4} \cdot \frac{2}{A} = \hbar\omega \left(\frac{3A}{2}\right)^{4/3} \frac{1}{2A} = \hbar\omega A^{1/3} \frac{1}{2} \left(\frac{3}{2}\right)^{4/3}$$

Finally we have

$$\frac{B}{A} = -V_0 + 41 \times 0.86$$

and we recover the volume term of the semi empirical

The IPM and the semi-empirical mass formula

- **If we go to next to leading order, keeping the terms in p_F^3, we obtain the surface term with the correct coefficient**
- **We can repeat the calculation at leading order but with N$\neq$Z, and obtain**

$$B = -AV_0 + \frac{\hbar\omega}{4}((p_F^\nu)^4 + (p_F^\pi)^4) = -AV_0 + \frac{\hbar\omega}{4}((3N)^{4/3} + (3Z)^{4/3})$$

- **Making a Taylor expansion around the minimum at N=Z and using the previously determined values we find an extra term of the form $(N\text{-}Z)^2/A$ with a coefficient a_{sym} = 16 MeV) which does not agree with the one resulting from the fit of the semi empirical mass formula to the experimental binding energies (a_{sym} = 23 MeV).**

The symmetry energy in the IPM

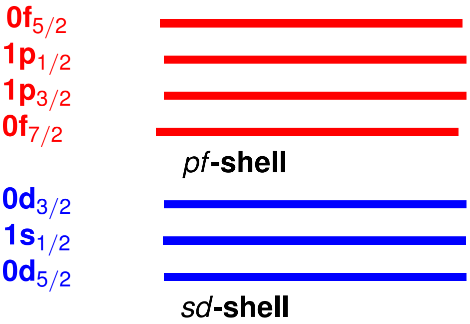
- This reflects the fact that the nuclear two body neutron-proton interaction is in average more attractive than the neutron-neutron and the proton proton ones, and it is related as well to the experimental evidence of the near equality of the neutron and proton radii for N$\neq$Z . Therefore we should use different values of $\hbar\omega$ and V_0's for protons and neutrons in the derivation, which complicates a lot the calculation because both effects go in opposite directions.

The limits of the IPM

- When a nucleus is such that it has both neutrons and protons outside closed shells, the IPM fails completely
- This is mainly due to the very strong residual interaction between neutrons and protons
- Dominated by its quadrupole quadrupole components
- Which may favor energetically that the nucleus acquire a permanent deformation and exhibit rotational spectra. This is a case of spontaneous symmetry breaking.
- In other cases collective states of vibrational type may also develop

The limits of the IPM; Doubly magic ^{40}Ca

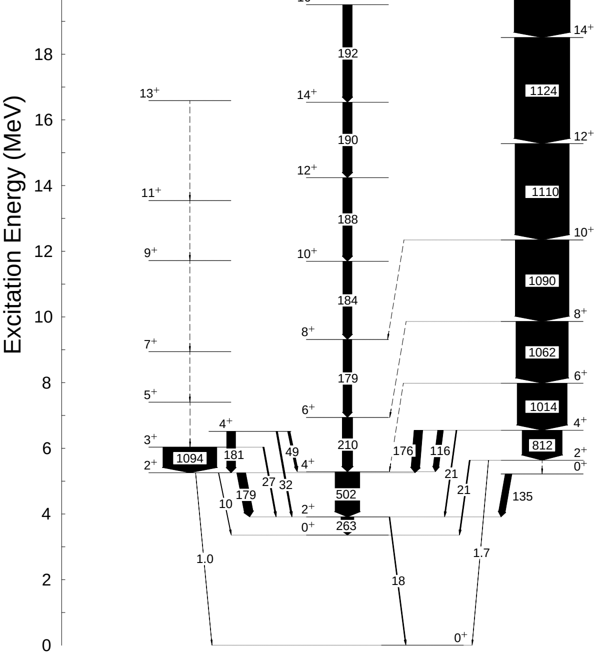
- **The single particle orbits around the Fermi level for ^{40}Ca are:**



- **The experimental gap between the *sd*-shell and the *pf*-shell is about 7 MeV**

The IPM predictions for the excitation spectrum of ^{40}Ca are:

- 0^+ ground state
- Quasi degenerated 1p-1h states of negative parity at about 7 MeV of excitation energy
- Quasi degenerated 2p-2h states of positive parity at about 14 MeV of excitation energy
- An so on ...
- But nature likes to play tricks!



