Entanglement in Quantum Mechanics

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Notes of the course given at the Master in Theoretical Physics at UAM and at the Department of Theoretical Physics of Univ. Zaragoza. Course 2009-2010. This is a draft.

I. EPR 1935

Can Quantum-Mechanical description of Physical Reality be considered complet? A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935).

In this famous article EPR argueed that QM is an incomplete theory of the physical reality.
A necessary requirement for a complete theory is that:
"every element of the physical reality must have a counterpart in the physical theory".
The criteria of assigning reality is:
If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity"
In other words. Let A denote an element of the theory representing a physical quantity. Then A represents an element of the physical reality if A has a definite value in a given state of the system before we can measure it, i.e. without disturbing the system.
An example: take a particle with wave function ($\hbar = 1$)

$$\psi(x) = e^{ip_0 x}$$

and consider the momenta operator

$$P = -i\frac{d}{dx}$$

so that

$$P\,\psi = p_0\,\psi$$

Since P has a value p_0 with probability 1, even before we measure it, then the momenta of the particle in the state ψ has a physical reality. On the other hand if we consider the position operator

$$Q\,\psi = x\,\psi(x) \neq x_0\,\psi, \qquad \forall x_0$$

it does not have a definite value when evaluated on the state ψ. In this case we can only talk about the probabity of finding the particle in a region $a \leq x \leq b$, which is proportional to $b - a$. Hence according of the definition given above, the coordinate Q has no a physical reality. This is of course the general situation for any pair of observables A and B that do not commute in the QM theory.
From these considerations EPR are lead to two exclusive possibilities:

- 1) The wave function of QM does not give a complete description of reality
- 2) If the wave function gives a complete description, then non commuting operators, corresponding to two physical quantities, cannot have simultanous reality.

Alternative 2) follows from the fact that if both quantities had simultanous reality, thus definite values, these values will be predictable for the wave function ψ, which is not the case since the two operators do not commute.
In more mathematical terms:

$$\text{If } AB - BA \neq 0 \implies \nexists\psi,\ A\,\psi = a\,\psi \quad \text{and} \quad B\,\psi = b\,\psi \Longrightarrow A\ \&\ B : \text{not real}$$

Next EPR show that one can assign two different wave functions ψ_k and ϕ_r to the same reality, such that these wave functions are eigenstates of two non commuting operators, which violates the conditions of alternative 2), implying that the wave function does not give a complete description of reality.

To do so EPR consider a system with two particles with wave function

$$\psi(x_1, x_2) = \sum_n u_n(x_1)\psi_n(x_2)$$

where $u_n(x_1)$ is a basis of eigenstates of some operator A with eigenvalues $a_1, a_2, \ldots$. This state is prepared during a time interval $0 < t < T$ and let evolve freely afterwards. At $t > T$ one measures the quantity A and obtains a value of a_k. According to QM this measurement produces a "reduction of the wave packet" so that the state of the first system becomes $u_k(x_1)$, while that of the second system becomes $\psi_k(x_2)$. The total wave function after the measurement is the product $u_k(x_1)\psi_k(x_2)$. One can alternatively choose another operator B with eigenfunctions $\phi_s(x_1)$ and eigenvalues $b_1, b_2, \ldots$, to make another expansion

$$\psi(x_1, x_2) = \sum_s v_s(x_1)\phi_s(x_2)$$

If the quantity B is measured and found to have a value b_r, then the first system is left in the state $v_r(x_1)$ and the second in the state $\phi_r(x_2)$. Hence as a consequence of two different measurements upon the first system, the second system is left in two different wave functions. EPR continue saying

Since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system....Thus it is possible to assign two different wave functions to the same reality.

The interesting situation is when the operators A and B, that are measured on the part 1 of the system, do not commute. EPR consider the wave function

$$\psi(x_1, x_2) = \int_{-\infty}^{\infty} dp\ e^{i(x_1 - x_2 + x_0)p}$$

where x_0 is a constant. If the operator A is choosen as the momenta of the first particle one has the following identifications:

$$A = -i\frac{d}{dx_1} \longrightarrow u_p(x_1) = e^{ipx_1}, \qquad \psi_p(x_2) = e^{-ip(x_2 - x_0)}$$

Hence if p is the eigenvalue of A, the particle 2 is left in an eigenstate of the operator

$$P = -i\frac{d}{dx_2}$$

with eigenvalue $-p$. Similarly if one chooses B as the position of the first particle one can write $\psi(x_1, x_2)$ as

$$\psi(x_1, x_2) = 2\pi\,\delta(x_1 - x_2 + x_0) = 2\pi\int_{-\infty}^{\infty} dx\ \delta(x - x_1)\delta(x - x_2 + x_0)$$

one obtains

$$B = q_1 \longrightarrow v_x(x_1) = \delta(x - x_1), \qquad \phi_x(x_2) = 2\pi\delta(x - x_2 + x_0)$$

If one measures B obtaining x_1, then the coordinate Q of the second particle takes the eigenvalue $x_2 - x_0$. Thus one has constructed two different eigenfunctions ψ_{-p} and $\phi_{x_2 - x_0}$ corresponding to two non commuting operators of the second particle

$$PQ - QP = -i$$

According to the EPR criteria the quantites P and Q are both elements of the same physical reality, that of the particle 2. But since P and Q do not commute, this is not possible according to the alternative 2) hence the asumption that ψ is complete must be wrong.

At the end of the paper EPR left open the question wether a complete description of the physical reality exists, stating that they believe is possible.

Comments

- EPR make the reasonable asumption of *local realism*, meaning that the measurements made one one part of the system does not affect the other part of the system when the interaction has ceased.
- The search of an alternative to QM gave rise to the so called hidden variables models. One of the most notable models of this sort is due to D. Bohm[9]
- In 1957 reformulated the EPR paradox using two spin 1/2 particles, which made the problem more easiy to grasp. This is the formulation used by Bell in his 1964 crucial paper[11].
- The EPR idea was suggested by Einstein to Rosenfeld while they were both hearing a talk of Bohr in the 1933 Solvay Conference on the meaning of QM. The conversation went as follows[2,3]

  Einstein asked Rosenfeld: What would you say of the following situation? Suppose two particles are set in motion towards each other with the same, very large, momentum, and that they interact with each other for a very short time when they pass at known positions. Consider now an observer who gets hold of one of the particles, far away from the region of interaction, and measures its momentum; then, from the conditions of the experiment, he will obviously be able to deduce the momentum of the other particle. If, however, he chooses to measure the position of the first particle, he will be able to tell where the other particle is. This is a perfect correct and straightforward deduction from the principles of quantum mechanics; but it is not very paradoxical? How can the final state of the second particle be influenced by a measurement performed on the first, after all the physical interaction has ceased between them?

  Einstein called this phenomena "Spooky action at a distance"
- Bohr gave in July 1935 a replied to EPR paper indicating an ambiguity regarding the expression "without in any way disturbing a system" which supports the criterion of physical reality. Bohr says

  Of course there is in a case like that just considered no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially the question of an influence on the very conditions which define the possible types of predictions regarding the future behaviour of the system

  The notion of complementary is used once more by Bohr to save QM from incompleteness.
- For an explanation of the EPR paradox from a philosophical view see[12].

II. SCHROEDINGER

Schrodinger coined the term entanglement ("verschränkung" in german) to describe this peculiar connection between quantum systems (Schrdinger, 1935; p. 555):

When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled.

He added (Schrdinger, 1935; p. 555):

Another way of expressing the peculiar situation is: the best possible knowledge of a whole does not necessarily include the best possible knowledge of all its parts, even though they may be entirely separate and therefore virtually *capable of being best possibly known, i.e., of possessing, each of them, a representative of its own. The lack of knowledge is by no means due to the interaction being insufficiently known at least not in the way that it could possibly be known more completely it is due to the interaction itself.*

Attention has recently been called to the obvious but very disconcerting fact that even though we restrict the disentangling measurements to one system, the representative obtained for the other system is by no means independent of the particular choice of observations which we select for that purpose and which by the way are entirely arbitrary. It is rather discomforting that the theory should allow a system to be steered or piloted into one or the other type of state at the experimenter's mercy in spite of his having no access to it.

III. VON NEUMANN WRONG PROOF

von Neumann gave in his book[7] a proof of the impossibility of hidden variables. This proof remained accepted by the Physics community, until 1966 when John Bell showed that von Neumann' s, and other related proofs due to Piron and Jauch, relied on unreasonable asumptions[8].

Bell illustrated the problem with a spin 1/2 particle. The quantum state of this particle is described by a two component spinor ψ. The observables are represented by 2×2 hermitean matrices

$$\mathcal{O} = \alpha \mathbf{1} + \vec{\beta} \cdot \vec{\sigma}$$

where $\mathbf{1}$ is the identity matrix, $\vec{\sigma}$, the Pauli matrices and $\alpha, \vec{\beta}$ are a real number and real vector respectively. To simplify the discussion one can choose the state ψ to be given by the eigenstate of σ^z with eigenvalue $+1$, i.e.

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The expectation of the observable $\mathcal{O}$ in this state is given by the QM formula

$$\langle \mathcal{O} \rangle_\psi = \frac{\langle \psi | \mathcal{O} | \psi \rangle}{\langle \psi | \psi \rangle} = \alpha + \beta_z$$

Bell then constructed a hidden variable theory which gives the same value for this expectation value. He uses the notion of "dispersion free" states (DF). A DF state is determined by the wave function, ψ, of the associated quantum state and a set of additional (hidden) variables λ. Both, ψ and λ determine the results of individual measurements. For the spin 1/2 example given above Bell chooses only a hidden variable λ in the interval $(-1/2, 1/2)$. The result of the measurement of the operator $\mathcal{O}$ in the DF state specified by ψ and λ is then choosen as

$$\langle \mathcal{O} \rangle_{\psi,\lambda} = \alpha + |\vec{\beta}| \, \mathrm{sign}(\lambda|\vec{\beta}| + \frac{1}{2}\beta_z) \, \mathrm{sign} X$$

where

$$X = \begin{cases} \beta_z & \text{if } \beta_z \neq 0 \\ \beta_x & \text{if } \beta_z = 0, \ \beta_x \neq 0 \\ \beta_y & \text{if } \beta_z = 0, \ \text{and } \beta_x = 0 \end{cases}$$

and

$$\mathrm{sign}\, X = \begin{cases} +1 & \text{if} X \geq 0 \\ -1 & \text{if} X < 0 \end{cases}$$

The QM state is obtained by a uniform averaging over λ. This averaging gives an expectation value that agrees with the QM prediction

$$\langle \mathcal{O} \rangle_\psi = \int_{-1/2}^{1/2} d\lambda \, \langle \mathcal{O} \rangle_{\psi,\lambda} = \alpha + \beta_z$$

Hence at this level one can indeed construct a hidden variable theory that gives the same predictions as QM. In the next paragraph of Bell' s paper, he considers the von Neumann proof of the impossibility of hidden variables. The essential asumption is:

Any real linear combination of any two Hermitean operators represents an observable, and the same linear combination of expectations values is the expectation value of the combination (von Neumann[7])

In mathematical terms this reads

$$\begin{aligned} \text{If } A^\dagger = A, B^\dagger = B \implies & (\mu_A A + \mu_B B)^\dagger = \mu_A A + \mu_B B,\ \forall\ \mu^*_{A,B} = \mu_{A,B} \\ \langle \mu_A A + \mu_B B\rangle_\psi \ \ = & \ \ \mu_A \langle A\rangle_\psi + \mu_B \langle B\rangle_\psi \end{aligned}$$

These eqs. are certainly true for QM states, but von Neumann also impose them on the hypothetical DF states, i.e.

$$\langle \mu_A A + \mu_B B\rangle_{\psi,\lambda} \ = \ \mu_A \langle A\rangle_{\psi,\lambda} + \mu_B \langle B\rangle_{\psi,\lambda} \tag{1}$$

In the example given above this eq. would imply

$$\langle \alpha \mathbf{1} + \vec{\beta}\cdot\vec{\sigma}\rangle_{\psi,\lambda} = \alpha\langle \mathbf{1}\rangle_{\psi,\lambda} + \vec{\beta}\langle\vec{\sigma}\rangle_{\psi,\lambda}$$

which cannot be satisfied because the RHS is a linear function in $\vec{\beta}$ while the LHS is non linear in it. Hence according to von Neumann this DF state is impossible.

