Quantum Primes

Workshop "Latorre Fest"

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Felices 3 x 4 x 5 ‘s

Where this story begun

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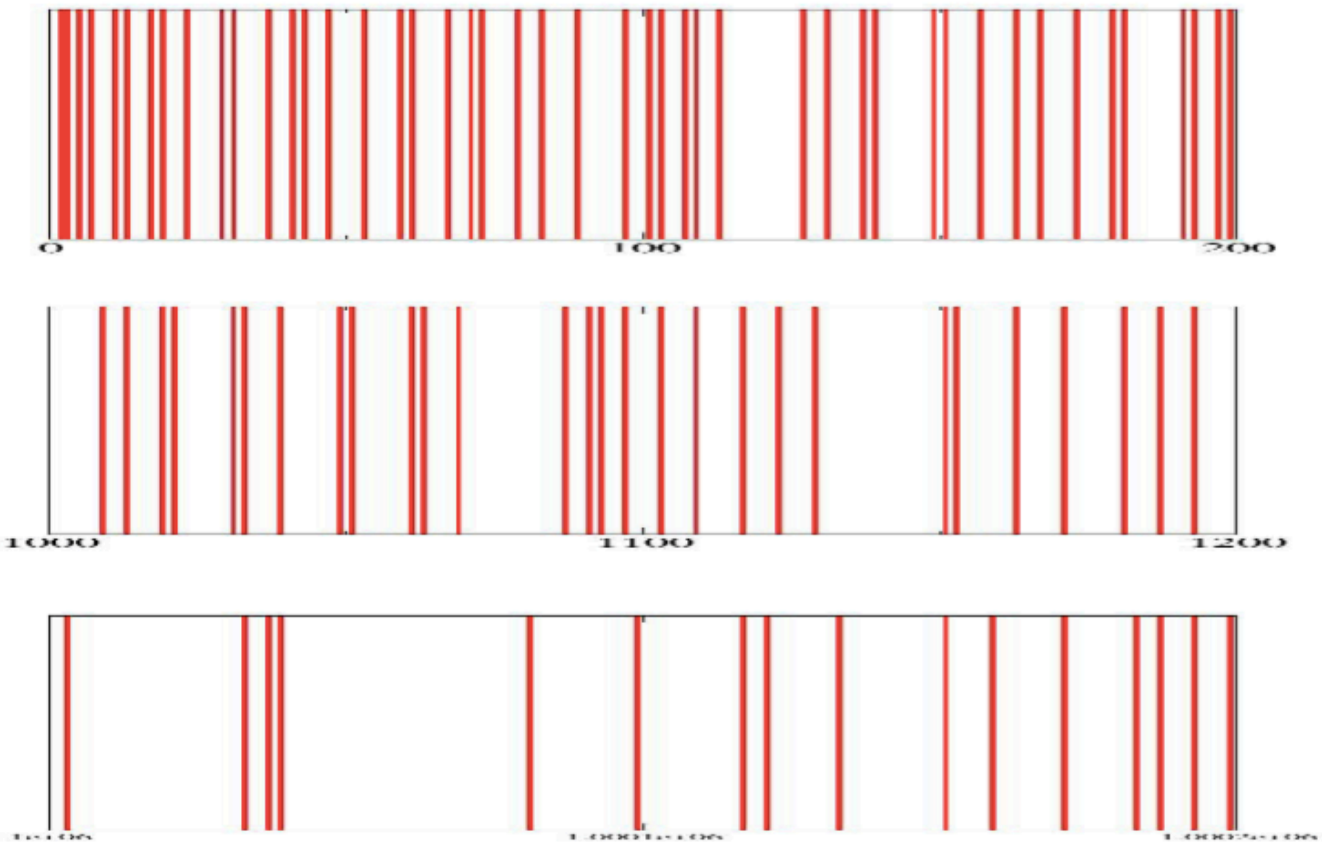
Outline

- Part I: Prime Numbers and Riemann Zeros
- Part II: Random Matrix Theory and Quantum Chaos
- Part III: The Prime State

Part I

Order and chaos in the primes



Order, Order !! Gauss Law

The number of primes less than N is approximately

$$\pi(N) \sim \frac{N}{\ln N}, \qquad N \to \infty$$

N	$\pi(N)$	$\frac{N}{\ln N}$
1000	168	145
10000	1229	1086
100000	9592	8686
1000000	78498	72382
10000000	664579	620420
100000000	5761455	5428681

Gauss Law = Prime Number Theorem

(Hadamard, de la Vallee-Poussin, 1896)

$$lim_{N\to\infty} \frac{\pi(N)}{Li(N)} = 1$$

Logarithmic integral function

$$Li(x) = \int_2^x \frac{dt}{\ln t} = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + \cdots$$

Density of primes $\frac{d\pi(x)}{dx} \sim \frac{1}{\ln x}$

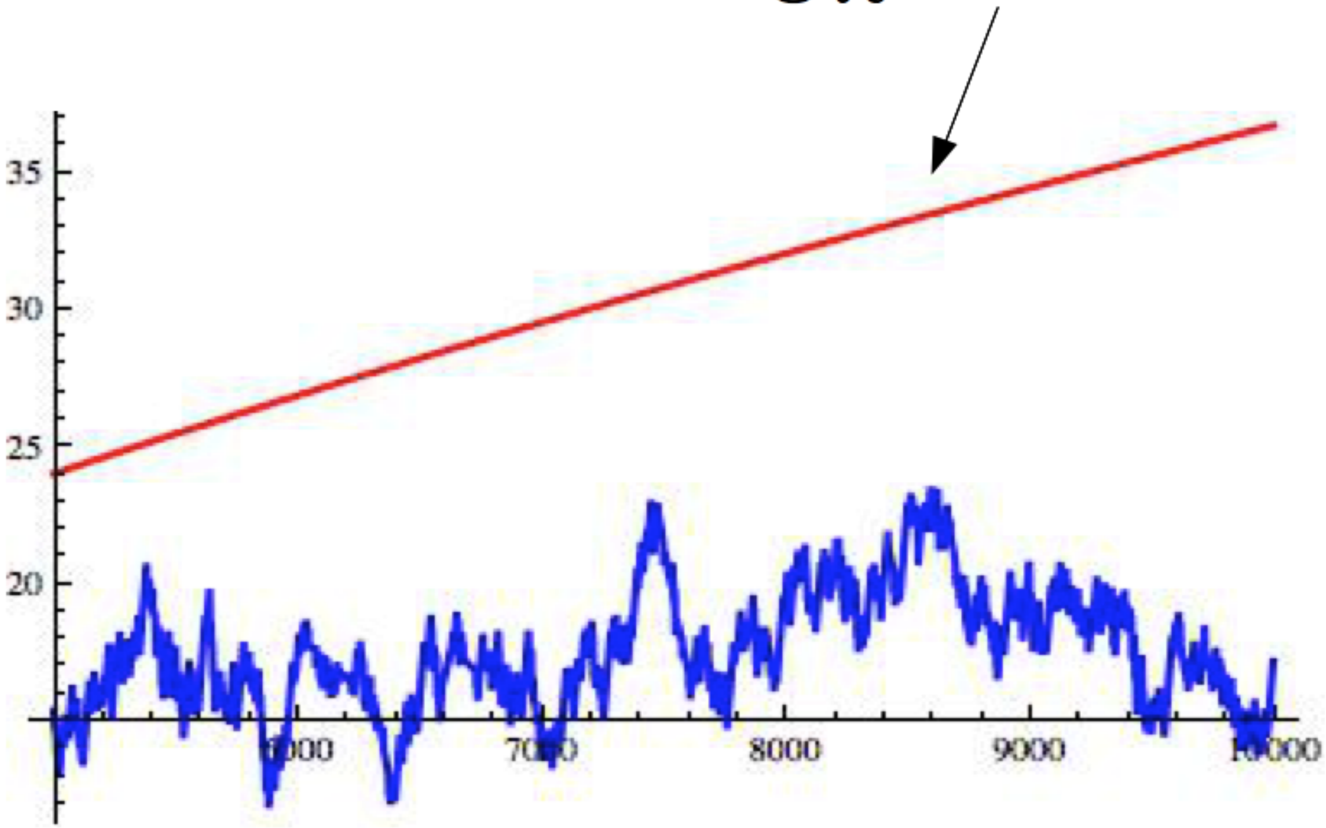
x	$\pi(x)$	$\pi(x) - x / \ln x$	$li(x) - \pi(x)$
10	4	−0.3	2.2
10^2	25	3.3	5.1
10^3	168	23	10
10^4	1,229	143	17
10^5	9,592	906	38
10^6	78,498	6,116	130
10^7	664,579	44,158	339
10^8	5,761,455	332,774	754
10^9	50,847,534	2,592,592	1,701
10^{10}	455,052,511	20,758,029	3,104
10^{11}	4,118,054,813	169,923,159	11,588
10^{12}	37,607,912,018	1,416,705,193	38,263
10^{13}	346,065,536,839	11,992,858,452	108,971
10^{14}	3,204,941,750,802	102,838,308,636	314,890
10^{15}	29,844,570,422,669	891,604,962,452	1,052,619
10^{16}	279,238,341,033,925	7,804,289,844,393	3,214,632
10^{17}	2,623,557,157,654,233	68,883,734,693,281	7,956,589
10^{18}	24,739,954,287,740,860	612,483,070,893,536	21,949,555
10^{19}	234,057,667,276,344,607	5,481,624,169,369,960	99,877,775
10^{20}	2,220,819,602,560,918,840	49,347,193,044,659,701	222,744,644
10^{21}	21,127,269,486,018,731,928	446,579,871,578,168,707	597,394,254
10^{22}	201,467,286,689,315,906,290	4,060,704,006,019,620,994	1,932,355,208
10^{23}	1,925,320,391,606,803,968,923	37,083,513,766,578,631,309	7,250,186,216
10^{24}	18,435,599,767,349,200,867,866	339,996,354,713,708,049,069	17,146,907,278
10^{25}	176,846,309,399,143,769,411,680	3,128,516,637,843,038,351,228	55,160,980,939
10^{26}	1,699,246,750,872,437,141,327,603	28,883,358,936,853,188,823,261	155,891,678,121
10^{27}	16,352,460,426,841,680,446,427,399	267,479,615,610,131,274,163,365	508,666,658,006