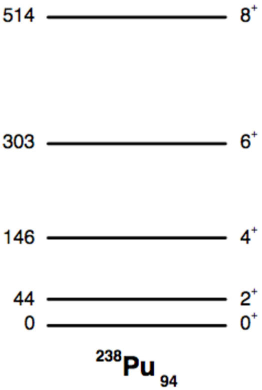
Add a 3^- at 3.74 MeV and a 5^- at 4.49 MeV

Spherical Mean Field *vs* Correlations

- It is evident that the IPM model fails completely in describing the low energy spectrum of ^{40}Ca, apart from its ground state
- The more so because the excited 0^+ at 3.74 MeV is the head of a triaxial rotational band, corresponding to a deformed β=0.3 intrinsic state. This band is of 4p-4h nature and should naively appear at 28 MeV
- In addition, the excited 0^+ at 5.21 MeV is the head of a super deformed band, β=0.6. This band is of 8p-8h nature and should naively appear at 56 MeV
- Shouldn't we speak of doubly magic STATES instead of doubly magic NUCLEI?
- All that brings us to the basic point; The dominance of correlations in the nuclear many body system

Collective behaviour in Atomic Nuclei

- As we depart from the magic numbers the correlations take over the IPM and the nucleus shows collective features
- When we have only neutrons or protons on top of a doubly magic closure, and several quasi degenerated orbits, the nucleus becomes superfluid.
- When we have neutrons and protons on top of a doubly magic closure, and several quasi degenerated orbits, most often the nucleus becomes deformed, and exhibits a level scheme in which the excitation energies go like J(J+1), as do the spectrum of a quantum rotor.
- In other, less numerous cases, nuclei show spectra which resemble to that of a quantum vibrator, with equi-spaced levels



Spectrum of a deformed heavy nucleus

How to deal with deformed nuclei

- To describe microscopically deformed nuclei is a very difficult task as we will see later
- Historically, the shortcut was to go semiclassical and submit that it made sense to move from the laboratory frame to an intrinsic frame in which the nucleons behave as independent particles in a (deformed) mean field. If the mean field is that of the IPM but with an anisotropic harmonic oscillator, we get the Nilsson model.
- The wave functions in this symmetry breaking mean field, must be thereafter complemented with the rotor eigenfunctions in order to restore rotational invariance, as we shall see soon.
- This is the basis of the unified model of Bohr and Mottelson that earned them the Nobel prize.

Deformed nuclei: The Nilsson model.

In modern terms, the Nilsson model is an approximation to the solution of the IPM plus a quadrupole-quadrupole interaction.

$$H = \sum_i h(\vec{r}_i) + \hbar\omega\kappa \sum_{i<j} Q_i \cdot Q_j$$

$$h(r) = -V_0 + t + \frac{1}{2} m\omega^2 r^2 - V_{so}\vec{l} \cdot \vec{s} - V_B l^2$$

Which amounts to linearizing the quadrupole quadrupole interaction, replacing one of the operators by the expectation value of the quadrupole moment (or by the deformation parameter).

Deformed nuclei; The Nilsson model

Thus, the resulting physical problem is that of the IPM subject to a quadrupole field, which, obviously breaks rotational symmetry.

$$H_{Nilsson} = \sum_i h(\vec{r}_i) - \frac{1}{3}\hbar\omega\delta Q_0(i)$$

In other words to diagonalizw the quadrupole operator $Q_0 = r^2\ Y_{20}$ in the basis of the IPM eigenstates. The resulting (Nilsson) levels are characterized by their magnetic projection on the symmetry axis m, also denoted K and the parity. Notice also that in order to keep the density of the nucleus constant $\hbar\omega$ must depend on δ as:

$$\omega(\delta)(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3)^{\frac{1}{6}} = \omega_0$$

ω_0 being the IPM value for this mass.