Bell critizes the asumption (1) on DF states as follows. First of all he recognizes that additivity of expectation values seems very reasonable. However if one considers the sum of two non commuting operators, like σ_x and σ_z their expectation value is not the sum of their eigenvalues. He concludes saying:

"There is no reason to demand it (i.e. condition (1)) individually of the hypothetical dispersion free states, whose function is to reproduce the measurable peculiarities of quantum mechanical states when averaged over" (Bell)

The DF states constructed above have additive expectation values only for commuting operators and they give consistent predictions for all possible measurements when averaged over λ. Finally, Bell states that the formal proof of von Neumann does not justify his informal conclusion:

"It is therefore not, as is often assumed, a question of reinterpretation of quantum mechanics-the present system of quantum mechanics would have to be objectively false in order that another description of the elementary process than the statistical one be possible (von Neumann)"

continuing with

"It was not the objective measurable predictions of quantum mechanics which ruled out hidden variables. It was the arbitrary assumption of a particular (and imposible) relation between the results of incompatible measurements either of which might be made on a given ocassion but only one of which can in fact be made" (Bell)

IV. BELL INEQUALITIES

In this historical paper Bell gave a precise mathematical formulation of the EPR paradox using a pair of entangled spin 1/2 particles. The EPR requirement of locality and causality which leads to the introduction of hidden variables is considered, and Bell derives an inequality for the observables of such a hypothetical theory. This inequality being violated in QM.

Bell starts from the Bohm and Aharonov formulation of the EPR paradox in terms of two spin 1/2 particles in a singlet state formed at $t = 0$ and which movely freely afterwards in opposite directions[10] . Two observers at points, say 1 and 2, then measure the spin of the particles with two Stern-Gerlach magnets which are oriented in the directions given by two unit vectors $\vec{a}$ and $\vec{b}$. If the magnet 1 detects a spin $\vec{\sigma}_1\cdot\vec{a} = +1$, then, according to QM, the magnet of the second observer must detect $\vec{\sigma}_2\cdot\vec{a} = -1$ and viceversa.

Next, Bell assumes the Einstein hypothesis:

"But on one supposition we should, in my opinion, absolutely hold fast: the real factual of the system S_2 is independent of what is done with system S_1, which is spatially separated from the former"

which is translated into the Bell statement

"It seems one at least worth considering, that if two measurements are made at places remote from one another the orientation of one magnet does not influence the results of the other. Since we can predict in advanced the result of measuring any choosen component of $\vec{\sigma}_2$, by previously measuring the same component of $\vec{\sigma}_1$, it follows that the result of any such measurement must actually be predetermined. Since the initial QM wave function does not determine the result of an individual measurement, this predetermination implies the possibility of a more complete specification of the state"

To complete the specification of the state one introduces a generic hidden variable λ, so that the result A of measuring $\vec{\sigma}_1 \cdot \vec{a}$ is determined by $\vec{a}$ and λ and the result B of measuring $\vec{\sigma}_2 \cdot \vec{b}$ is determined by $\vec{b}$ and λ. The results A and B can only takes two discrete values

$$A(\vec{a}, \lambda) = \pm 1, \qquad B(\vec{b}, \lambda) = \pm 1 \tag{2}$$

The crucial asumption is that B does not depend on $\vec{a}$, nor A depends on $\vec{b}$. The hidden variable λ has a probability distribution $\rho(\lambda)$ so that the expectation value of the product $\vec{\sigma}_1 \cdot \vec{a}\, \vec{\sigma}_2 \cdot \vec{b}$ is

$$P(\vec{a}, \vec{b}) = \int d\lambda\, \rho(\lambda)\, A(\vec{a}, \lambda)\, B(\vec{b}, \lambda) \tag{3}$$

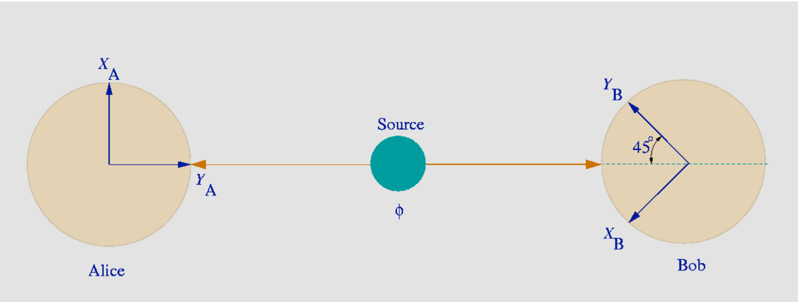

FIG. 1: Plot of (4) and (6) as a function of θ. Notice that they coincide only for $\theta = 0, Pi/2, Pi$ and at $\theta = 0, \pi$, the QM expression (4) is stationary unlike the eq.(6).

In QM the corresponding expectation value is given by

$$\langle \vec{\sigma}_1 \cdot \vec{a}\, \vec{\sigma}_2 \cdot \vec{b} \rangle = -\vec{a} \cdot \vec{b} \tag{4}$$

Later on, Bell will prove that this QM expression cannot be recovered from eq. (3), for generic values of the vectors $\vec{a}, \vec{b}$. Before that, Bell illustrates how a hidden variable model can reproduce the measurements made on a single particle. He uses as hidden variable a unit vector $\vec{\lambda}$. If $\vec{p}$ denotes the polarization of a pure spin, then the probability distribution $\rho(\lambda)$ is choosen as

$$\rho(\vec{\lambda}) = \frac{1}{2\pi} \quad \text{if} \;\; \vec{\lambda} \cdot \vec{p} > 0, \qquad \rho(\vec{\lambda}) = 0 \qquad \text{if} \;\; \vec{\lambda} \cdot \vec{p} < 0$$

That is, $\vec{\lambda}$ is uniformly distributed on the hemisphere whose north pole is $\vec{p}$. The measure $d\lambda$ is given by the solid angle on the sphere, so that the denominator of $\rho(\lambda)$ is half of the total solid angle 4π. The resultf os the measure of $\vec{\sigma} \cdot \vec{a}$ is taken as

$$A(\vec{a},\vec{\lambda}) = \text{sign}(\vec{\lambda}\cdot\vec{a}')$$

where $\vec{a}'$ is a unit vector which depends on $\vec{a}$ and $\vec{\lambda}$ in a way which specified later on. The average value of A over $\vec{\lambda}$ gives the expectation value

$$P(\vec{a}) = \int d\lambda\,\rho(\lambda)\,A(\vec{a},\vec{\lambda}) = 1 - \frac{2\theta'}{\pi}$$

where θ' is the angle between $\vec{a}'$ and $\vec{p}$ (see fig for the derivation). On the other hand the QM value of this expectation value is

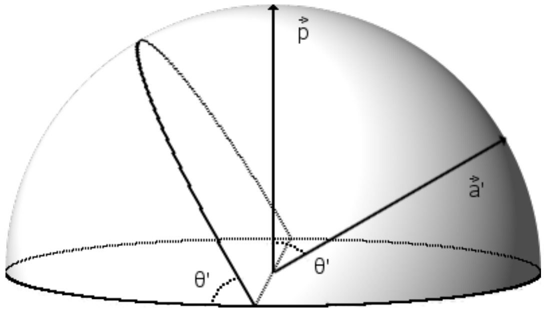

FIG. 2: Bell

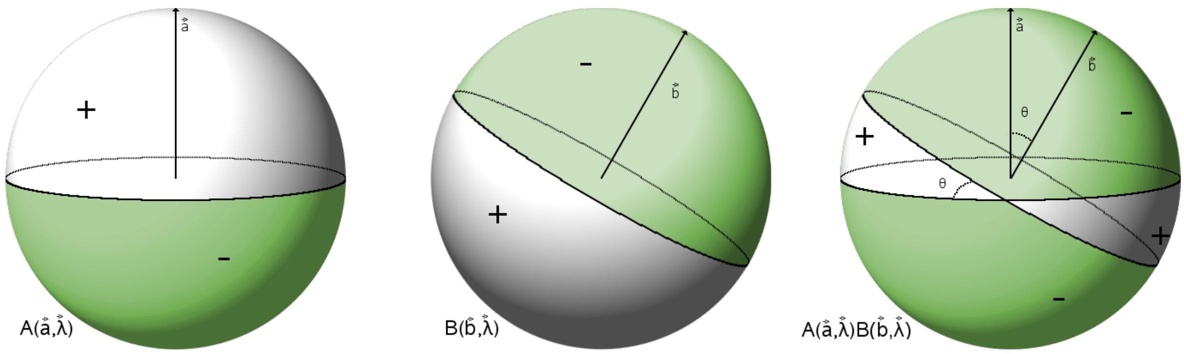

FIG. 3: Bell

$$\langle\vec{\sigma}\cdot\vec{a}\rangle = \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{p}$. Hence setting

$$1 - \frac{2\theta'}{\pi} = \cos\theta$$

one obtains the desired result: the statistical prediction of QM of a single particle are reproduced by a local random variable. Similarly, it is possible to reproduce the QM eq. (4) in the following cases:

$$\begin{aligned} P(\vec{a},\vec{a}) &= -P(\vec{a},-\vec{a}) = -1 \\ P(\vec{a},\vec{b}) &= 0 \quad \text{if } \vec{a}\cdot\vec{b} = 0 \end{aligned} \tag{5}$$

It suffices to choose $\vec{\lambda}$ uniformly distributed on the sphere and take

$$\begin{aligned} A(\vec{a},\vec{\lambda}) &= \text{sign}(\vec{a}\cdot\vec{\lambda}) \\ B(\vec{b},\vec{\lambda}) &= -\text{sign}(\vec{b}\cdot\vec{\lambda}) \end{aligned}$$

which gives (see fig)

$$P(\vec{a},\vec{b}) = -1 + \frac{2\theta}{\pi} \tag{6}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. This eq. reproduces (5) in the cases $\theta = 0, \pi/2, \pi$, but not for other values of θ as shown in fig. 8:

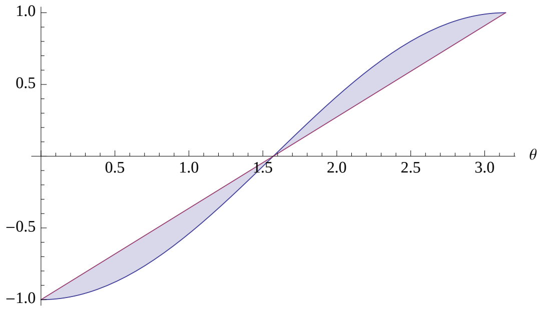

FIG. 4: Plot of (4) and (6) as a function of θ. Notice that they coincide only for $\theta = 0, Pi/2, Pi$ and at $\theta = 0, \pi$, the QM expression (4) is stationary unlike the eq.(6).

