Holographic Entanglement Entropy
and
Emergent Spacetime

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Superstring theory as quantum gravity

⇒ ∃ Many evidences e.g. microscopic derivation of BH entropy.

[Strominger-Vafa 96,...]

However, we still do not have any good controls of quantum gravity in non-trivial spacetimes such as time-dependent and highly curved ones, which are very important in e.g. cosmology.

⇒ We need to understand the non-perturbative dynamics of string theory.
Many of string theorists have noticed that the holographic principle can be a very important clue for this purpose.


However, we do not understand the mechanism of holographic principle so deeply as to apply the holography to other different backgrounds such as the flat space or de-Sitter space.

⇒ We need to understand how the holography works.

(This is the main motivation of this talk.)
Claim: The *quantum entanglement* can be a key concept to understand the holography.

\[ \text{Boundary} \Downarrow \text{Quantum many-body system} \quad |\Psi\rangle \quad \uparrow \text{Boundary} \]

\[ \text{Holography} = (\text{Quantum}) \text{ gravity} \quad g_{\mu\nu} \quad \uparrow \]

\[ \text{Entanglement entropy (EE)} = \text{Area of minimal surface} + \text{quantum corrections} \]

\[ \text{[Ryu-TT 06]} \]
Advantages of EE

• EE is defined for *any quantum many-body systems*. \(\Rightarrow\) Universal
  (In cond-mat, EE = a quantum order parameter)

• In the presence of quantum corrections, the metric may not be a good description of the spacetime. But, the EE is robust.

• EE can *capture spacetime topologies*. For example,
Holographic Entanglement Entropy

(2-1) Definition of Entanglement Entropy

Divide a quantum system into two subsystems $A$ and $B$.

Define the reduced density matrix $\rho_A$ for $A$ by taking trace over the Hilbert space of $B$.

Now the entanglement entropy $S_A$ is defined by the von-Neumann entropy:

$$S_A = - \text{Tr}_A \rho_A \log \rho_A .$$
EE in QFTs includes UV divergences.

In a d+1 dim. QFT (d>1) with a UV relativistic fixed point, the leading term of EE at its ground state behaves like

$$S_A \sim \frac{\text{Area}(\partial A)}{\alpha^{d-1}} + (\text{subleading terms}),$$

where $\alpha$ is a UV cutoff (i.e. lattice spacing). [d=1: log div.]

Intuitively, this property is understood like:

Most strongly entangled
(2-3) Holographic Entanglement Entropy (HEE) [Ryu-TT 06]

\[ S_A = \frac{\text{Area}(\gamma_A)}{4G_N} \]

\( \gamma_A \) is the minimal area surface (codim.=2) such that

\[ \partial A = \partial \gamma_A \quad \text{and} \quad A \sim \gamma_A. \]

Note: In time-dependent b.g., we need to employ the covariant version [Hubeny-Rangamani-TT 07].
An Intuitive Interpretation of HEE

Here we employ the global coordinate of AdS space and take its time slice at $t=t_0$.

The information in $B$ is encoded here.
In spite of a heuristic argument [Fursaev, 06], there has been no complete proof. But, so many evidences and no counter examples.

A Partial List of Evidences

- **Area law** [Ryu-TT 06]
- **AdS3/CFT2** [Ryu-TT 06, (CFT: Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]
- **Strong subadditivity** [Headrick-TT 07]
- **Disconnected subsystems** [Headrick 10]
- **Quantum Quenches**
  - [Abajo-Arrastia-Aparcio-Lopez 10, (CFT: Calabrese-Cardy 05)]
- **Agreements on the coefficients of log term in 4d CFT** \(\sim a+c\)
  - [Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10]
- **A direct proof when A = round ball** [Casini-Huerta-Myers 11]
- **Cadney-Linden-Winter inequality (and Monogamy)**
  - [Hayden-Headrick-Maloney 11]
- **Agreements with exact results in gauge theories** [Jafferis-Pufu 12]
(2-4) General Behavior of HEE

\[ S_A = \frac{\pi^{d/2} R^d}{2 G_N^{(d+2)} \Gamma(d/2)} \left[ p_1 \left( \frac{l}{a} \right)^{d-1} + p_3 \left( \frac{l}{a} \right)^{d-3} \right. \]

\[ \cdots + \left\{ \begin{array}{l}
  p_{d-1} \left( \frac{l}{a} \right) + p_d \quad \text{(if } d = \text{ even)} \\
  p_{d-2} \left( \frac{l}{a} \right)^2 + q \log \left( \frac{l}{a} \right) \quad \text{(if } d = \text{ odd)}
\end{array} \right. \]

where \( p_1 = (d-1)^{-1}, p_3 = -(d-2)/(2(d-3)), \ldots \)

\[ \ldots q = (-1)^{(d-1)/2} (d-2)!!/(d-1)!! . \]