Wikipedia

Chaos in the primes

The fluctuations of $\pi(x)$ around $Li(x)$ are expect to be bounded by

$$|Li(x)-\pi(x)|<\frac{1}{8\pi}\sqrt{x}\log x$$



This statement is equivalent to the Riemann hypothesis (RH)

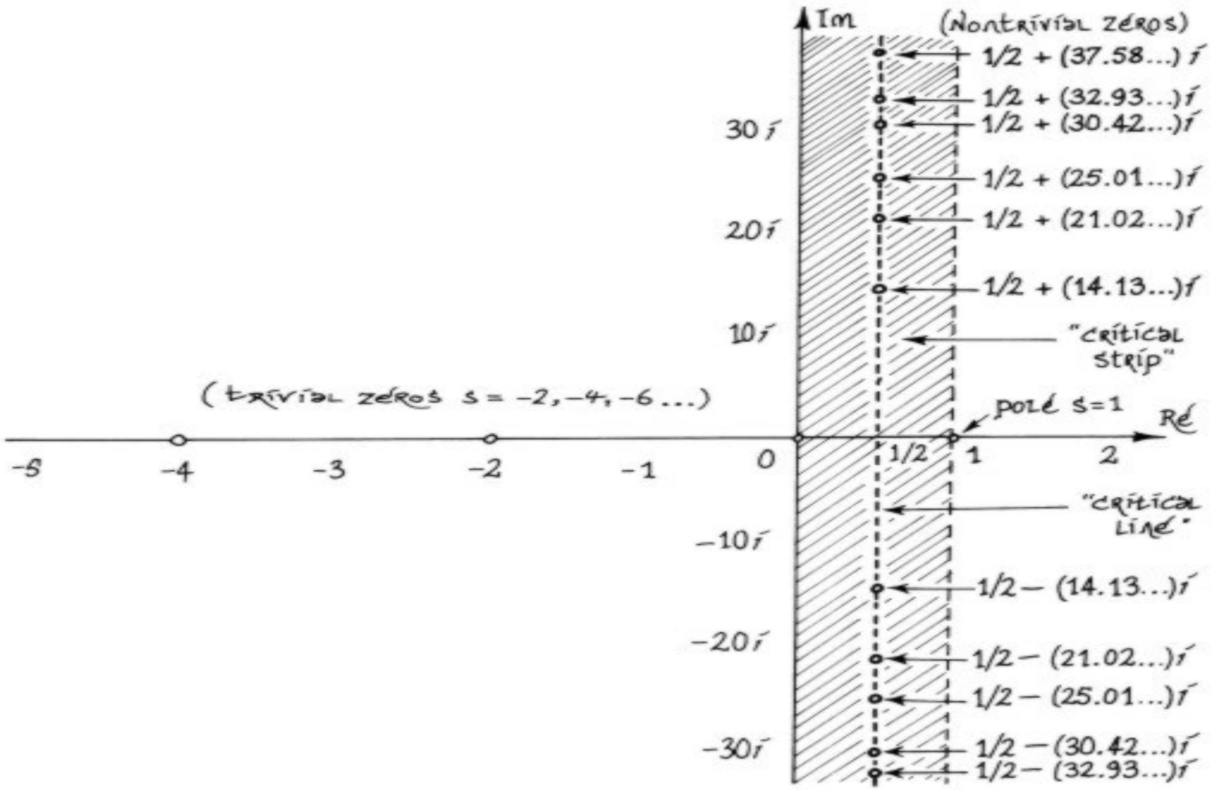
Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=2,3,..} \frac{1}{1-p^{-s}}$$

Euler formula

$Re\, s > 1$

RH: Non trivial zeros of $\zeta(s)$ have real part equal to 1/2



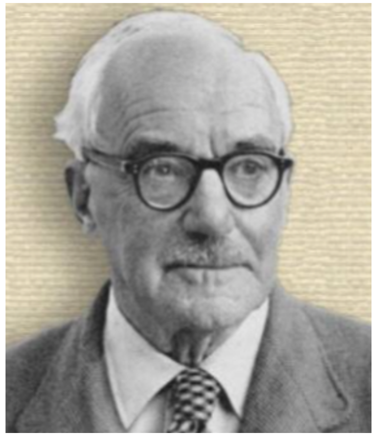
Twin primes

(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), ...