Having explain with these examples the meaning of eq.(3), Bell derives an inequality satisfied by a generic distribution $\rho(\lambda)$. First of all one recalls that

$$\int d\lambda\, \rho(\lambda) = 1$$

Consider now the case $\vec{a} = \vec{b}$ where $P = -1$, which as a consequence of eqs.(2) implies

$$P(\vec{a},\vec{a}) = -1 \Longrightarrow A(\vec{a},\lambda) = -B(\vec{a},\lambda) \tag{7}$$

except at a set of points λ of zero measure. Plugging this eq. into (3) one finds

$$P(\vec{a},\vec{b}) = -\int d\lambda\, \rho(\lambda)\, A(\vec{a},\lambda)\, A(\vec{b},\lambda) \tag{8}$$

Consider now another unit vector $\vec{c}$ and the expression

$$\begin{aligned} P(\vec{a},\vec{b}) - P(\vec{a},\vec{c}) &= -\int d\lambda\, \rho(\lambda)\, [A(\vec{a},\lambda)\, A(\vec{b},\lambda) - A(\vec{a},\lambda)\, A(\vec{c},\lambda)] \\ &= \int d\lambda\, \rho(\lambda)\, A(\vec{a},\lambda)\, A(\vec{b},\lambda)[A(\vec{b},\lambda)\, A(\vec{c},\lambda) - 1] \end{aligned}$$

Since $|A(\vec{a},\lambda)| = 1$ one fids

$$|P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \;\leq\; \int d\lambda\, \rho(\lambda)\, [1 - A(\vec{b},\lambda)\, A(\vec{c},\lambda)]$$

where the second term of the RHS is $P(\vec{b},\vec{c})$, thus

$$1+P(\vec{b},\vec{c}) \;\geq\; |P(\vec{a},\vec{b})-P(\vec{a},\vec{c})| \tag{9}$$

Unless P is constant the RHS of this eq. is of order $|\vec{b}-\vec{c}|$, for small values of $|\vec{b}-\vec{c}|$, thus $P(\vec{b},\vec{c})$ is not stationary at the minimun value (-1 at $\vec{b}=\vec{c}$) and cannot reproduce the QM result (4). Indeed, let us replace $P(\vec{a},\vec{b})$ by its QM value (4), i.e.

$$P(\vec{a},\vec{b}) \to P(\vec{a},\vec{b})_{\rm QM} = \langle \vec{\sigma}_1\cdot\vec{a}\,\vec{\sigma}_2\cdot\vec{b}\rangle = -\vec{a}\cdot\vec{b}$$

and choose a basis for the vectors $\vec{a},\vec{b},\vec{c}$,

$$\begin{aligned}
\vec{a} &= (0,0,1)\\
\vec{b} &= (\sin\theta_b,0,\cos\theta_b)\\
\vec{c} &= (\cos\phi_c\sin\theta_c,\sin\phi_c\sin\theta_c,\cos\theta_c)
\end{aligned}$$

defining

$$\begin{aligned}
{\rm Bell}(\theta_b,\theta_c,\phi_c) &\equiv 1+P_{\rm QM}(\vec{b},\vec{c})-|P_{\rm QM}(\vec{a},\vec{b})-P_{\rm QM}(\vec{a},\vec{c})| \\
&= 1-(\cos\phi_c\sin\theta_c\sin\theta_b+\cos\theta_c\cos\theta_b)-|\cos\theta_b-\cos\theta_c|
\end{aligned} \tag{10}$$

As shown if fig. there are values of the angles where Bell <0, so that the Bell inequality is violated by QM. The conclusions drawn from this result are

In a theory in which parameters are added to quantum mechanics to determined the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another intstrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant (Bell).

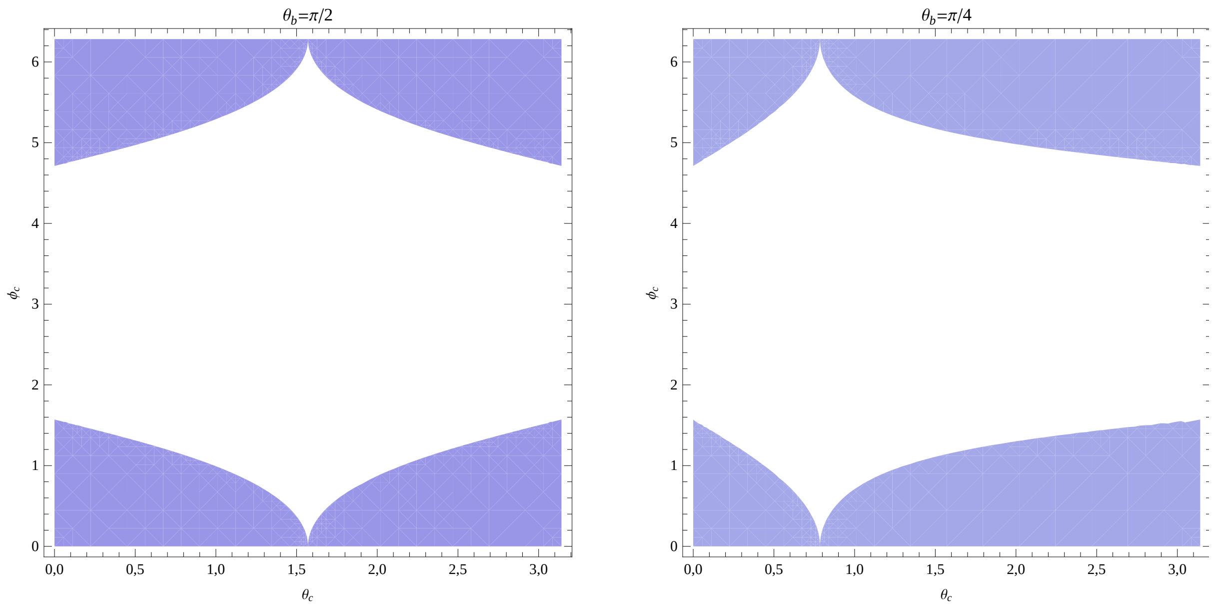

FIG. 5: Plot of ${\rm Bell}(\theta_b,\theta_c,\phi_c)$ for $\theta_b=\pi/2$ (left) and $\theta_b=\pi/4$ (right) as a function of θ_c and ϕ_c.

A. An intuitive argument (Mermin)

A simple way to see how the Bell inequality is violated consist in choosing the three magnets where the vectors $\vec{a}, \vec{b}$ and $\vec{b}, \vec{c}$ are separated by a small angle $\theta << 1$, while $\vec{a}$ and $\vec{c}$ are separated by 2θ, i.e.

$$\begin{aligned} \vec{a} &= (0,0,1) \\ \vec{b} &= (\sin\theta, 0, \cos\theta) \\ \vec{c} &= (\sin 2\theta, 0, \cos 2\theta) \end{aligned}$$

In this situation the correlations can be approximated as

$$P(\vec{a},\vec{b}) \sim -1+\epsilon, \quad P(\vec{b},\vec{c}) \sim -1+\epsilon, \quad P(\vec{a},\vec{c}) \sim -1+\epsilon' \qquad (\epsilon, \epsilon' << 1) \tag{11}$$

Bell's inequality (9) implies

$$\epsilon \geq |\epsilon - \epsilon'| \Longrightarrow \epsilon' \leq 2\epsilon$$

This result may arise in a situation with movies (coin-flips in the Mermin' s example). Suppose that Alice (A) and Bob (B) are aficionados to movies and that they like the same movies 99 % of the times. Suppose now that Claire (C) goes with Bob to see the same movies and that they also like them 99 % of the times. What can one say when Alice and Claire wach these movies? Since A and B do not like 1 out of 100 movies and the same happens for B and C, it is clear that A and C may not like at most 2 of the movies. This example corresponds to the case $\epsilon = 1/100$ and $\epsilon' \leq 2/100$ above.

In the quantum case eqs.(11) are replaced by

$$P_{\rm QM}(\vec{a},\vec{b}) = P_{\rm QM}(\vec{b},\vec{c}) = -\cos\theta \sim -1 + \frac{\theta^2}{2}, \quad P_{\rm QM}(\vec{a},\vec{c}) = -\cos 2\theta \sim -1 + \frac{(2\theta)^2}{2} \qquad (\theta << 1) \tag{12}$$

so that the Bell inequality (9) is badly violated

$$\frac{\theta^2}{2} \not\geq |\frac{\theta^2}{2} - \frac{(2\theta)^2}{2}| = \frac{3\theta^2}{2}$$

To continue with the movie example, we may take $\epsilon = \theta^2/2 = 1/100$ as the probablity that A and B or B and C dot not like the same movies. It then follows that for "quantum movies" A and C will not share their enjoyment in 4 out of 100 movies them.

V. THE CHSH PAPER

In 1969 Clauser, Horne, Shimony and Holt (CHSH) proposed an experimental test of local hidden variable theories following Bell' s approach to this problem. Instead of a pair of spin 1/2 entangled particles they suggested the use of polarized entangled photons emitted in an atomic cascade of Calcium atoms. The experimental set up would test not the Bell inequality but a modification of it which is nowadays called CHSH inequality. The final experiment was performed by the group of Alain Aspect in 1981 (see below).

The theoretical framework is as in Bell' s paper. There is a source that produces pair of particles in an entangled state. There are two observers, Alice and Bob, separated by some distance, who measure a discrete property of the particles they received with two only possible outcomes ± 1. The outcome of Alice is called $A(a) = \pm 1$, where a denotes adjustable parameters of her apparatus. Similarly, Bob' s outcome is denoted as $B(b)$, where b has a similar meaning. Of course the parameters a and b are set by Alice and Bob at will. In a theory of local hidden variables the results of Alice and Bob depend on a generic variable λ, such that $A(a, \lambda)$ and $B(b, \lambda)$ are deterministic variables. Locality requires that $A(a, \lambda)$ (resp. $B(b, \lambda)$) is independent of b (resp. a). Finally it is also assumed that the normalized probability density of $\rho(\lambda)$ is independent of a and b since the experimental set up must not affect the preparation of the entangled pairs.