A universal quantity which characterizes odd dim. CFT. \( \Rightarrow \) F-theorem in 3 dim. CFTs

[Renormalized EE]

[Liu-Mezei 12, Myers-Singh 12]

\[ \text{Central charges} \]

[Renormalized EE]

[Liu-Mezei 12, Myers-Singh 12]

\[ \text{Area law div.} \]

[Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Huerta-Myers 11]

\[ \text{Proof: Casini-Huerta 12} \]

\[ \text{See also Klebanov-Nishioka-Pufu-Safdi 12, Closset-Dumitrescu-Festuccia-Komargodski-Seiberg 12} \]
(2-5) Construction of an entangled pair in AdS/BCFT

\[ \left\{ (t, x, z) \in AdS_3 \mid -t^2 + x^2 + \left( z - r_D \sinh \frac{\rho_*}{R} \right)^2 \geq \left( r_D \cosh \frac{\rho_*}{R} \right) \right\} \]

\[ \Leftrightarrow \text{A pair of CFTs on semi-infinite intervals} \]

\[ \text{Two BCFTs (z=0)} \]

A and B are causally disconnected!
In addition to the standard AdS boundary $M$, we include an extra boundary $Q$, such that $\partial Q = \partial M$.

$$I_E = -\frac{1}{16\pi G_N} \int_N \sqrt{g} \left( R - 2\Lambda - L_{\text{matter}} \right) - \frac{1}{8\pi G_N} \int_Q \sqrt{h} \left( K - L_{\text{matter}}^Q \right).$$

EOM at boundary leads to the Neumann b.c. on $Q$:

$$K_{ab} - Kh_{ab} = 8\pi G_N T^Q_{ab}. \tag{16}$$

Conformal inv. $\Rightarrow T^Q_{ab} = -Th_{ab}$. 

\[\text{AdS/BCFT Proposal \ [TT 11]}\]
In the BCFT side, these two BCFTs are entangled with each other. The entanglement entropy between them is calculated as

$$S_A = \frac{R}{G_N} \int_{r_{De}}^{z_{IR}} \frac{dz}{R \frac{z}{z}}$$

Note: The g-theorem prohibits the entanglement between regions which are causally connected. (No wormholes !)
(3-1) Motivation

1\textsuperscript{st} law of thermodynamics: \( T \cdot dS = dE \)

Temp. Information Energy

\( \Rightarrow \) Can we find an analogous relation in any quantum systems which are far from the equilibrium?

Something like: \( \text{Tent} \cdot dSA = dEA \) ??

Information in A Energy in A = EE
(3-2) Holographic Calculation

Consider an asymptotically AdS\(_{d+2}\) background

\( (= \text{an excited state in CFT}_{d+1}) : \)

\[
\begin{align*}
ds^2 &= \frac{R^2}{z^2} \left( -f(z) dt^2 + g(z) dz^2 + \sum_{i=1}^{d} dx_i^2 \right), \\
f(z) &= 1 - mz^{d+1} + ..., \\
g(z) &= 1 + mz^{d+1} + .... \\
\Rightarrow \varepsilon = T_{tt} = \frac{dR^{d+1} m}{16 \pi G_N}.
\end{align*}
\]

We do not care the details of IR.

energy density
Holographic Entanglement Entropy Analysis

If we assume a small subsystem A with the size $l$ such that

$$ml^{d+1} << 1,$$

then we can show

$$T_{ent} \cdot \Delta S_A = \Delta E_A,$$

where

$$\Delta S_A = S_A - S_A^{\text{PureAdS}}, \quad \Delta E_A = \int_A dx^d T_{tt}.$$

The `entanglement temperature’ is given by

$$T_{ent} = \frac{c}{l}.$$

The constant $c$ is universal in that it only depends on the shape of the subsystem A:

$$e.g. \quad c = \frac{d + 2}{2\pi} \quad \text{when} \ A = \text{a round sphere.}$$
Claim

Consider an exited state in a given CFT and the entanglement entropy for a subsystem A. If the size of A (= l) satisfies

$$\varepsilon \cdot l^{d+1} \ll \frac{R^d}{G_N} \approx O(N^2),$$

then the following ‘1st law’ like relation is satisfied:

$$T_{\text{ent}} \cdot \Delta S_A = \Delta E_A, \quad T_{\text{ent}} \equiv \frac{c}{l}. \quad \text{Info. Energy}$$

The AdS/CFT predicts a universal value of c for strongly coupled large N gauge theories.