$\pi_2(x) =$ number of primes $p \leq x$ such that $p + 2$ is also prime

Hardy - Littlewood conjecture (1923)

$$\pi_2(x) \sim 2\, C_2 \,\frac{x}{(\ln x)^2}$$



Twin prime constant $$C_2 = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) = 0.6601618\ldots$$

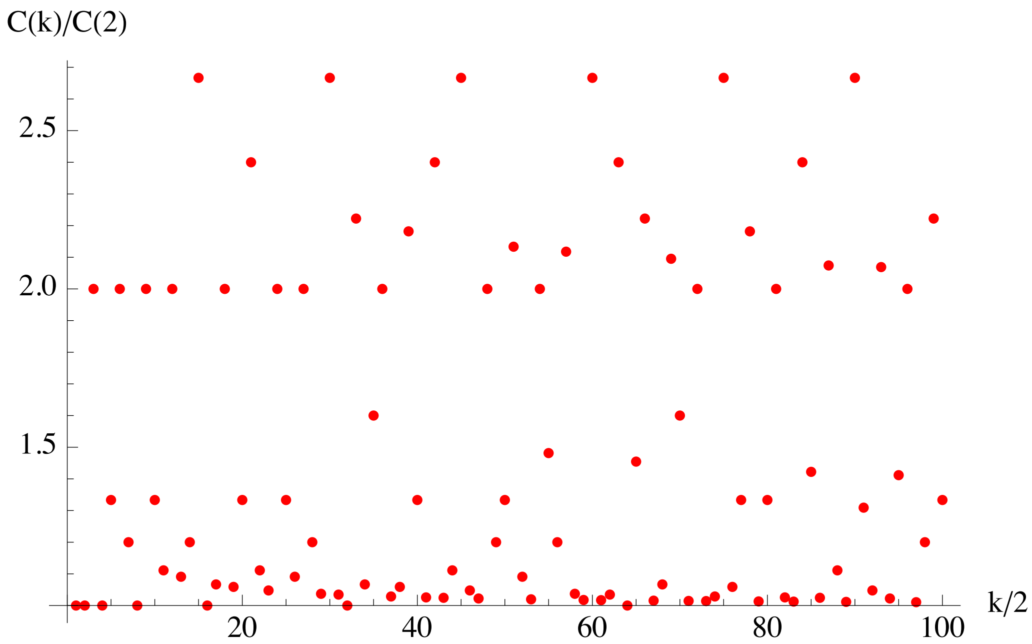
Hardy-Littlewood conjecture

$\pi_2(k,x) =$ number of primes $p \leq x$ such that $p+k$ is also prime

$$\pi_2(k,x) \sim C(k) \frac{x}{(\ln x)^2}$$

$$C(k) = \begin{cases} 2\,C_2 \prod_{p|k} \frac{p-1}{p-2} & if\ k : even \\ 0 \ \ if\ k : odd \end{cases}$$

HL conjecture : pairwise correlations between the primes



Averaged form of the Hardy-Littlewood conjecture

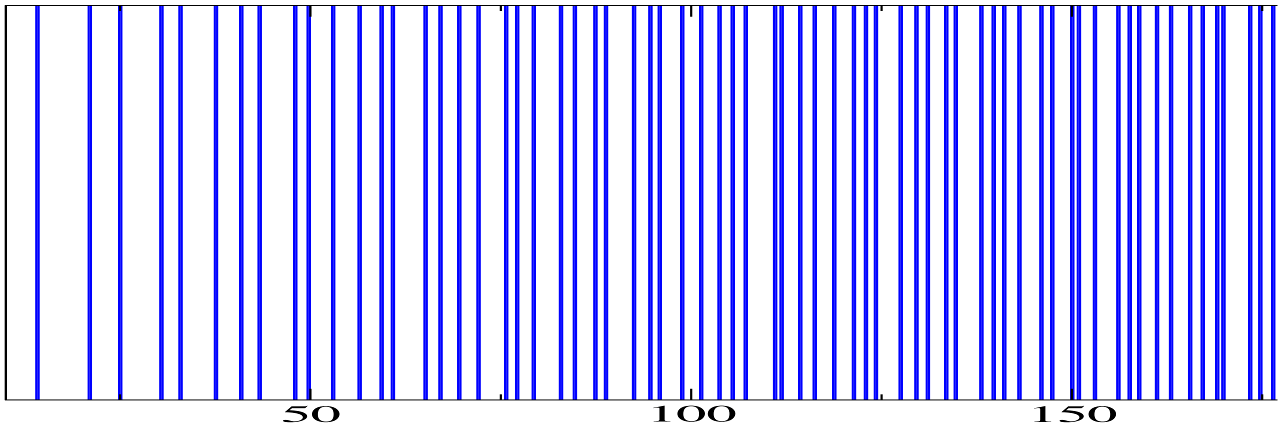
$$\sum_{k=1}^{K} C(k) \sim K - \frac{1}{2}\log K, \qquad K >> 1$$

$$\langle C(k) \rangle \sim 1 - \frac{1}{2k}$$ (Keating)

If primes were pairwise uncorrelated $C(k) \sim 1$

Part II

The Riemann zeros as spectrum $\zeta\left(\frac{1}{2}+i\,E_n\right)=0$



$$N_R(E)=\langle N(E)\rangle + N_{fl}(E)$$

Average $$\langle N(E)\rangle \approx \frac{E}{2\pi}\left(\log\frac{E}{2\pi}-1\right)+\frac{7}{8}+O(E^{-1})$$

Fluctuation $$N_{fl}(E)=\frac{1}{\pi}Arg\,\varsigma(\frac{1}{2}+i\,E)=O(\log E)$$

Pairwise correlations

Density of “zeros” $$d(E) = \sum_n \delta(E - E_n)$$

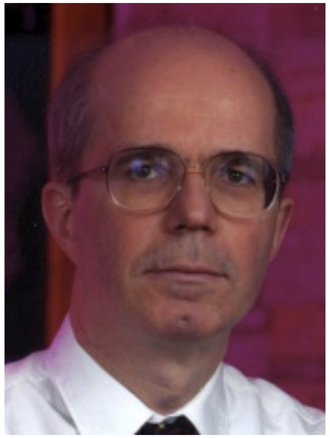
Average density $$\bar{d}(E) = \frac{d\langle N(E)\rangle}{dE} = \frac{1}{2\pi}\ln\frac{E}{2\pi}$$

Pair correlation $$R_2(\varepsilon) = \langle d(E - \varepsilon/2)d(E + \varepsilon/2)\rangle$$

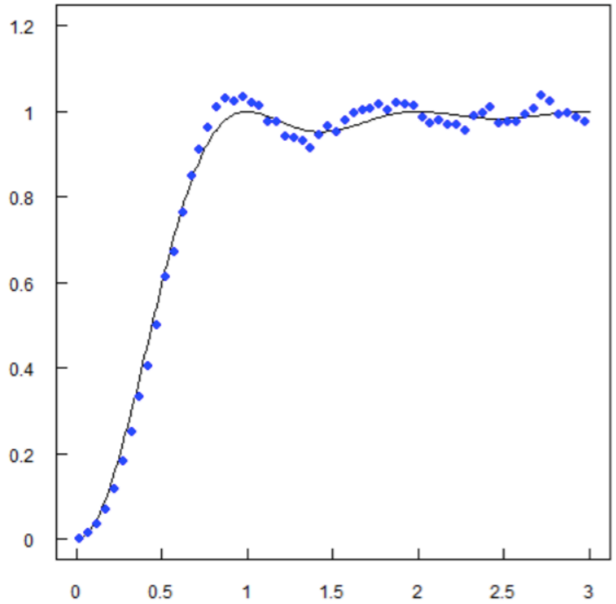
Montgomery conjecture (1973)

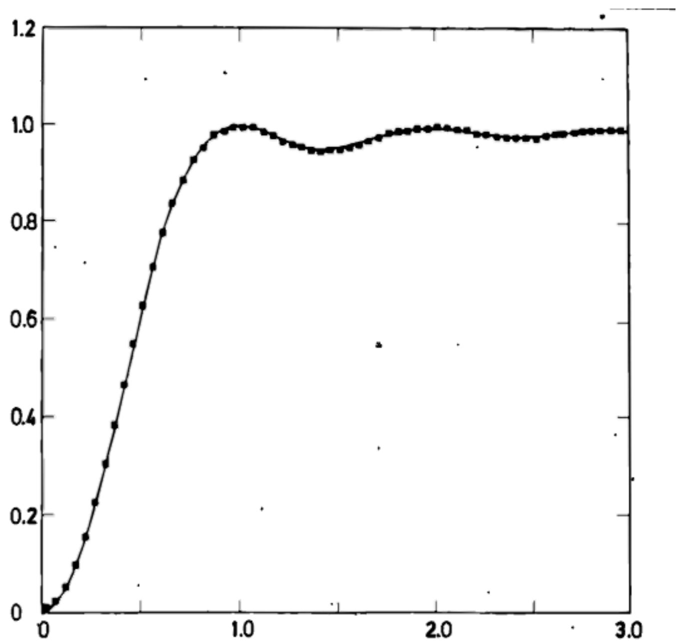
In the limit $E \to \infty$

$$R_2(\varepsilon) \to R_{GUE}(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2 \qquad x = \frac{\varepsilon}{\bar{d}(E)}$$