The correlation function is therefore given by Bell' s formula:

$$P(a,b) = \int_\Gamma d\lambda\, \rho(\lambda)\, A(a,\lambda)\, B(b,\lambda) \tag{13}$$

where Γ is the total λ space. Next, CHSH write the following eq.

$$\begin{aligned} |P(a,b) - P(a,c)| &= \left| \int_\Gamma d\lambda\, \rho(\lambda)(A(a,\lambda)\, B(b,\lambda) - A(a,\lambda)\, B(c,\lambda)) \right| \\ &\leq \int_\Gamma d\lambda\, \rho(\lambda)\, |A(a,\lambda)\, B(b,\lambda) - A(a,\lambda)\, B(c,\lambda)| \\ &= \int_\Gamma d\lambda\, \rho(\lambda)\, |A(a,\lambda)\, B(b,\lambda)|\, |[1 - B(b,\lambda)\, B(c,\lambda)| \\ &= \int_\Gamma d\lambda\, \rho(\lambda)[1 - B(b,\lambda)\, B(c,\lambda)] = 1 - \int_\Gamma d\lambda\, \rho(\lambda) B(b,\lambda)\, B(c,\lambda) \end{aligned} \tag{14}$$

If we follow Bell' s paper and replace $B(b,\lambda)$ by $-A(b,\lambda)$ (recall eq. (7)), we would require the inequality (9). However CHSH critizes this point saying:

Here we avoid Bell's experimentally unrealistic restriction that for some pair of parameters b' and b there is a perfect correlation (CHSH)

Indeed, setting $B(b,\lambda) = -A(b,\lambda)$ implies that Alice and Bob can achieve a perfect maching between their devices. To overcome this problem, CHSH introduce another set of parameters d for Alice' s apparatus (called b' in their paper) and such that $P(d,b) = 1 - \delta$ where $0 \leq \delta \leq 1$ (actually, one can take $0 \leq \delta \leq 2$ which covers all possibilities). Then one can divide the parameter space Γ into two regions

$$\Gamma = \Gamma_+ \cup \Gamma_-, \qquad \Gamma_\pm = \{\lambda \,|\, A(d,\lambda) = \pm B(b,\lambda)\}$$

So that

$$P(d,b) = \int_\Gamma d\lambda\, \rho(\lambda)\, A(d,\lambda)\, B(b,\lambda) = \int_{\Gamma_+} d\lambda\, \rho(\lambda) - \int_{\Gamma_-} d\lambda\, \rho(\lambda) = 1 - 2 \int_{\Gamma_-} d\lambda\, \rho(\lambda) \tag{15}$$

which leads to

$$P(d,b) = 1 - \delta \Longrightarrow \int_{\Gamma_-} d\lambda\, \rho(\lambda) = \frac{\delta}{2} \tag{16}$$

Choosing $\delta = 0$ would imply that $A(d,\lambda) = B(b,\lambda)$, $\forall \lambda$, which corresponds to a perfect anticorrelation between Alice and Bob devices. In Bell' s notation this reads $\vec{d} = -\vec{b}$.

Returning to the last term in eq.(14) one can derive

$$\begin{aligned} \int_\Gamma d\lambda\, \rho(\lambda) B(b,\lambda)\, B(c,\lambda) &= \int_{\Gamma_+} d\lambda\, \rho(\lambda) A(d,\lambda)\, B(c,\lambda) - \int_{\Gamma_-} d\lambda\, \rho(\lambda) A(d,\lambda)\, B(c,\lambda) \\ &= \int_\Gamma d\lambda\, \rho(\lambda) A(d,\lambda)\, B(c,\lambda) - 2 \int_{\Gamma_-} d\lambda\, \rho(\lambda) A(d,\lambda)\, B(c,\lambda) \\ &\geq P(d,c) - 2 \int_{\Gamma_-} d\lambda\, \rho(\lambda) |A(d,\lambda)\, B(c,\lambda)| \\ &= P(d,c) - 2 \int_{\Gamma_-} d\lambda\, \rho(\lambda) = P(d,c) - \delta = P(d,b) + P(d,c) - 1 \end{aligned} \tag{17}$$

Hence from eq.(14) one gets

$$|P(a,b) - P(a,c)| \leq 2 - P(d,b) - P(d,c) \tag{18}$$

This is the so called CHSH inequality which is a generalization of the Bell's inequality which is recovered choosing $d = -b$ with $P(d,b) = 1$ i.e.

$$|P(a,b) - P(a,c)| \leq 1 + P(b,c) \tag{19}$$

A more symmetric way to write eq.(18) is

$$|P(a,b) - P(a,c)| + P(d,b) + P(d,c) \leq 2$$

Notice that a and d describes two possible Alice' s devices , while b and d describes those of Bob. So after a relabelling of symbols one gets

$$P(a,b) + P(a,b') + |P(a',b) - P(a',b')| \leq 2$$

One can derive a more general form of this identity, using the fact that $B(b,\lambda) = \pm 1$ which implies

$$B(b,\lambda) + B(b',\lambda) \ \text{ or } \ B(b,\lambda) - B(b',\lambda) = 0$$

So

$$\begin{aligned} -2 \leq A(a,\lambda)(B(b,\lambda) + B(b',\lambda)) + A(a',\lambda)(B(b,\lambda) - B(b',\lambda)) \leq 2 \\ -2 \ \leq A(a,\lambda)B(b,\lambda) + A(a,\lambda)B(b',\lambda) + A(a',\lambda)B(b,\lambda) - A(a',\lambda)B(b',\lambda)) \ \leq 2 \end{aligned} \tag{20}$$

Multipliying this expression by $\rho(\lambda)$ and integrating over λ one finds

$$-2 \leq P(a,b) + P(a,b') + P(a',b) - P(a',b') \leq 2 \tag{21}$$

This formula was obtained by Clauser and Horn[14].

A. An experimental proposal

In the experimental set up proposed by CHSH the measuring devices consist of a filter followed by a detector (see fig. 6). A source S provides entangled photons which move in opposite directions. One uses linear polarization filters whose orientations are set by the parameters a and b, given typically by two angles. If the photon that hits Alice's filter emerges afterwards, one assigns a value $A(a) = +1$, otherwise $A(a) = -1$. These two possibilities are denoted as $A(a)_\pm$. Same thing for the photons hiting Bob' s detector, i.e. $B(b)_\pm$. There is also the option to remove, Alice' s or Bob' s filters. These options are denoted by the convention $a = \infty$ and $b = \infty$ respectively. In either case the photon will go straight to Alice' s or Bob' s detectors, so $A(\infty) = +1$ or $B(\infty) = +1$ (i.e. $A(\infty)_+$ or $B(\infty)_+$). In fig. we show all the outcomes.

There are four possible experiments one can performed depending on wether the filters are present or not (see fig.). In the experiment 1, when the two filters are present, one counts how many photons are detected by the coincidence monitor CM. The rate of coincidence is denoted by $R(a,b)$. In the experiment 2, Bob's filter has been removed and one counts the number of coincidence photons in CM, whose rate is denoted by $R_1(a)$, which obviously only depends on Alice' s filter parameter a. Similarly, in experiment 3, one removes Alice' s detector, so that the coincidence rate $R_2(b)$ only depends on Bob' s filter parameter b. Finally, in experiment 4 the two filters have been removed and the coincidence rate R_0 is independent of both a and b.

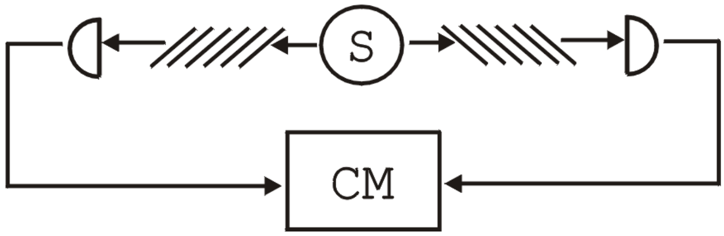

FIG. 6: The source S produces pairs of "photons", sent in opposite directions. Each photon encounters a single channel (e.g. "pile of plates") polariser whose orientation can be set by the experimenter. Emerging signals are detected and coincidences counted by the coincidence monitor CM).

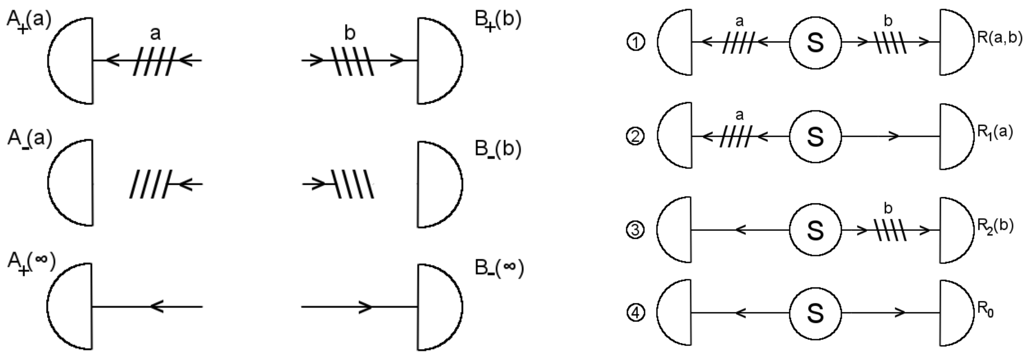

FIG. 7: probs

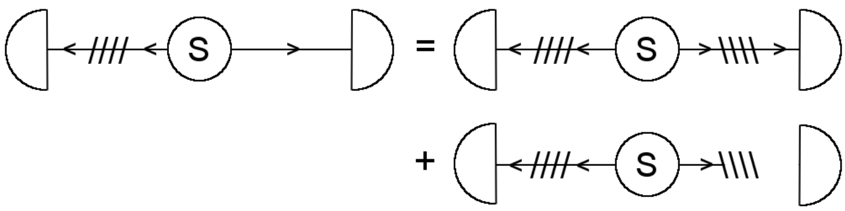

FIG. 8: probs

One would like to use the coincidences rates $R(a, b), R_1(a), R_2(b), R_0$, to check wether the CHSH equation (21) is violated or not. This requires to relate the correlation function $P(a, b)$ to latter quantities. This is done as follows. Let us denote by $w[A_\pm(a), B_\pm(b)]$, the probabilites that $A(a) = \pm 1$ and $B(b) = \pm 1$. One clearly has

$$P(a, b) = w[A_+(a), B_+(b)] - w[A_+(a), B_-(b)] - w[A_-(a), B_+(b)] + w[A_-(a), B_-(b)] \tag{22}$$

and of course

$$1 = w[A_+(a), B_+(b)] + w[A_+(a), B_-(b)] + w[A_-(a), B_+(b)] + w[A_-(a), B_-(b)] \tag{23}$$

The rate $R(a, b)$ must be proportional to $w[A_+(a), B_+(b)]$, with the proportionality factor being given by R_0,

$$w[A_+(a), B_+(b)] = \frac{R(a, b)}{R_0} \tag{24}$$

Similarly, the rate $R_1(a)$ must be proportional to the sum of the possible outcomes when Bob's filter is present

$$\frac{R_1(a)}{R_0} = w[A_+(a), B_+(\infty)] = w[A_+(a), B_+(b)] + w[A_+(a), B_-(b)] = \frac{R(a,b)}{R_0} + w[A_+(a), B_-(b)] \tag{25}$$

Same thing for $R_2(b)$,

$$\frac{R_2(b)}{R_0} = w[A_+(\infty), B_+(b)] = w[A_+(a), B_+(b)] + w[A_-(a), B_+(b)] = \frac{R(a,b)}{R_0} + w[A_+(a), B_-(b)] \tag{26}$$

Using eqs.(23- 26) one finds

$$P(a,b) = 1 + \frac{4R(a,b) - 2R_1(a) - 2R_2(b)}{R_0} \tag{27}$$

which expresses $P(a,b)$ in terms of measurables quantities. Plugging (27) into (21) yields finally the inequality

$$-1 \leq \frac{R(a,b) + R(a,b') + R(a',b) - R(a',b') - R_1(a) - R_2(b))}{R_0} \leq 0 \tag{28}$$

This is basically the inequality that CHSH proposed to test in an experiment with photons. By that time there were already some experiments which could be in principle used for this purpose: Wu and Shaknov examined in 1950 the polarization correlations of γ rays emitted during positron-electron pair annihilation[16]. This experiment was inspired by an earlier paper of Wheeler in 1946[17], who suggested that two photons emitted in that annihilation are polarized at right angles to each other. However Wu and Shaknov results could not test (28) due to experimental difficulties with high energy photons.