Note: For more general critical points with dynamical exponent z, we have

$$T_{\text{ent}} = c \cdot l^{-z}.$$
(3-3) Comments on Negative Specific Heat

Consider the gravity dual of D3-brane shells [Kraus-Larsen-Trivedi 98].

We can excite a black 3-brane in flat spacetime which has a negative specific heat.

This branch breaks the strong subadditivity $\partial^2_l S^A \leq 0$ and does not contribute to HEE. However, its behavior $S^\text{Finite}_A \propto l^6$, agrees with the entropy of brane 3-brane if we assume $T_{BR} \sim 1/l$.

These suggest: the positivity of specific heat $\Leftrightarrow$ strong subadditivity. ($\sim$2nd law of thermodynamics)
Emergent Metric from Quantum Entanglement

(4-1) Basic Outline

In principle, we can obtain a metric from a CFT as follows:

\[ \text{a CFT state } \Rightarrow \text{Information (\sim EE) } = \text{Minimal Areas } \Rightarrow \text{metric} \]

\[ |\Psi\rangle \quad S_A \quad \text{Area}(\gamma_A) \quad g_{\mu\nu} \]

One candidate of such frameworks is so called the entanglement renormalization (MERA) [Vidal 05 (for a review see 0912.1651)] as pointed out by [Swingle 09]. [cf. Emergent gravity: Raamsdonk 09, Lee 09]
Recently, there have been remarkable progresses in numerical algorithms for quantum lattice models, based on so called tensor product states.

This leads to various nice variational ansatzs for the ground state wave functions in various quantum many-body systems.

⇒ An ansatz is good if it respects the quantum entanglement of the true ground state.
Ex. Matrix Product State (MPS)

\[ |\Psi\rangle = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_n} \text{Tr}[M(\sigma_1)M(\sigma_2)\cdots M(\sigma_n)] \left| \sigma_1, \sigma_2, \ldots, \sigma_n \right\rangle \]

\[ \alpha = \begin{array}{c} \alpha_1 \ \alpha_2 \ \alpha_3 \ \cdots \ \alpha_n \\ \sigma_1 \ \sigma_2 \ \sigma_n \end{array} \]

Spin chain

\[ M_{\alpha\beta}(\sigma) \]

\[ \alpha_i = 1, 2, \ldots, \chi, \]

\[ \sigma_i = \uparrow \text{ or } \downarrow. \]

[DMRG: White 92,..., Rommer-Ostlund 95,...]
MPS and TTN are not so suitable near quantum critical points (CFTs) because their entanglement entropies are too small:

\[ S_A \leq 2 \log \chi \quad (\ll \log L \sim S_A^{CFT}). \]

In general,

\[ S_A \sim N_{\text{int}} \cdot \log \chi, \]

\[ N_{\text{int}} = \min[\# \text{Intersections of } \gamma_A]. \]
**MERA** (Multiscale Entanglement Renormalization Ansatz): An efficient variational ansatz to find CFT ground states have been developed recently. [Vidal 05 (for a review see 0912.1651)].

To respect its large entanglement in a CFT, we add *(dis)entanglers.*

**Diagram:**

Unitary transf. between 2 spins
Calculations of EE in 1+1 dim. MERA

\( S_A \propto \text{Min}[\#\text{Bonds}] \propto \log L \)

\( \Rightarrow \) agrees with 2d CFTs.
A conjectured relation to AdS/CFT [Swingle 09]

Min[# Bonds]

\( u = 0 \)
\( u = -1 \)

\( u = -\infty (\equiv u_{IR}) \)

Equivalent?

\( \text{Min[Area]} \)

\( \gamma \)

\( A \)

\( \gamma_A \)

\( \text{AdS}_{d+2} \)

\( \text{CFT}_{d+1} \)

\[ \text{Metric} = ds^2 + \frac{e^{2u}}{\varepsilon^2} (-dt^2 + d\vec{x}^2) = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}, \]

where \( z = \varepsilon \cdot e^{-u} \).
Now, to make the connection to AdS/CFT clearer, we would like to consider the MERA for quantum field theories.