Odlyzo numerics (1989)





Prime Numbers | **Riemann zeros**

$\pi_2(k, x)$ | $R_2(\varepsilon)$

Averaged HL conjecture

$$\langle C(k) \rangle \sim 1 - \frac{1}{2k}$$

Keating (1993)

Montgomery conjecture

$$R_2(\varepsilon) \to R_{GUE}(x)$$

Full HL conjecture

$$C(k)$$

Bogomolny, Keating (1996)

Berry, Bogomolny, Keating,…

$$R_2(\varepsilon) = R_{GUE}(x) + R_c(x)$$

TWIN PRIME CORRELATIONS FROM THE PAIR CORRELATION OF RIEMANN ZEROS

J. P. KEATING AND D. J. SMITH

arXiv:1903.07057v1

Full HL conjecture ⇄ $R_2(\varepsilon) = R_{GUE}(x) + R_c(x)$

Heuristic equivalence between the two conjectures

Berry's analogy in Quantum Chaos (80's)

Prime Numbers	Riemann zeros
Classical chaotic Hamiltonian	Quantum Hamiltonian
p : primitive periodic orbit log p: period	E_n Eigenenergies
Breaks time reversal	GUE statistics
Trace formula	Gutzwiller formula

Spectral approach to prove the Riemann Hypothesis

Part III

Quantum Computation and prime numbers
(JI Latorre, GS, 2013)

Classical computer

n bits $x = x_0 2^0 + x_1 2^1 + \ldots x_{n-1} 2^{n-1}, \quad x_i = 0,1, \quad x = 0,1,\ldots 2^n - 1$

Quantum computer

n qubits $|x\rangle = |x_{n-1},\ldots,x_0\rangle = |x_{n-1}\rangle \otimes \ldots \otimes |x_0\rangle$

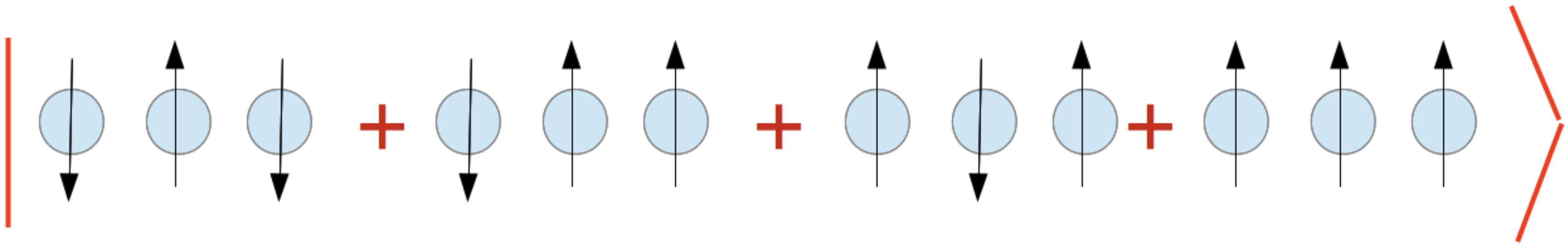
The Prime State

$$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{p<2^n \in Primes} |p\rangle$$

$\pi(2^n)$ is the prime counting function

Ex. n=3

$$|P(3)\rangle = \frac{1}{\sqrt{4}}\left(|2\rangle + |3\rangle + |5\rangle + |7\rangle\right)$$



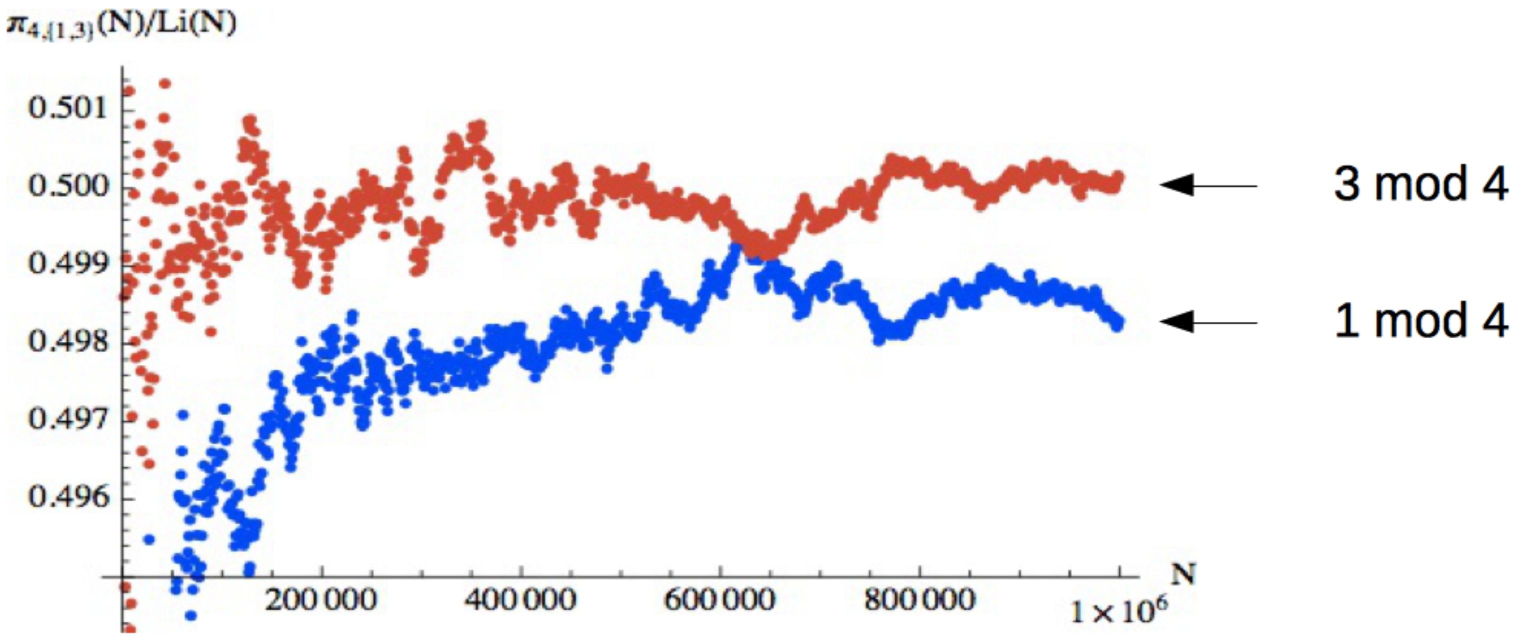
Efficient construction of the prime state:

Grover algorithm with an oracle that implements classical primality tests.

Quantum counting algorithm provides an estimate of $\pi(2^n)$ that can be used to falsify the RH but not to prove it.