Therefore, for δ=0 we recover the eigenvalues and eigenvectors of the IPM $|pljm\rangle$, E(nlj). For $\delta \neq 0$ the eigenstates of the problem do not have a well defined value of j, only p, m, and the parity, are good quantum numbers. The eigenvalues depend on p and m, but also in some auxiliary labels related to the solutions in the limit in which the spin-orbit interaction is negligible, the so called asymptotic quantum numbers. Obviously, these states are not physical anymore.

Deformed nuclei; The Nilsson model

The formulae below make it possible to build the relevant matrices, which upon diagonalization produce the Nilsson diagrams, which are just the eigenvalues plotted as a function of the deformation parameter δ.

$$\langle pl|r^2|pl\rangle = p + 3/2 \quad : \quad \langle pl|r^2|pl+2\rangle = -[(p-l)(p+l+3)]^{1/2}$$

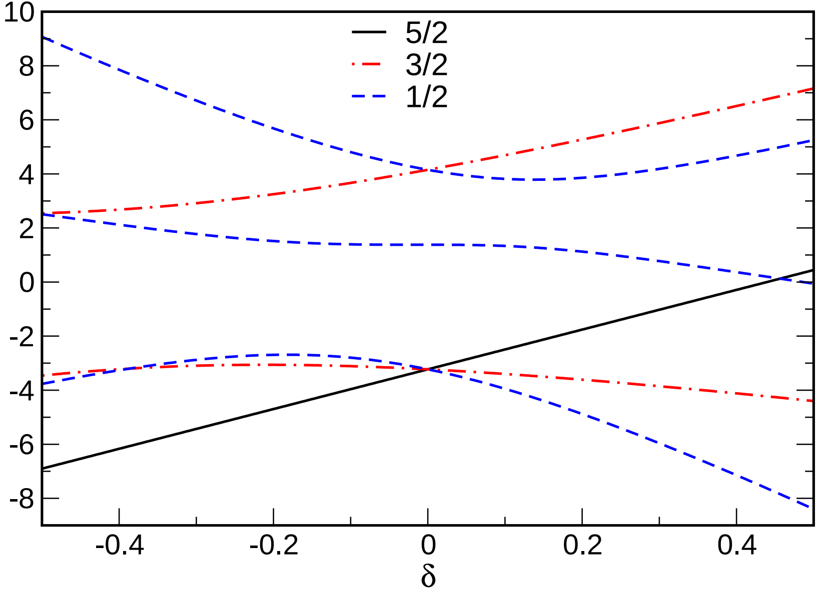
$$Q_0 = 2r^2 C_2 = 2r^2\sqrt{4\pi/(2l+1)}\,Y^{20} \quad : \quad \langle jm|C_2|jm\rangle = \frac{j(j+1)-3m^2}{2j(2j+2)}$$

$$\langle jm|C_2|j+2m\rangle = \frac{3}{2}\left\{\frac{[(j+2)^2-m^2][(j+1)^2-m^2]}{(2j+2)^2(2j+4)^2}\right\}^{1/2}$$

$$\langle jm|C_2|j+1m\rangle = -\frac{3m[(j+1)^2-m^2]^{1/2}}{j(2j+4)(2j+2)}$$

Diagramas de Nilsson para la capa p=2

Nilsson Diagrams for the p=2 HO shell



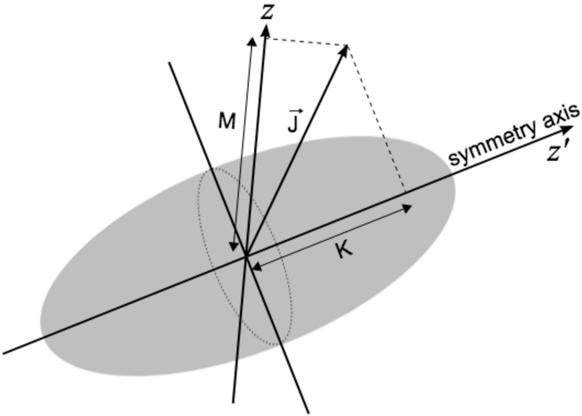
Intrinsic and Laboratory frame wave functions

The intrinsic wave functions provided by the Nilsson model are Slater determinants built putting the neutrons and the protons in the lowest Nilsson levels (each one has degeneracy two, $\pm$m). Therefore:

For even-even nuclei, K=0,

For odd nuclei K is equal to the m-value of the last half occupied orbit,

For odd-odd, there are different empirical rules, not always very reliable.