CHSH suggested that a more promissing experiment was the one performed by Kocher and Commins in 1967 who used a pair of entangled photons generated in the atomic cascade of Calcium: $6\,^1S_0 \to 4^1P \to 4^1S_0$. Here the photons are visible so Polaroid type filters could be used (see fig for the experimental set up and the cascade process).

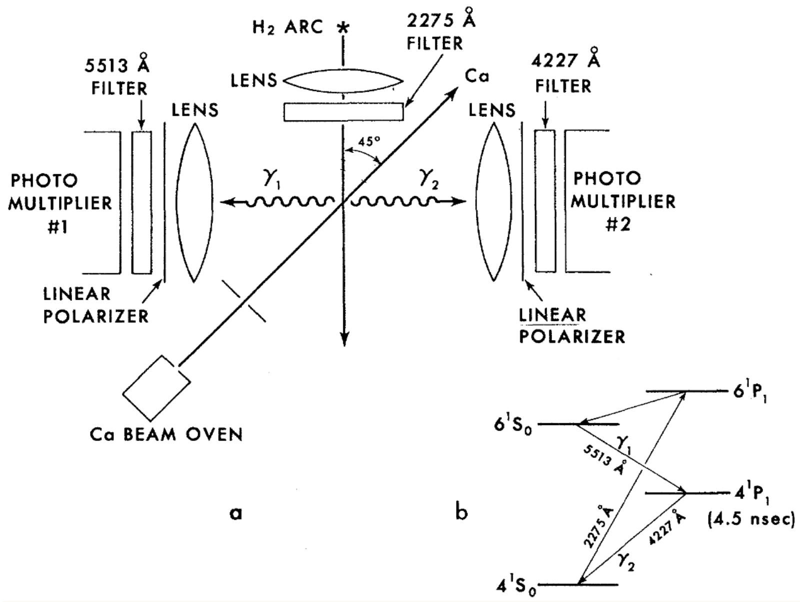

FIG. 9: a) Experimental set up proposed by Kocher and Commins, b) atomic cascade process $6\,^1S_0 \to 4^1P \to 4^1S_0$ (figure taken from[18]).

The data obtained by Kocher and Commins were not sufficient to test CHSH inequality because the polarizers were not highly efficient and were placed only in the relative orientations 0° and 90○. CHSH proposed a modification of the KC experiment by observing at two appropiate relative orientations of the polarizers, and also including the removal of one of them and the other. Quantum Mechanics implies the violation of the inequality 21 for certain relative orientations.

If a and b denote the angles of the polarizers, then by rotational invariance $R(a,b)$ only depends on the difference $\phi = a - b$, while $R_1(a)$ and $R_2(b)$ are independent on a and b. The QM predictions for these quantities are

$$\begin{aligned} \frac{R(\phi)}{R_0} &= \frac{1}{4}(\epsilon_M^I + \epsilon_m^I)(\epsilon_M^{II} + \epsilon_m^{II}) + \frac{1}{4}(\epsilon_M^I - \epsilon_m^I)(\epsilon_M^{II} - \epsilon_m^{II})\, F(\theta)\cos 2\phi \\ \frac{R_1}{R_0} &= \frac{1}{2}(\epsilon_M^I + \epsilon_m^I), \qquad \frac{R_2}{R_0} = \frac{1}{2}(\epsilon_M^{II} + \epsilon_m^{II}) \end{aligned} \tag{29}$$

where ϵ_M^i is the efficiency of the polarizer i $(i = I, II)$ for light polarized parallel to the polarizer axis and ϵ_m^i is that for light perpendicularly polarized. The parameter θ is the half-angle characterizing the cone formed by the point source and the filter-detector assemblies, and ϕ the angle between polarizer axes. CHSH then argueed that special choices of the polarization vectors a, b, a', b' leads to a maximal violation of the inequality (21). To see this fact let us write (28) as

$$-1 \le S \equiv \frac{R(\vec{a},\vec{b}) - R(\vec{a},\vec{b'}) + R(\vec{a'},\vec{b}) + R(\vec{a'},\vec{b'}) - R_1(\vec{a'}) - R_2(\vec{b}))}{R_0} \le 0 \tag{30}$$

and choose the polarization vectors as in fig 10 , so that ϕ is the angle for the pairs $(\vec{a},\vec{b}), (\vec{a'},\vec{b}), (\vec{a'},\vec{b'})$, and 3ϕ for the pair $(\vec{a},\vec{b'})$. Eq.(30) becomes

$$-1 \le S(\phi) = \frac{3R(\phi) - R(3\phi) - R_1 - R_2}{R_0} \le 0 \tag{31}$$

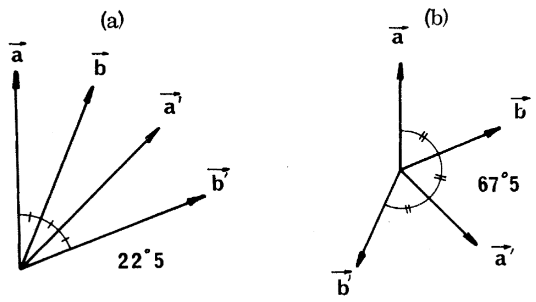

FIG. 10: Choice of the four polarization vectors in eq.(30) (fig. taken from[19])).

Let us take two examples of parameters entering in eqs.(29) and (31). Data A are real ones and were obtained by Aspect in 1981[19], while data B are speculative. The QM prediction for the function $S(\phi)$ are shown in fig.11. For the data A the inequality (31) is violated in four regions of the angle ϕ. The maximal violations takes place at the following angles

$$\phi = 22.5^\circ, \quad 67.5^\circ, \quad 112.5^\circ, \quad 157.5^\circ \tag{32}$$

which corresponds in radians to $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$. The data B however does not violate (31), although the maximal values of $|S(\phi)|$ occur at the angles (32). The later is a general result of the function $S(\phi)$. This example shows that the parameters entering in an experiment must be carefully choose, i.e. efficiencies must be good, in order to show a violation of Bell' s inequalities.

$$\begin{aligned} \text{Data}\, A: \ \epsilon_M^I &= 0.971, \quad \epsilon_m^I = 0.029, \quad \epsilon_M^{II} = 0.968, \quad \epsilon_m^{II} = 0.028, \quad F(\theta) = 0.984 \\ \text{Data}\, B: \ \epsilon_M^I &= 0.900, \quad \epsilon_m^I = 0.100, \quad \epsilon_M^{II} = 0.900, \quad \epsilon_m^{II} = 0.100, \quad F(\theta) = 0.900 \end{aligned} \tag{33}$$

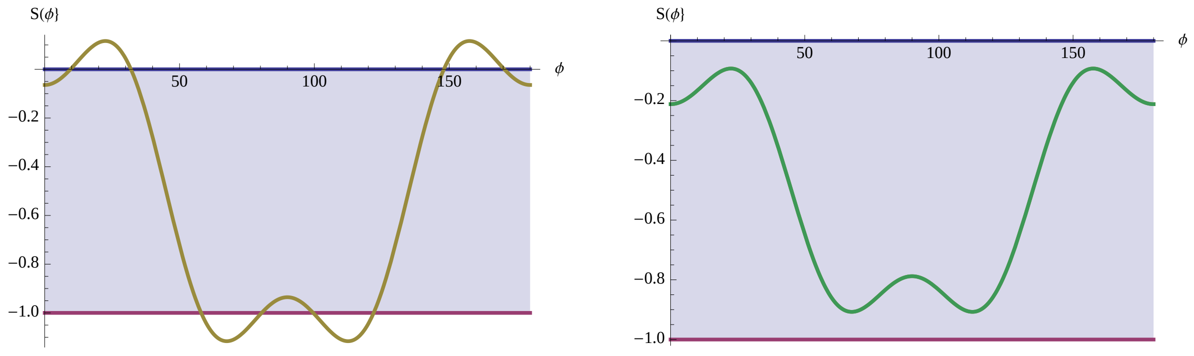

FIG. 11: The function $S(\phi)$ defined in eq.(31) is plotted for the data A (left) and data B (right) given in eq. (33).

VI. EXPERIMENTAL TESTS OF REALISTIC LOCAL THEORIES VIA BELL' S THEOREM (ASPECT ET AL (1981)[19–21]

In the years 1981-1982 Aspect and his group conducted three experiments which tested some version of the Bell inequalities with a very high precision. These were not the first experiments done on this problem but they were very conclusive in those days.

First experiment: one channel

The first experiment tested the CHSH inequality derived above[?] . The experimental set up was the calcium cascade $4p^2\,{}^1S_0 \to 4s4p\,{}^1P_1 \to 4s^2\,{}^1S_0$. This cascade yields two visible photons correated in polarization with frequencies $\nu_1 = 551.3$ nm and $\nu_2 = 422.7$ nm (see fig 12. The efficiencies of the polarizers are given by data A in eq.(33). The polarizers only transmit photons of a given polarization while the photons with the other polarization are lost.

The coincidence rate $R(\phi)$ obtained experimentally agree with the QM prediction (see fig. 14). The experimental and QM theoretical values of the quantity $S(\phi)$ defined in eq. (31)

$$S_{\rm exp}(22.5^\circ) = 0.126 \pm 0.014, \qquad S_{\rm QM}(22.5^\circ) = 0.118 \pm 0.005 \tag{34}$$

This emplies that the CHSH inequality is violated by 9 standard deviations $(0.126/0.014 = 9)$ and that it agrees with the QM prediction.

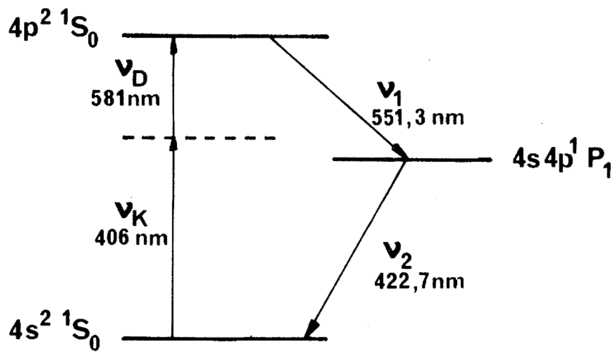

FIG. 12: Energy levels of calcium used in Aspect experiment in 1981. The atoms are pumped to the upper levels by nonlinear absorption of photons with frequencies ν_K, ν_D, and emits the photons with frequencies ν_1, ν_2 correlated in polarizations. Fig taken from[19].