Prime numbers in arithmetics progressions

The primes are equally distributed in the progressions 1 mod 4 and 3 mod 4

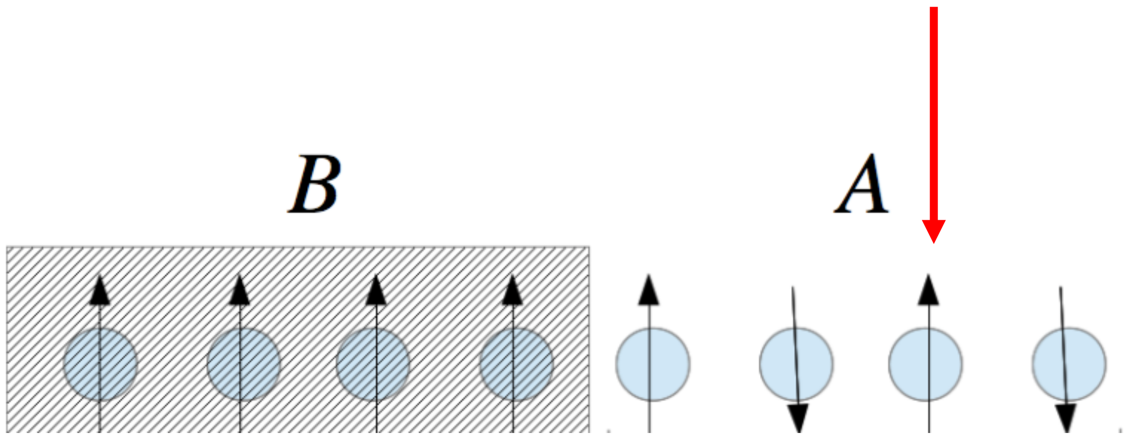


Prime number theorem for progressions $\pi_{4,b}(x) \to \frac{x}{2\ln x}, \qquad b = 1,3$

Chebyshev bias as magnetization

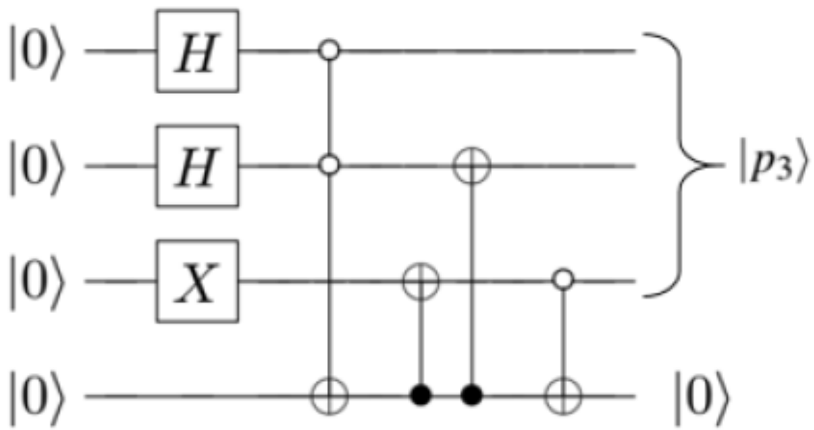
Chebyshev bias: $\Delta(x) = \pi_{4,3}(x) - \pi_{4,1}(x)$

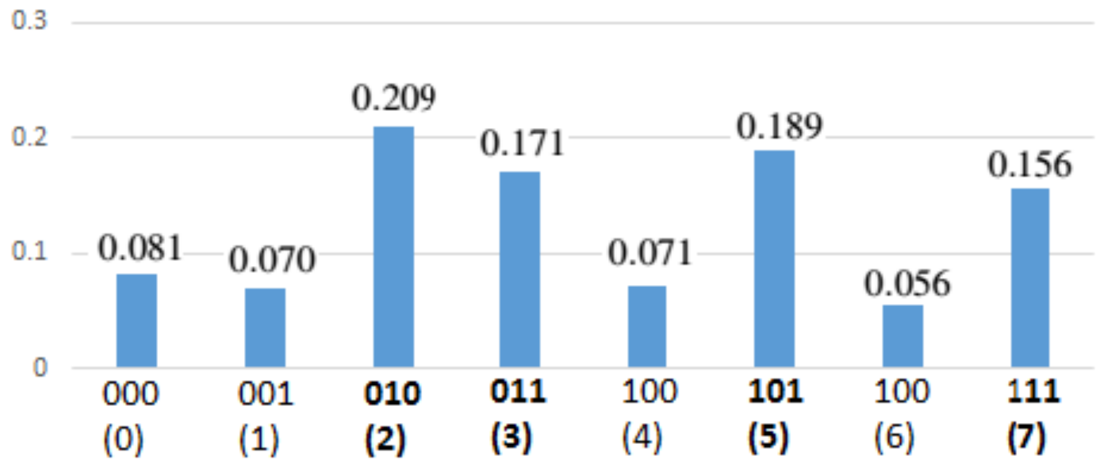
Magnetization of the qubit 1 $\langle \sigma_1^z \rangle = -\frac{\Delta(N)+1}{\pi(N)}, \quad N = 2^n$



IBM quantum computer and the Prime state

(Diego García-Martín, GS, 2018)



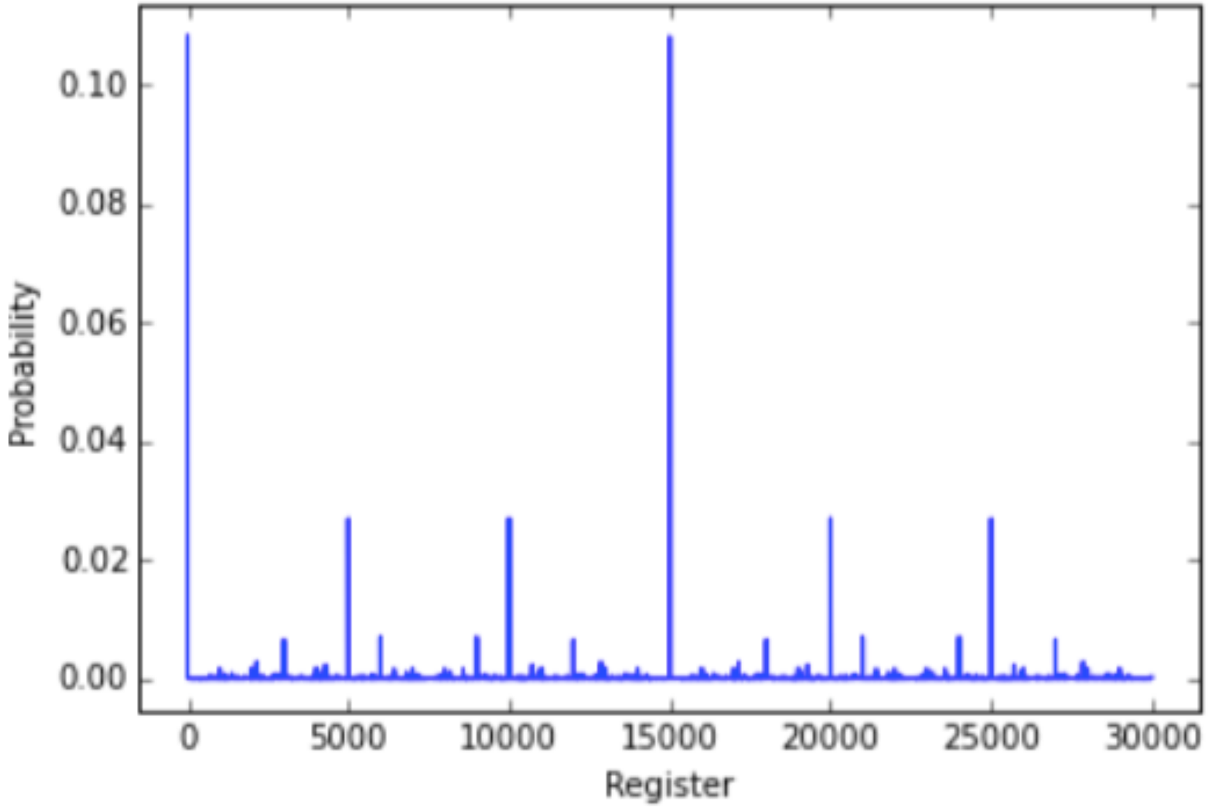


$$\langle \sigma_z^1 \rangle_{exp} = -0.183 \pm 0.029 \qquad \langle \sigma_z^1 \rangle_{th} = -0.500$$