The intrinsic reference frame

Intrinsic and Laboratory frame wave functions

The laboratory frame wave functions are obtained rotating the intrinsic frame with the Wigner matrices, i.e. correspond to the solutions of the rigid rotor problem. This leads to the following formula for the excitation energies of an axial rotor:

$$E(J) = \sum_i \epsilon_i(Nilsson) + \frac{\hbar^2}{2\mathcal{I}}\, J(J+1) + a\,(-1)^{J+1/2}\delta(K, 1/2)$$

where a is the so called decoupling parameter and can be computed with the Nilsson wave functions.

In fact, the hypothesis underlying this formula is that the hamiltonian can be written in terms of the intrinsic coordinates $\vec{r'}$, which refer to the intrinsic frame of reference, and by the three Euler angles (θ, ψ, ϕ) that define the orientation of the reference frame.

$$H(\vec{r}_i) = H_{intrinsic}(\vec{r'}_i) + H_{coll.}(\theta, \psi, \phi) + H_{coupling}$$

And that $H_{coupling}$ can be neglected, which is only the case for well and rigidly deformed nuclei.

The optimal $H_{intrinsic}$ can be obtained from the effective nucleon-nucleon interaction using the Deformed Hartree-Fock approximation.

Examples of the use of the Nilsson diagrams.

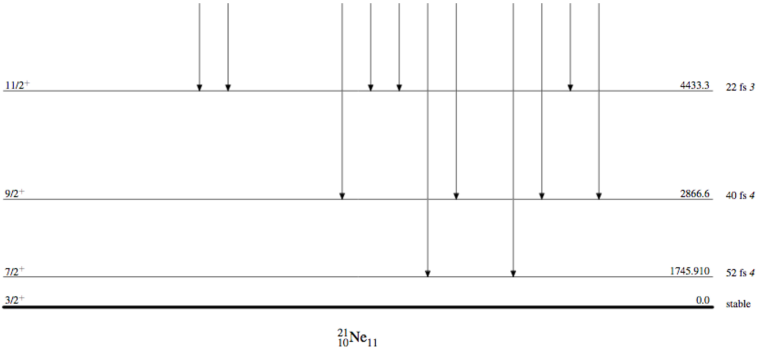
- The Nilsson states maintain a fourfold degeneracy, the states with $\pm$m have the same energy. And the neutrons and protons as well, given that we are not taking into account the Coulomb interaction.
- Obviously, all the even-even nuclei have M=K=0 in their ground states. Curiously, this leads to the prediction of J=0^+ for the ground states of all the deformed even-even nuclei, the same result that we had obtained for the superfluid ones due to the pairing interaction.

Deformed nuclei in the sd-shell

- ^{20}Ne, has two neutron and two protons more than doubly magic ^{16}O, and has K=0 and a rotational spectrum, with excited states 2^+, 4^+, 6^+, 8^+, that follow the J(J+1) formula. For a K=0 band, the states with odd angular momentum are not allowed by symmetry requirements.
- The lowest state of each J-value is called ”yrast” from the danish, the one which rotates more rapidly. In the case of a rotor, the yrast band is the rotational band. Notice that another fingerprint of nuclear deformation is that the in-band E2 electromagnetic transitions are very much enhanced. The Nilsson model can explain this behaviour trivially.

Deformed nuclei in the sd-shell

- Think now of ^{19}F. In the IPM plus pairing one would predict that its ground state is $5/2^+$. On the contrary, the Nilsson model prediction is K=1/2, and thus a rotational band J=$1/2^+$, $5/2^+$, $3/2^+$, etc. Notice the effect of the K=1/2 extra term in the energy formula above. These predictions agree with the experimental data.
- Another example is ^{21}Ne. Now K=3/2 and the ground state spin J=$3/2^+$. The yrast band should have the normal J(J+1) behaviour. And indeed it does.