A simpler form of the CHSH identity is[15]

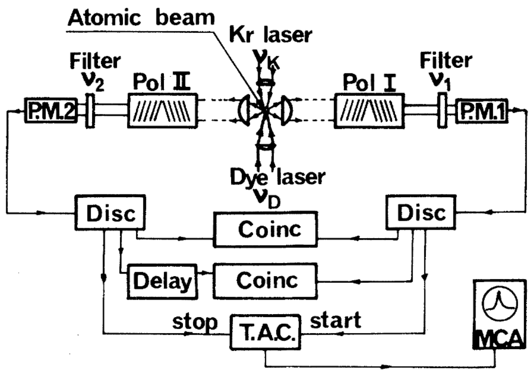

FIG. 13: Schema of the apparatus and electronics of Aspect first experiment. Fig taken from[19].

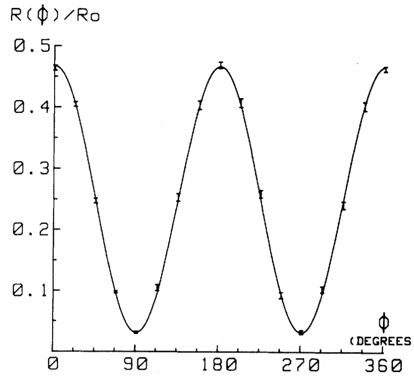

FIG. 14: Normalized coincidence rate as a function of the relare polarizer orientation. Tne solid curve id the theoretical prediction. Fig taken from[19].

$$\delta \equiv \frac{|R(22.5^\circ) - R(67.5^\circ)|}{R_0} - \frac{1}{4} \leq 0 \tag{35}$$

To derive this eq. one writes eq.(31) in two ways

$$\begin{aligned} -1 &\leq \tfrac{3R(22.5^\circ) - R(67.5^\circ) - R_1 - R_2}{R_0} &\leq 0 \\ 0 &\leq \tfrac{-3R(67.5^\circ) - R(22.5^\circ) + R_1 + R_2}{R_0} &\leq 0 \end{aligned} \tag{36}$$

Addig these two eqs. we get (35). The experimental and QM values are given by

$$\delta_{\rm exp} = 5.72 \times 10^{-2} \pm 0.43 \times 10^{-2}, \qquad \delta_{\rm QM} = 5.8 \times 10^{-2} \pm 0.2 \times 10^{-2}$$

which are in full agreement with QM. This result violates the inequality (35) by more than 13 standard deviations (i.e. $5.72/0.43 = 13.2$).

Second experiment: two channel

In a second experiment Aspect' s group tested another CHSH identity using two-channel polarizers instead of one-channel as in the first experiment. These polarizers distinguish the polarization of the photons and in this sense they are the optical analog of the Stern-Gerlach filters. Now one can measrue the probabilities $P_{\pm\pm}(\vec{a}, \vec{b})$ of obtaining the result ± 1 along the direction $\vec{a}$ (particle 1) and ± 1 along the direction $\vec{b}$ (particle 2). The quantity

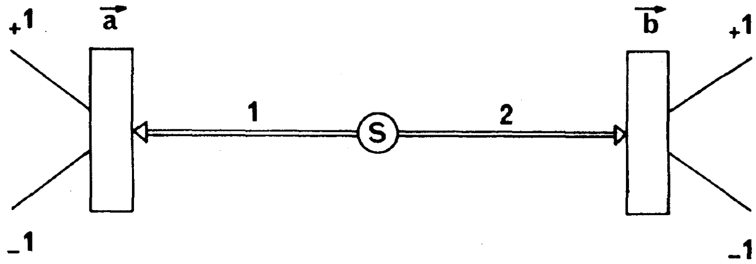

FIG. 15: EPR gedanken experiment with two channels. The spin (or polarization components) of 1 and 2 are measured along $\vec{a}$ and $\vec{b}$.

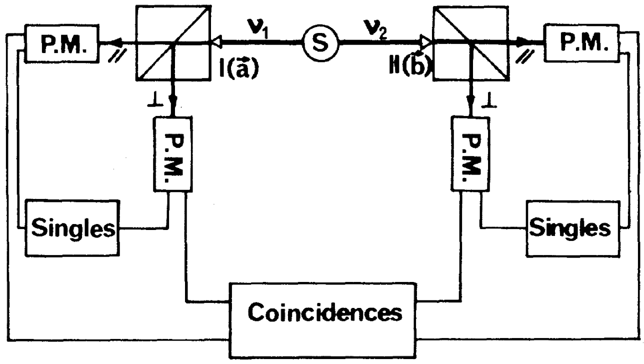

FIG. 16: Two-channel set up of EPR experiment. The two polarimeters I and II, in orientations $\vec{a}$ and $\vec{b}$, perform true dichotomic measurements of linear polarizations of photons ν_1 and ν_2. The counting electronics monitors the singles and the coincidences.

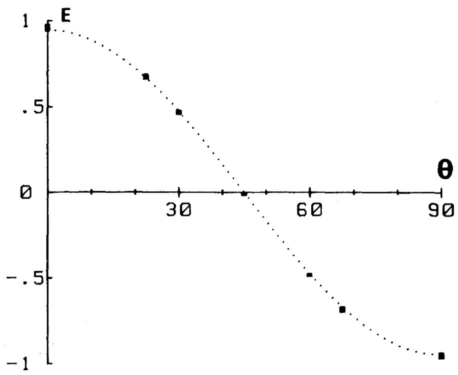

FIG. 17: Correlation $E(\vec{a}, \vec{b})$ as a function of the relative angle of polarimeters. The dotted curve is the QM prediction.

$$E(\vec{a}, \vec{b}) = P_{++}(\vec{a}, \vec{b}) + P_{--}(\vec{a}, \vec{b}) - P_{+-}(\vec{a}, \vec{b}) - P_{-+}(\vec{a}, \vec{b})$$

is the correlation coefficient of the measurements of the two particles, which satisfies the inequality

$$-2 \leq S \leq 2$$

where

$$S = E(\vec{a},\vec{b}) - E(\vec{a},\vec{b'}) + E(\vec{a'},\vec{b}) + E(\vec{a'},\vec{b'})$$

The maximal QM departure is $S = \pm 2\sqrt{2}$. The experimental determination of $E(\vec{a},\vec{b})$ is obtained using the four coincidence rates $R_{\pm\pm}(\vec{a},\vec{b})$

$$E(\vec{a},\vec{b}) = \frac{R_{++}(\vec{a},\vec{b}) + R_{--}(\vec{a},\vec{b}) - R_{+-}(\vec{a},\vec{b}) - R_{-+}(\vec{a},\vec{b})}{R_{++}(\vec{a},\vec{b}) + R_{--}(\vec{a},\vec{b}) + R_{+-}(\vec{a},\vec{b}) + R_{-+}(\vec{a},\vec{b})}$$

The theoretical and experimental values of S at the usual angles $22.5^\circ, 67.5^\circ$ are given by

$$S_{\text{exp}} = 2.697 \pm 0.01 \qquad S_{\text{QM}} = 2.70 \pm 0.05$$

The violation of the inequality ois 83% of the maximal violation predicted by QM (i.e. $100 \times (2.697-2)/(2\sqrt{2}-2) = 84.$)

Third experiment: Time varying analyzers

Finally, in the third Aspect' s experiment the new element of the experimental set up was to change the direction of the polarizers during the time of flyi of the photons from the source to the detectors. In all previous experiments by Aspect and others the configuration were static. This left open the possibility for some sort of interaction between the polarizers. If the configuration is change during the flight then the locality conditions impose by Bell to derive his inequalities will become a consequence of Einstein' causality, preventing any faster-than-light influence. The CHSH inequality $S \leq 0$ was violated by 5 standard deviations.

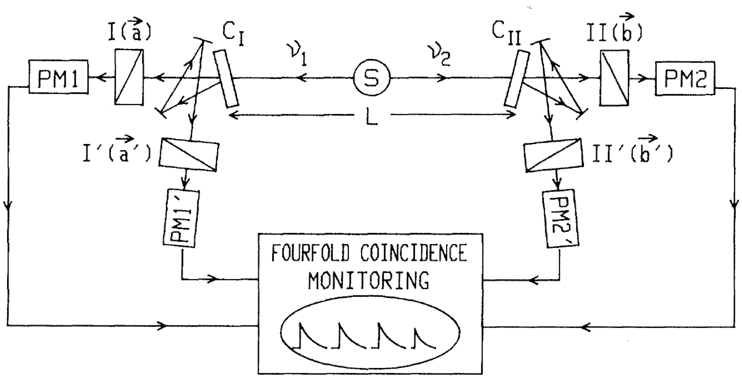

FIG. 18: Timing experiment with optical switches

VII. THE CANARY ISLANDS EXPERIMENT

A long distance entanglement experiment was performed by Ursin et al in 2007[22] between the Canary island of La Palma and Tenerife. An entangled pair was generated in La Palma island where one of the photons was measured. The other one was sent via an optical free-space link to Tenerife, 144 Kms away, where the Optical Ground Station of the ESA acted as receiver.