Quantum Fourier transform of the Prime state

Eduard Ribas Fernández , J.I. Latorre, TFG (18)

$$U_{QFT}|\mathbb{P}_N\rangle = \frac{1}{\sqrt{N\pi(N)}}\sum_{k=0}^{N-1}\left(\sum_{p\in\mathbb{P}_N} e^{\frac{2\pi i}{N}kp}\right)|k\rangle$$



$$P(N) \propto \pi(N)^2$$

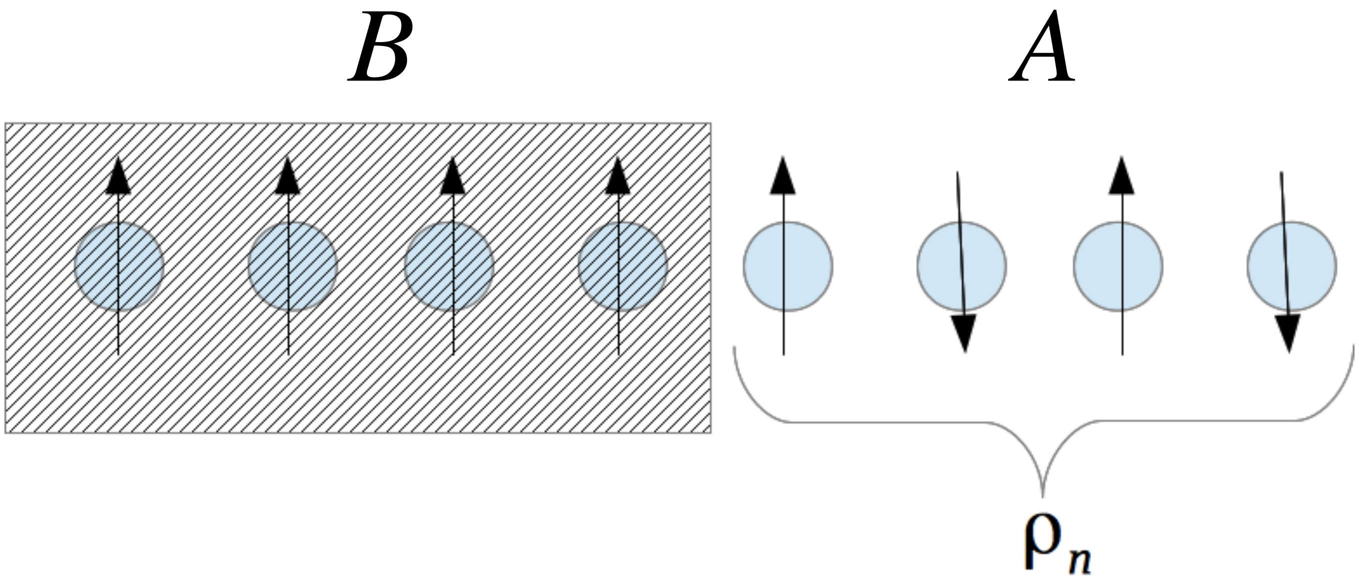
$$P\left(\frac{N}{2}\right) \propto \pi(N)^2 - 4\,\pi(N) + 4$$

$$P\left(\frac{N}{4}\right) \propto \Delta(N)^2 + 1$$

.

Entanglement in the Prime state

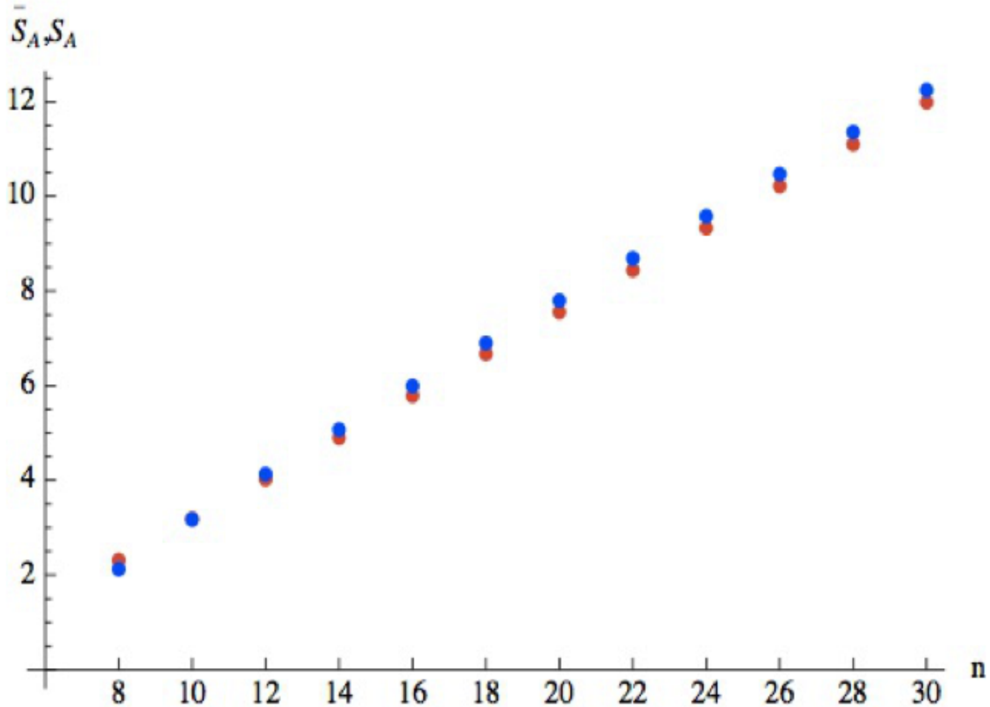
Density matrix of the Prime state



$$\rho_n = -Tr_B \left| P(n) \right\rangle \left\langle P(n) \right|$$

$$S_n = -Tr_A (\rho_n \log \rho_n)$$

Entanglement entropy of the Prime state



ρ exact
approx

Volumen law entropy

$$S_A \approx c_\pi \frac{n}{2} + cte$$

$$c_\pi \approx 0.886(1)$$

Scaling of entanglement entropy

$$S \sim n - const$$

Random states

$$S \sim .8858\, n + const$$

Prime state

$$S \sim n^{\frac{d-1}{d}} + const$$

Area law in d-dimensions

$$S \sim \frac{c}{3} \log n + const$$

Critical scaling in d=1
at quantum phase transitions

$$S \sim \log(\xi) = const$$

Finitely correlated states
away from criticality

Entropies for Random Ensembles   Lubkin (78), Don Page (93)

$$|\psi\rangle = \sum_C \psi(C)\,|C\rangle$$

$\psi(C)$ (real or complex) uniformly distributed in the Hilbert space

$$\langle S_{vN}\rangle = \langle S_n\rangle = \ln(|\mathcal{H}_A|) \sim V_A$$

Entropies for Random Positive Ensembles   Grover, Fisher (14)

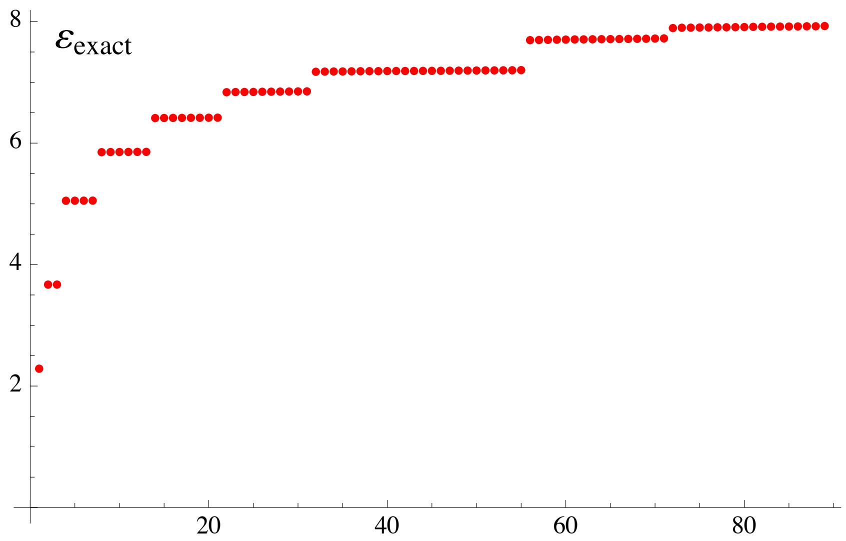
$$|\,|\psi|\,\rangle = \sum_C |\psi(C)|\,|C\rangle$$