**Continuous MERA (cMERA)**

\[ \left| \Psi(u) \right\rangle = P \cdot \exp \left( -i \int_{u_{IR}}^{u} ds [K(s) + L] \right) \cdot \left| \Omega \right\rangle , \]

True ground state (highly entangled)

IR state (no entanglement)

⇒ Real space renormalization flow: length scale \( \sim \varepsilon \cdot e^{-u} \).

\( K(u) : \) disentangler, \( L: \) scale transformation

**Conjecture**

\[ d + 1 \text{ dim. cMERA} = \text{gravity on AdS}_{d+2} \]

\[ z = \varepsilon \cdot e^{-u} . \]
We focus on gravity duals of translational invariant static states, which are not conformal in general.

We conjecture that the metric in the extra direction is given by using the Bures metric (or Fisher information metric):

\[
g_{uu} du^2 = N \cdot \left( 1 - \left| \langle \Psi(u) | e^{iLdu} | \Psi(u + du) \rangle \right|^2 \right).\]

\[
N^{-1} \equiv \int dx^d \cdot \int_0^{\Lambda e^u} dk^d = \text{The total volume of phase space at energy scale } u.
\]
Bures Metric

The Bures distance between two states is defined by

\[ D(\psi_1, \psi_2) = 1 - \left| \langle \psi_1 | \psi_2 \rangle \right|^2. \]

More generally, for two mixed states \( \rho_1 \) and \( \rho_2 \),

\[ D(\rho_1, \rho_2) = 1 - \text{Tr} \sqrt{\sqrt{\rho_1 \rho_2} \sqrt{\rho_1}}. \]

When the state depends on the parameters \( \{\xi_i\} \),

the Bures metric (Fisher information metric) is defined as

\[ D[\psi(\xi), \psi(\xi + d\xi)] = g_{ij} d\xi^i d\xi^j. \]

\[ \Rightarrow \text{ Reparameterization invariant (in our case: coordinate } u) \]
The operation $e^{ildu}$ removes the coarse-graining procedure to extract the strength of unitary transformations (disentanglers).

⇒ Our metric = the density of disentanglers

= the metric $g_{uu}$ in the gravity dual

Understandable from the HEE:

$$S_A \sim \int_{u_{IR}}^{0} du \sqrt{g_{uu}} \cdot e^{(d-1)u}$$
(4-5) Emergent Metric in a (d+1) dim. Free Scalar Theory

Hamiltonian:  \[ H = \frac{1}{2} \int dk^d \left[ \pi(k)\pi(-k) + (k^2 + m^2)\phi(k)\phi(-k) \right]. \]

Ground state  \[ |\Psi\rangle : a_k |\Psi\rangle = 0. \]

Moreover, we introduce the `IR state'  \[ |\Omega\rangle \] which has no real space entanglement.

\[ a_x |\Omega\rangle = 0, \quad a_x = \sqrt{M} \phi(x) + \frac{i}{\sqrt{M}} \pi(x), \]

i.e.  \[ |\Omega\rangle = \prod_x |0\rangle_x \]

\[ \Rightarrow S_A = 0. \]
For a free scalar theory, the ground state corresponds to

\[ \hat{K}(u) = \frac{i}{2} \int dk^d \left[ \chi(u) \Gamma(ke^{-u}/M) a_k^+ a_{-k}^+ + (h.c.) \right], \]

where \( \Gamma(x) \) is a cut off function: \( \Gamma(x) = \theta(1 - |x|) \).

\[ \chi(s) = \frac{1}{2} \cdot \frac{e^{2u}}{e^{2u} + m^2/M^2}, \quad (\text{for } m = 0, \ \chi(u) = 1/2.) \]

For the excited states, \( \chi(s) \) becomes time-dependent.

One might be tempted to guess

\[ ds^2_{Gravity} = g_{uu} \, du^2 + \frac{e^{2u}}{\epsilon^2} \cdot d\bar{x}^2 - g_{tt} \, dt^2 \]

Density of bonds

\[ \sqrt{g_{uu}} \propto |\chi(u)| \ ? \]

Indeed, the previous proposal for guu lead to \( g_{uu} = \chi(u)^2 \).
Explicit metric

\[ \frac{d s_{\text{Gravity}}^2}{d t} = g_{uu} d u^2 + \frac{e^{2u}}{\varepsilon^2} \cdot d \vec{x}^2 - g_{tt} d t^2 \]

(i) Massless scalar (E=k)

\[ g_{uu} = \frac{1}{4}, \quad g_{tt} = g_{xx} \quad \Rightarrow \text{the pure } AdS \]