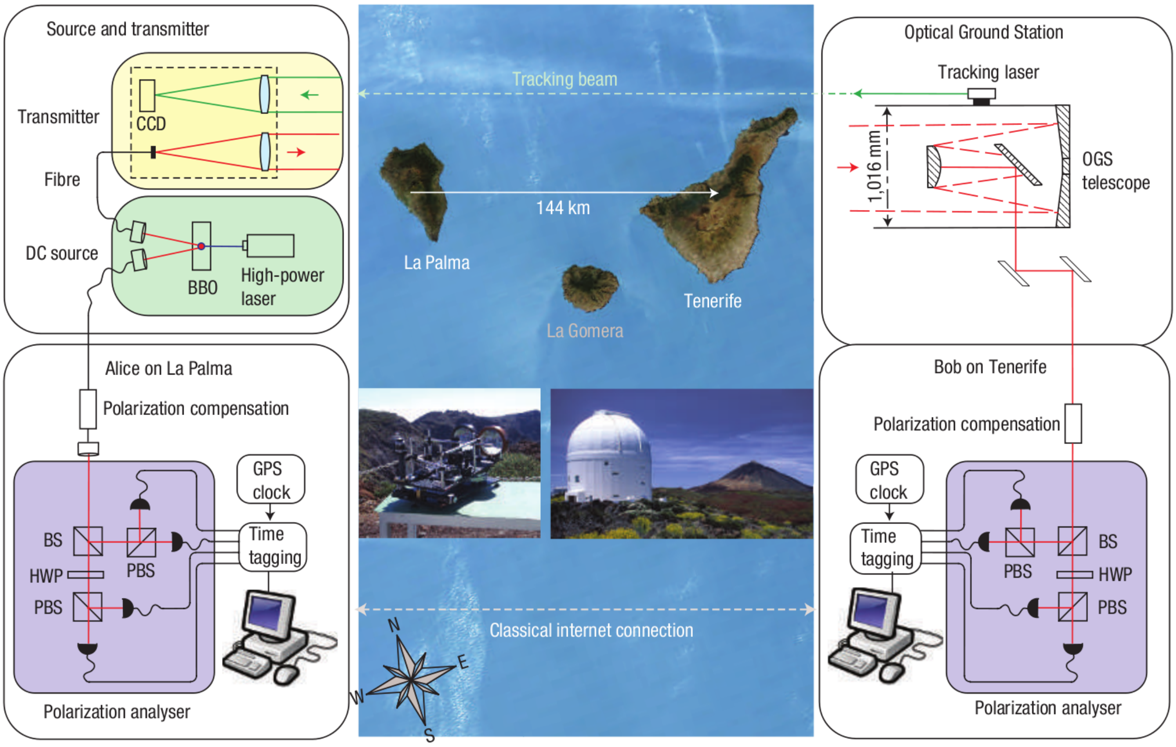

FIG. 19: Picture taken from[22]

VIII. TEST OF NON-LOCAL REALISM

Physical realism suggests that the results of observations are a consequence of properties carried by physical systems. The original experiments by Aspect at al. were refined progressively in the last two decades in order to take into account the so called ”detection” and ”locality” loopholes. All the experiments performed so far have close these loopholes which implies that the violation of the local realism is a well established fact. Hence a possible alternative to hidden variable theories must violate one of these two asumptions. One of them is due to Legget who has proposed a non-local realistic theory[23]. That theory leads to a new type of inequalities involving a more complex combination of polarizations between Alice and Bob detectors. Legget-type inequalities were tested experimentally in reference[24], showing their violation and confirming the validity of Quantum Mechanics.

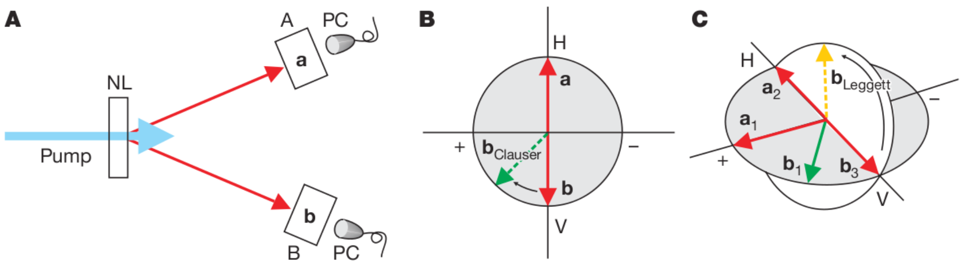

FIG. 20: (A) Experimental set up to test hidden-variables theories. Measurements directions to test (B) Bell inequalities and (C) Legget-type inequalities. Picture taken from[24]

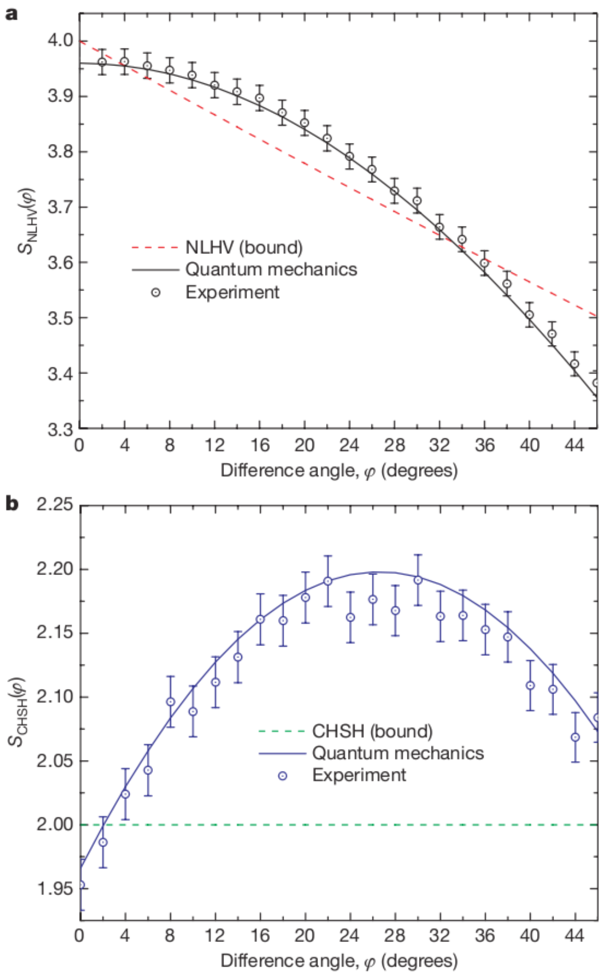

FIG. 21: Experimental violation of inequalities for non-local hidden variable theories. Top: Legget.-type inequality, Bottom: CHSH inequality. Picture taken from[24]

IX. BEYOND BELL' S THEOREM: GHZ (1989)

In 1989 Greenberg, Horne and Zeilinger wrote a short proceedings paper entitled "Going beyond Bell's theorem" where they show that the violation of the EPR arguments can be stronger for quantum states involving 3 or more particles[25]. GHZ start by reviewing the Bell' s theorem observing that the Bell' s inequality says nothing when two polarization vectors are parallel, i.e.

$$1 + P(\vec{b}, \vec{c}) \geq |P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \stackrel{\vec{a}=\vec{b}}{\Longrightarrow} 1 + P(\vec{a}, \vec{c}) \geq |1 + P(\vec{a}, \vec{c})|$$

where one has used that $P(\vec{a}, \vec{a}) = -1$. Indeed for this case Bell constructed in his 1964 paper a hidden variable model which yields the same results as the quantum model for this situation. This case is called by GHZ superclassical and its existence is what lead Bell to consider inequalities to show the departure of QM from the classical EPR ideas. In a later publication, which includes Shimony, there is a more extended version of this idea[26], which was also discussed by Mermin[27]. In reference[?] it is said:

Unlike Bell's original theorem and variants of it, GHZ's demostration of the incompatibility of quantum mechanics with EPR' s propositions concerns only perfecf correlations rather that statistical correlations, and completely dispenses with inequalities.

The GHSZ paper first summarize the EPR argument using the same terminology as in 1935. The basic premisses are:

(i) *Perfect correlation*: If the spins of particles 1 and 2 are measured along the same direction, then with certainty the outcomes will be found to be opposite.

(ii) *Locality:* "Since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system".

(iii) *Reality:* "If, without in any way disturbing a system, we can predict with certainty (i.e. probability equal to unity) the value of a physical quantity , then there exists an element of physical reality corresponding to this physical property".

(iv) *Completeness:* "Every element of the physical reality must have a counterpart in the [complete] physical theory".

Next GHSZ review the EPR argument as follows:

1) Measure a component of the spin of particle 1, in a certain direction.

2) By perfect correlation (i), we can predict with certainty the result of measuring the same component of the spin for particle 2.

3) By locality (ii), the measurement on particle 1 cannot cause any real change on particle 2.

4) By reality (iii), the chosen spin component of particle 2 is an element of the physical reality.

5) This argument is valid for any component of the spin, hence all the components are elements of physical reality.

6) There is no quantum state of a spin 1/2 particle in which all the components have a definite value.

7) By (iv) quantum mechanics cannot be a complete theory, at least for a spin 1/2 particle in a singlet state. There are elements of physical reality for which quantum mechanics has no counterpart.

The condition (i) is consistent with conditions (ii-iv) for the spin 1/2 state. However they show next and example of perfect correlation which is not compatible with conditions (ii-iv). The state considered by GHZ is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle_1 |+\rangle_2 |-\rangle_3 |-\rangle_4 - |-\rangle_1 |-\rangle_2 |+\rangle_3 |+\rangle_4\right] \tag{37}$$

Let now place Ster-Gerlach analyzers to measure the spin component along the directions $\vec{n}_i$ $(i = 1, \ldots, 4)$. As in the spin 1/2 case one defines the expectation value of the of the product of the outcome of these measurements, whose QM value is given by

$$E^{\Psi}(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4) = \cos\theta_1\, \cos\theta_1\, \cos\theta_1\, \cos\theta_1 - \sin\theta_1\, \sin\theta_2\, \sin\theta_3\, \sin\theta_4 \times \cos(\phi_1 + \phi_2 - \phi_3 - \phi_4)$$

where (θ_i, ϕ_i) are the polar and azimuthal angles of $\vec{n}_i$. For simplicity it is enough to consider the vectors $\vec{n}_i$ to belong to the $x - y$ axis,

$$E^{\Psi}(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4) = -\cos(\phi_1 + \phi_2 - \phi_3 - \phi_4)$$

There are two cases of perfect correlation,

$$\text{If } \phi_1 + \phi_2 - \phi_3 - \phi_4 = 0 \implies E^{\Psi}(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4) = -1 \tag{38}$$

$$\text{If } \phi_1 + \phi_2 - \phi_3 - \phi_4 = \pi \implies E^{\Psi}(\vec{n}_1, \vec{n}_2, \vec{n}_3, \vec{n}_4) = +1 \tag{39}$$

Now the EPR's premises can be adapted as follows

(i) *GHZ-Perfect correlation* : If the Stern-Gerlach analyzers are set to the angles satisfying (38) or (47), then knowledge of the outcomes for three particles determines a prediction with certainty of the ourtcome of the fourth.

(ii) *GHZ- Locality:* "Since at the time of measurement the four particles presumably do not interact, no real change can take place in any one of them in consequence of anything that may be done on the other three".