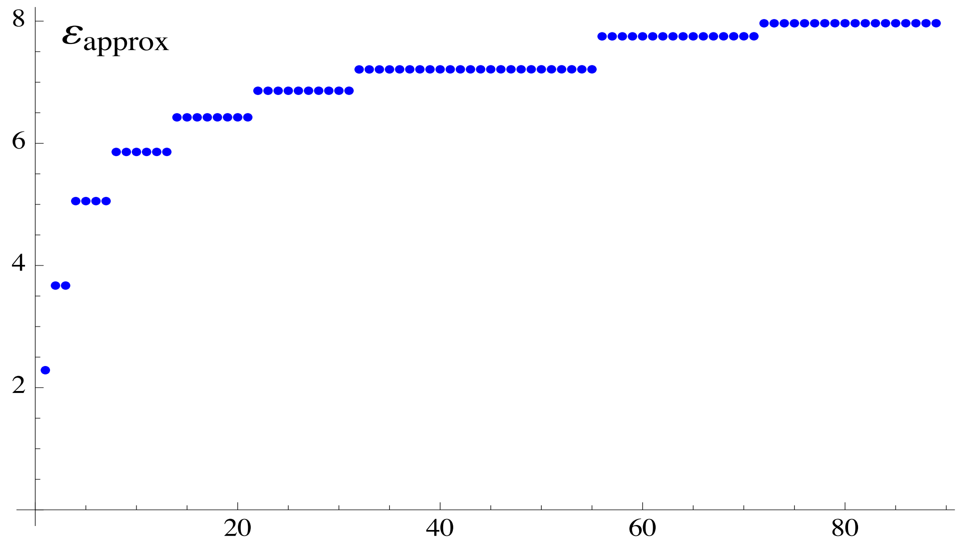
$$S_n(\langle\rho_A\rangle) = \begin{cases} \frac{n}{n-1}\ln\left(\frac{\pi}{2}\right) \text{ if } n > 1, \\ \left(1-\frac{2}{\pi}\right)\ell_A^d \text{ if } n = 1, \\ \ell_A^d \text{ if } n \leq 1, \end{cases}$$

Entanglement Spectrum

λ_i eigenvalues of ρ_A

$$\varepsilon_i = -\log \lambda_i$$





The prime state and Hardy-Littlewood conjecture

Approximate reduced density matrix

$$\bar{\rho}_A = \frac{1}{d}\,(I + \ell_m\, C_m)$$

C_m Toeplitz matrix using the Hardy-Littlewood constants

$$C_m(i,j) = C(2(i-j)), \qquad i,j = 1,\cdots,d = 2^{m-1}$$

Quantum Entanglement ⟺ Pairwise Correlations

The entanglement spectrum follows from the spectrum of C_m

square free number $n = p_1 \, p_2 \cdots p_r \, , \qquad p_i \neq p_j$

Moebius function $\mu(n) = (-1)^r \quad or \quad 0 \;\; else$

Positive Eigenvalues: $\gamma_j = 2^m \left(\frac{\mu(j)}{\phi(j)} \right)^2 , \; j = 1,3,5, \dots$

Degeneracy: $\phi(j)$ Euler totient function

n	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
$\phi(n)$	1	2	4	6	6	10	12	8	16	18	12	22	20	18	28	30	20	24	36	24
$\widehat{\gamma}_n$	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{36}$	0	$\frac{1}{100}$	$\frac{1}{144}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{324}$	$\frac{1}{144}$	$\frac{1}{484}$	0	0	$\frac{1}{784}$	$\frac{1}{900}$	$\frac{1}{400}$	$\frac{1}{576}$	$\frac{1}{1296}$	$\frac{1}{576}$

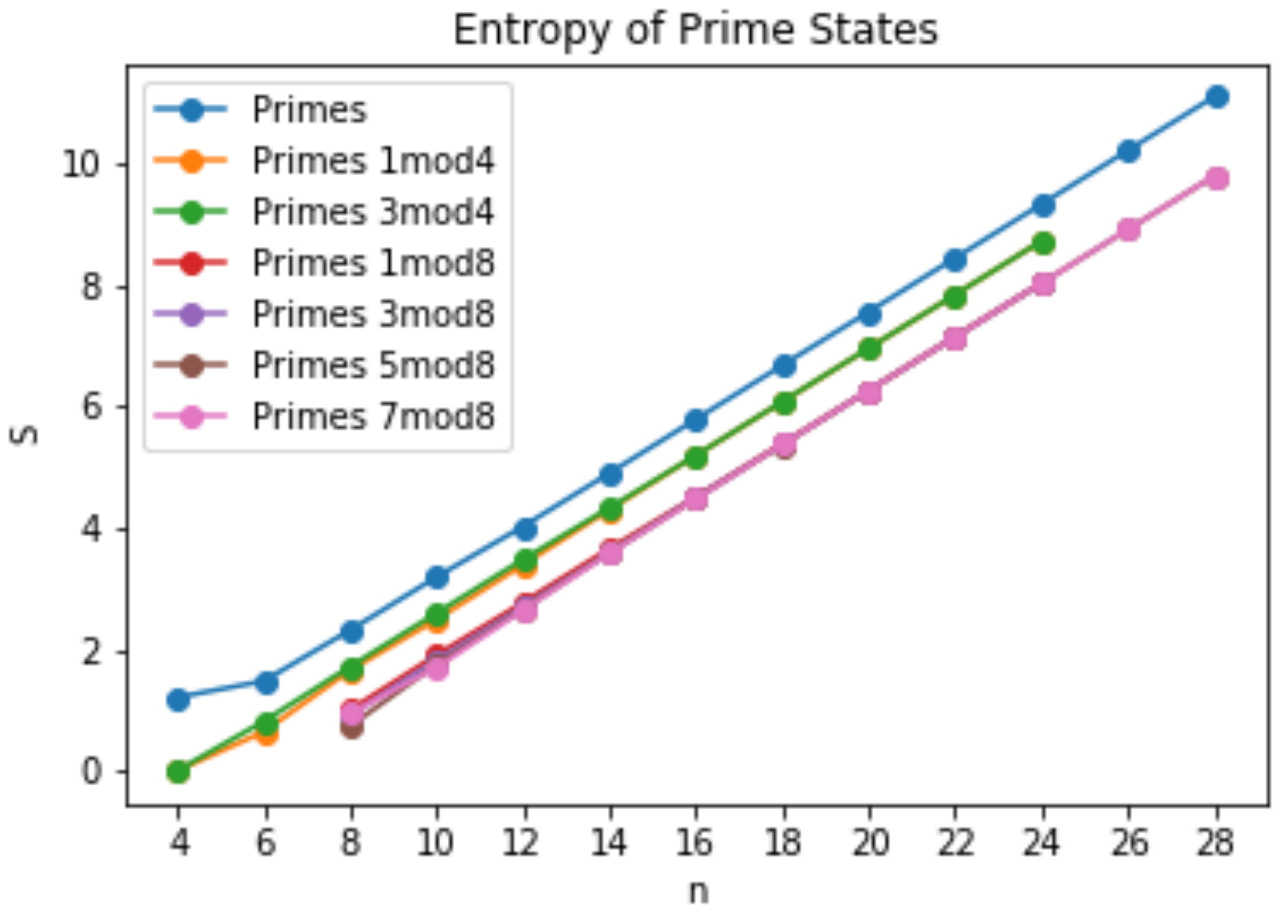
Conjecture: $$c_\pi = H_2\left(\frac{3}{\pi^2}\right)$$