The yrast band of ^{21}Ne. The 5/2$^+$ state which does not appear in the plot is at about 400 keV of excitation energy.

The ultimate goal of nuclear theory

- Is to link the theory of quarks and gluons (QCD) with the spectroscopy of nuclei across the full nuclear chart
- By the winding road of Effective Theories
- Dealing only with our elementary ”particles”; heavily dressed protons and nucleons
- And to understand the emergence of coherent (collective) modes in the nuclear many body system
- In addition to the complicated structure of the two body nucleon nucleon interaction, three body terms seem to be necessary as well, due to the composite nature of the nucleons. But there is hope that its influence be limited to the correct making up of the spherical mean field.

A Tower of Effective interactions; resolution scales

- From quarks and gluons (QCD) to neutrons and protons in vacuum
- From bare nucleons to regularized quasi-nucleons in the nucleus
- From the full Fock space to the Shell Model valence spaces
- In each one of these steps the interaction gets renormalized
- Remember the case of a diatomic non-polar molecule. When the electronic degrees of freedom are averaged out, the Coulomb potential is substituted by a Lennard-Jones one which only depends on the distance R between the two nuclei
- $$V(R) = \frac{A}{R^{12}} - \frac{b}{R^6}$$

Indeed,

- To have a good microscopic description of the nucleus is an important scientific goal by itself
- In addition, high quality nuclear wave functions are an essential ingredient in Nuclear Astrophysics
- And in other fundamental processes like neutrinoless $\beta\beta$ decays, direct dark matter searches, etc
- And for the description of nuclear reactions, nuclear fission etc.

The Nuclear A-body Problem

- **The challenge is to find $\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_A)$ such that**
- **HΨ=EΨ, with**
- **H=**$\displaystyle\sum_i^A T_i + \sum_{i,j}^A V_{2b}(\vec{r}_i, \vec{r}_j) + \sum_{i,j,k}^A V_{3b}(\vec{r}_i, \vec{r}_j, \vec{r}_k)$
- **The knowledge of the eigenvectors Ψ and the eigenvalues E make it possible to obtain electromagnetic moments, transition rates, weak decays, cross sections, spectroscopic factors, etc.**

Is an approximation to the exact solution of the nuclear A-body problem using effective interactions in restricted spaces

The single particle states (i,j, k, ...), which are the solutions of the IPM, provide as well a basis in the space of the occupation numbers (Fock space). The many body wave functions are Slater determinants:

$$\Phi = a^{\dagger}_{i_1}, a^{\dagger}_{i_2}, a^{\dagger}_{i_3}, \ldots a^{\dagger}_{i_A}|0\rangle$$

We can distribute the A particles in all the possible ways in the available single particle states. This provides a complete basis in the Fock space. The number of Slater determinants will be huge but not infinite because the theory is no longer valid beyond a certain cutt-off.

A formal solution to the A-body problem

Therefore, the ”exact” solution can be expressed as a linear combination of the basis states:

$$\Psi = \sum_{\alpha} \Phi_{\alpha}$$

and the solution of the many body Schödinger equation

$$H\Psi = \ E\Psi$$

is transformed in the diagonalization of the matrix:

$$\langle \Phi_{\alpha} | H | \Phi_{\beta} \rangle$$

whose eigenvalues and eigenvectors provide the ”physical” energies and wave functions

Beyond the IPM; The Interacting Shell Model (ISM)

- A Shell Model calculation amounts to diagonalizing the effective nuclear hamiltonian in the basis of all the Slater determinants that can be built distributing the valence particles in a set of orbits which is called valence space. The orbits that are always full form the core.
- If we could include all the orbits in the valence space (a full No Core calculation) we should get the ”exact” solution.

- The effective interactions are obtained from the bare nucleon-nucleon interaction by means of a regularization procedure aimed to soften the short range repulsion. In other words, using effective interactions we can treat the A-nucleon system in a basis of independent quasi-particles. As we reduce the valence space, the interaction has to be renormalized again in a perturbative way.