(ii) Lifshitz scalar (E=k^\nu)

\[ g_{uu} = \frac{\nu^2}{4} \quad \Rightarrow \text{the Lifshitz geometry} \]

(iii) Massive scalar

\[ g_{uu} = \frac{e^{4u}}{4(e^{2u} + m^2 / \Lambda^2)^2}, \quad g_{tt} = g_{xx}, \]

\[ \Rightarrow d s^2 = \frac{d z^2}{z^2} + \left( \frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2} \right)(d \vec{x}^2 - d t^2). \]

Capped off in the IR z<1/m
(4-6) Excited states after quantum quenches

\[
(A_k a_k + B_k a_{-k}^+) |\Psi\rangle = 0, \quad (|A_k|^2 - |B_k|^2 = 1).
\]

To realize these states, we need to extend the ansatz such that

\[
A_k = \frac{1}{2} \left( \left( \frac{k^2 + m0^2}{k^2} \right)^{1/4} + \left( \frac{k^2}{k^2 + m0^2} \right)^{1/4} \right) e^{ikt},
\]

\[
B_k = \frac{1}{2} \left( \left( \frac{k^2 + m0^2}{k^2} \right)^{1/4} - \left( \frac{k^2}{k^2 + m0^2} \right)^{1/4} \right) e^{-ikt}.
\]

To realize these states, we need to extend the ansatz such that

\[
\hat{K}(u) = \frac{i}{2} \int dk^d \Gamma(k e^{-u} / M) \left[ g(u)a_k^+ a_{-k}^+ + g^*(u)a_k a_{-k} \right] \]

\[
\Rightarrow \quad \text{SU(1,1) Bogoliubov transf. } \quad \text{Mk}(u)
\]

\[
(A_k(u), B_k(u)) = (\alpha_k, \beta_k) \cdot M_k(u).
\]
For a given UV state $|\Psi\rangle$ or equally $M_k(0)$, the intermediate state $|\Psi(u)\rangle$ or $M_k(u)$ is determined up to an ambiguity.

This stems from the phase factor ambiguity of wave function:

$$\left( A_k a_k + B_k a_k^+ \right) |\Psi\rangle = 0 \quad \Rightarrow \quad e^{i\theta_k(t)} \cdot \left( A_k a_k + B_k a_k^+ \right) |\Psi\rangle = 0.$$  

**Our conjecture:**

the phase ambiguity $\theta_k(t)$

$\Leftrightarrow$ the choice of the time slice

$$F(t,u) = \text{const.}$$
Time dependent metric from the Quantum Quench

We can also (analytically) confirm the linear growth: $SA \propto t$. This is consistent with the known CFT (2d).

[Calabrese-Cardy 05]

Towards Holographic Dual of Flat Space

If we consider the (almost) flat metric

\[ ds^2 = e^{2u} du^2 + e^{2u} dx^2 \Rightarrow g_{uu} = e^{2u}, \]

the corresponding dispersion relation reads

\[ \chi(u) = \frac{1}{2} \cdot \left( \frac{k \partial_k E_k}{E_k} \right) \bigg|_{k = \Lambda e^u} = e^u \quad \Rightarrow \quad E_k = e^k. \]

This leads to the highly non-local Hamiltonian:

\[ H = \int dx^d \phi(x) e^{\sqrt{-\partial^2}} \phi(x). \]

[cf. Li-TT 10]
Conclusions

- The entanglement entropy (EE) and its holographic counterpart (HEE) offers us a powerful method to extract universal properties in quantum systems and gravity theories.

- We pointed out a relation of entanglement entropy and energy, which looks like a 1-st law of thermodynamics. The `entanglement temperature’ turns out to be universal.

- We gave an interpretation of AdS/CFT in terms of cMERA and found supporting evidences. Especially, we proposed a metric in the extra dimension purely in terms of QFT data.
Future problems

- How to calculate gtt? Boosting the subsystem? Finite temp.?
- The effect of Large N limit in cMERA?
  (largen N limit ⇔ locality ⇒ saturation of entropy bound?)
  [see also Swingle 12]
- Time slices and diff. inv. in cMERA?
- Free field theories ⇒ Higher spin gauge theory?
- The origin of Gravitational force from CFT side? Entropic?
- Condensed matter applications [See also Ugajin’s poster]
  :
  :
  :
Quantum Many-body Systems
(Cond-mat, QFTs, CFTs, .....)

AdS/CFT
(Holography)

Quantum gravity
String theory

HEE, BH info.

EE, ES,
Tensor networks,
etc.

Quantum Information
Theory