$$H_2(x) = -x\, log_2\, x\; - (1-x)\, log_2(1-x)$$

$\frac{3}{\pi^2} = \frac{1}{2\,\zeta(2)}$: density of odd square free integers

$$c_\pi^{num} = 0.886(1), \quad c_\pi^{th} = 0.886082\,...$$

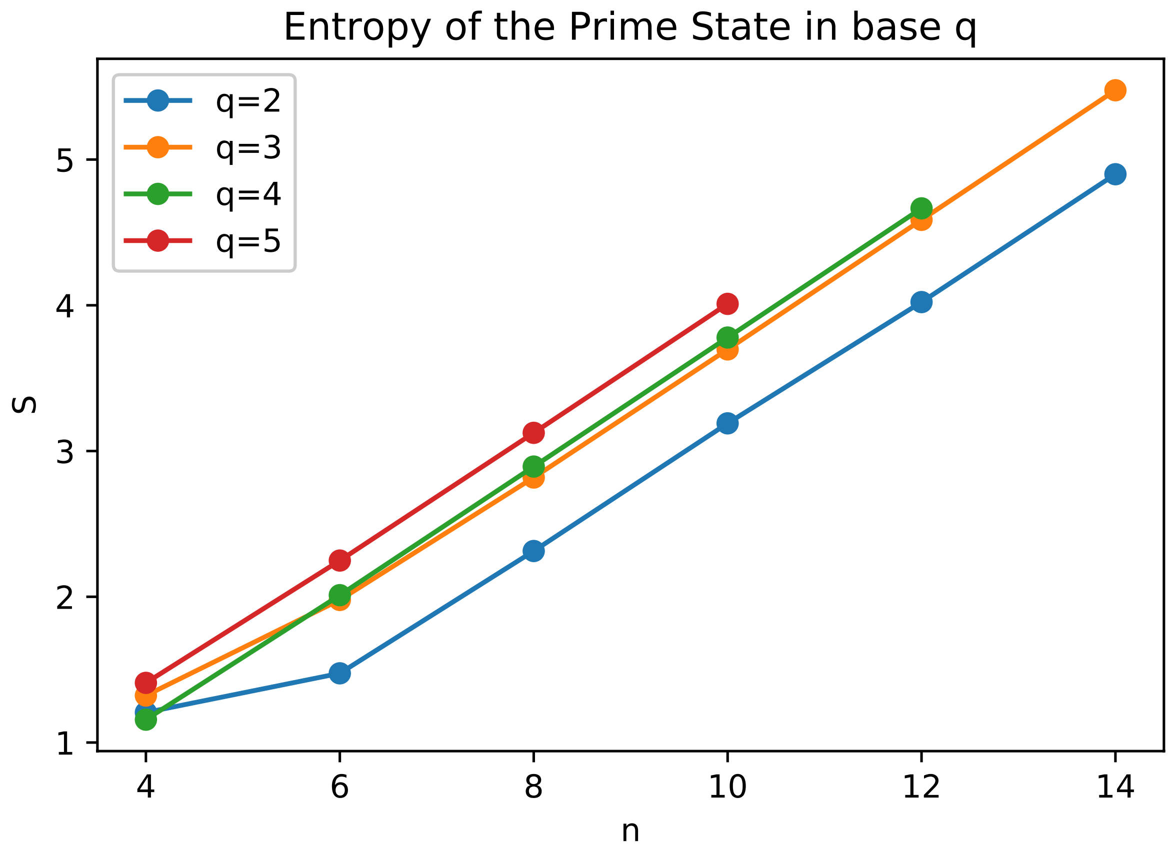
Entropies of primes in arithmetics progressions



Prime 1 mod 4
Prime 3 mod 4

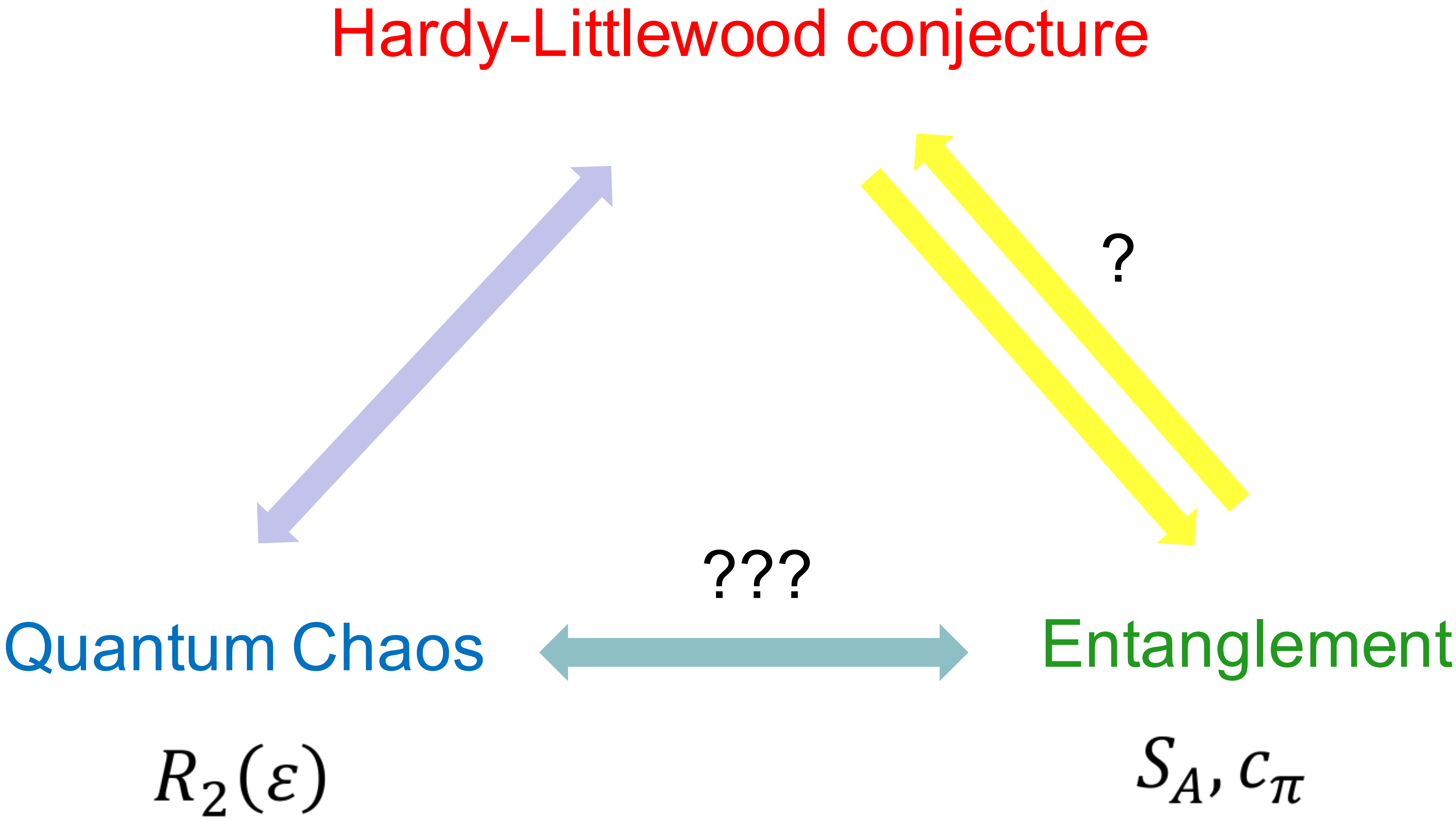
$$S_A \approx c_\pi \frac{n}{2} + cte$$

Entropies of primes using qudits



c_π : entanglement prime constant

CONCLUSIONS


Hardy-Littlewood conjecture
?
???
Quantum Chaos
Entanglement
$R_2(\varepsilon)$
S_A, c_π

References

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D. García-Martín and G.S., J. Applied Maths and Phys. 2018

Congratulations

Congratulations

The best is yet to come