(iii) *Reality:* As in EPR

(iv) *Completeness:* as in EPR

At this stage one would like to reproduce this perfect correlated situation using a hidden variable theory. This was possible for the spin 1/2 case, so let' s see if it is possible for the four spin case. Following Bell' s approach, one introduces four functions $A_\lambda(\phi_1), B_\lambda(\phi_2), C_\lambda(\phi_3), D_\lambda(\phi_4)$ describing the outcomes of the four possible measurements of the spins along the directions ϕ_i and a generic hidden variable λ. The premisse (i) GHZ is satisfied provided

$$\text{If } \phi_1 + \phi_2 - \phi_3 - \phi_4 = 0 \implies A_\lambda(\phi_1)B_\lambda(\phi_2)C_\lambda(\phi_3)D_\lambda(\phi_4) = -1 \tag{40}$$

$$\text{If } \phi_1 + \phi_2 - \phi_3 - \phi_4 = \pi \implies A_\lambda(\phi_1)B_\lambda(\phi_2)C_\lambda(\phi_3)D_\lambda(\phi_4) = +1 \tag{41}$$

Now one can show that these two eqs. are imposible to satisfy by any set of function A, B, C, D. Consider four instances of these eqs.

$$A_\lambda(0)B_\lambda(0)C_\lambda(0)D_\lambda(0) = -1 \tag{42}$$
$$A_\lambda(\phi)B_\lambda(0)C_\lambda(\phi)D_\lambda(0) = -1 \tag{43}$$
$$A_\lambda(\phi)B_\lambda(0)C_\lambda(0)D_\lambda(\phi) = -1 \tag{44}$$
$$A_\lambda(2\phi)B_\lambda(0)C_\lambda(\phi)D_\lambda(\phi) = -1 \tag{45}$$

From eqs. (42) and (43) one gets

$$A_\lambda(\phi)C_\lambda(\phi) = A_\lambda(0)C_\lambda(0)$$

and from the (42)and (44)

$$A_\lambda(\phi)D_\lambda(\phi) = A_\lambda(0)D_\lambda(0)$$

Combining these two eqs. one gets

$$C_\lambda(\phi)/D_\lambda(\phi) = C_\lambda(0)/D_\lambda(0)$$

and using that $D_\lambda(\phi) = \pm 1$

$$C_\lambda(\phi)D_\lambda(\phi) = C_\lambda(0)D_\lambda(0)$$

Taking eq. (45)

$$A_\lambda(2\phi)B_\lambda(0)C_\lambda(0)D_\lambda(0) = -1$$

which combined with (42)

$$A_\lambda(2\phi) = A_\lambda(0), \qquad \forall\phi \tag{46}$$

This eq. is already strange for a physics point of view because it says that the result of the measurement of $A_\lambda(\phi)$ is independent on the direction. The full contradiction is brought about when considering another instance of eq.(41)

$$A_\lambda(\theta+\pi)B_\lambda(0)C_\lambda(\theta)D_\lambda(0) = 1$$

which together with (43) yields

$$A_\lambda(\theta+\pi) = -A_\lambda(\theta), \qquad \forall\phi$$

which cannot be satisfied since according to (46) $A_\lambda(\phi)$ is constant. This shows that premises (i-iv) does not hold for the 4 particles example. GHZ arguee that the same holds for 3 particles since the argument of $B_\lambda(\phi_2)$ is kept fixed.

The most significant feature of the new argument is that the EPR program cannot handle even perfect correlations for systems with 3 or more particles.

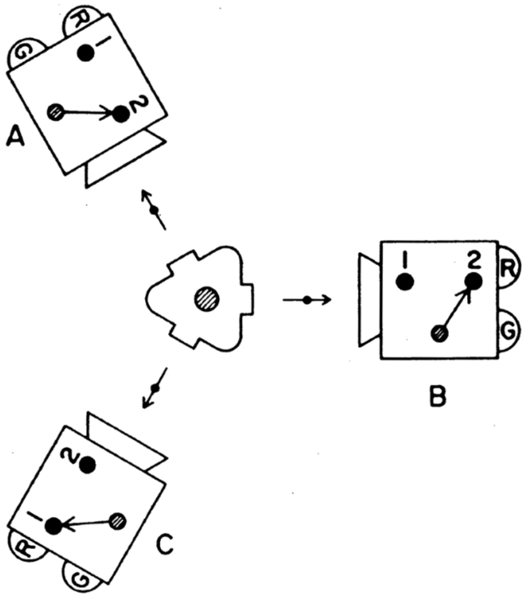

FIG. 22: Mermin gendanken experiment of the GHZ state

A. Mermin picture of GHZ

In 1990 Mermin gave a suggestive version of the GHZ state using the set up shown in fig. ??. At the center of the system one produces a certain state whose properties are specified below. The devices A, B, C have a switch which can be set to two possible positions 1 or 2. Each device measures the state of the particle arriving on it and the result flashes a light which can be green (G) or red (R).

There are 8 possible experimental set ups given by the 2^3 positions of the switches. Among them we shall only consider the ones given by $111, 122, 212, 221$.

Now the state produced at the center of the experiment satisfies the following rules:

- Rule 1: For the cases 122, 212, 221, the number of red lights is odd, namely: RRR, RGG, GRG, GGR.
- Rule 2: In the case 111, the number of red lights is even, namely: GGG, RRG, RGR, GRR

Rule 1 implies that the knowledge of two outputs yields with certainty the result on the third one. Indeed one has

$$RR \to R, \qquad GG \to R, \qquad GR, RG \to G$$

This is an example of perfect correlation analogous to the EPR gedanken experiment. Following EPR, we might think that each particle carries a set of instructions which gives the the outcome of a measurement. There are four possible set of instructions per particle

$$\begin{pmatrix} R \\ R \end{pmatrix}, \begin{pmatrix} G \\ G \end{pmatrix}, \begin{pmatrix} R \\ G \end{pmatrix}, \begin{pmatrix} G \\ R \end{pmatrix}$$

where the upper (resp. lower) component gives the outcome if the switch is set to 1 (resp. 2). For three particles there are a total of $64 = 2^6$ instructions, but only 8 satisfy the Rule 1, namely

$$\begin{pmatrix} R & R & R \\ R & R & R \end{pmatrix}, \begin{pmatrix} R & G & G \\ R & G & G \end{pmatrix}, \begin{pmatrix} G & R & G \\ G & R & G \end{pmatrix}, \begin{pmatrix} G & G & R \\ G & G & R \end{pmatrix}$$
$$\begin{pmatrix} R & R & R \\ G & G & G \end{pmatrix}, \begin{pmatrix} R & G & G \\ G & R & R \end{pmatrix}, \begin{pmatrix} G & R & G \\ R & G & R \end{pmatrix}, \begin{pmatrix} G & G & R \\ R & R & G \end{pmatrix}$$

The trouble appears when considering Rule 2. Since the number of R' s on the upper row of these matrices is always odd we see that the Rule 2 is never satisfied. This proof in a very simple case that the perfect correlations for this state cannnot be supported by a hidden variable theory.

It remains to show that this GHZ state really corresponds to a Quantum Mechanical state. This state is give by three spin 1/2 particles

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (| \uparrow\uparrow\uparrow\rangle - | \downarrow\downarrow\downarrow\rangle) \tag{47}$$

The switches 1 and 2 of each device correspond to the Pauli matrices σ_x and σ_y respectively.

$$\text{switch } 1 \leftrightarrow \sigma_x \qquad \text{switch } 2 \leftrightarrow \sigma_y$$

while the G and R outocomes correspond to the measurements 1 and -1 respectively.

Using the properties

$$\sigma_x \,| \uparrow\rangle = | \downarrow\rangle, \qquad \sigma_x \,| \downarrow\rangle = | \uparrow\rangle, \qquad \sigma_y \,| \uparrow\rangle = i| \downarrow\rangle, \qquad \sigma_y \,| \downarrow\rangle = -i| \uparrow\rangle$$

One can easily show that

$$\sigma_x \,\sigma_y \,\sigma_y \,|GHZ\rangle = \sigma_y \,\sigma_x \,\sigma_y \,|GHZ\rangle = \sigma_y \,\sigma_y \,\sigma_x \,|GHZ\rangle = |GHZ\rangle$$

Hence the GHZ state diagonalizes simultaneously the operators which are in one-to-one correspondence with the sep ups 122, 212, 221. Since their eigenavalue is always one it means that the measurement always gives an even number of -1' s, hence and odd number of 1' s, namely R flashes. This is the rule 1. However for the operator associated to the set up 111, one obtains

$$\sigma_x \,\sigma_x \,\sigma_x \,|GHZ\rangle = -|GHZ\rangle$$

whice means that the number of -1 's measures by each of the σ_x aparatus is odd, namely an even number of R flashes, which is nothing but the Rule 2. Mermin emphasizes the simplicity and beauty of the anticommuting nature of the Pauli matrices as the source of the impossibility of the existence of an set of EPR instruction set.

Finally we shall mention as a curiosity the analogy made by Aravind between the GHZ state and the Borromean knot (see fig?). In this knot three circles are linked but not two of them.

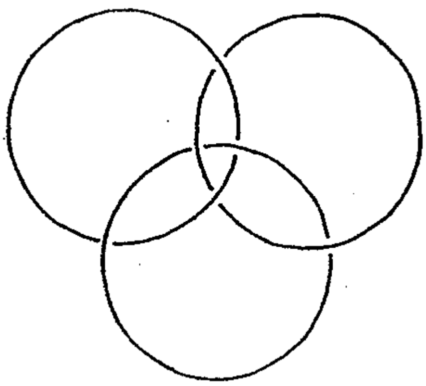

FIG. 23: GHZ figurative representation using the Borromean ring

X. NO CLONING THEOREM 1982

In 1982 Wootters and Zurek published a one page paper in Nature entitled "A single quantum cannot be cloned"[29]. This paper starts from the consideration of a photon with a precise polarization which encounters an excited atom which can then emit a second photon with the same polarization by stimulated emission. The authors then ask wether this or another process could be used to amplify a quantum state, that is to produce several copies of it. This problem was triggered by a recent paper which suggest that possibility which would then allow for faster-than-light communication[?] (see reference[?] for an interesting story concering these papers).

The proof of the non-cloning theorem is extremely simple. Suppose that system A, has a quantum state $|\psi\rangle_A$, that one wants to copy. For that purpose we introduce another system B in an initial state $|e\rangle_B$. A copy of the quantum state ψ amounts to the operation

$$|\psi\rangle_A\,|e\rangle_B \to |\psi\rangle_A\,|\psi\rangle_B$$

which copies the state ψ in the auxiliary system B. In Quantum Mechanics this transformation should be represented by a linear, and in fact, unitary operator, i.e.

$$U\;|\psi\rangle_A\,|e\rangle_B = |\psi\rangle_A\,|\psi\rangle_B$$

For only one state ψ there is no problem in constructing such a map. The problem arises when one tries to copy another state, say ϕ, i.e.

$$U\;|\phi\rangle_A\,|e\rangle_B = |\phi\rangle_A\,|\phi\rangle_B$$

Using the unitarity of U one gets

$$_B\langle e|\,_A\langle\phi|U^\dagger\,U|\psi\rangle_A\,|e\rangle_B = \langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2$$

hence

$$\langle\phi|\psi\rangle = 0 \quad \text{or} \quad 1$$

So the two states are either identical or orthogonal. However if ψ and ϕ are not orthogonal they cannot be copied by the same operation U. The reason for this no go theorem is the linearity of the unitary transformation. To see this more clearly let us consider a two dimensional quantum system with a orthonormal basis $|0\rangle, |1\rangle$. One can certainly find a U which doubles these two states, i.e.

$$\begin{aligned} U\;|0\rangle_A\,|e\rangle_B &= |0\rangle_A\,|0\rangle_B \\ U\;|1\rangle_A\,|e\rangle_B &= |1\rangle_A\,|1\rangle_B \end{aligned}$$

and this is equivalent to a classical copying machine. Let us now try to copy a generic state $a|0\rangle + b|1\rangle$, i.e.

$$U\;(a|0\rangle_A + b|1\rangle_A)\,|e\rangle_B = (a|0\rangle_A + b|1\rangle_A)(a|0\rangle_B + b|1\rangle_B)$$

by the linearity of U this is

$$a\,U\;|0\rangle_A\,|e\rangle_B + b\,U\;|1\rangle_A\,|e\rangle_B = a|0\rangle_A|0\rangle_B + b|1\rangle_B)|1\rangle_B)$$

The RHS of these two eqs. only agree provided $ab = 0$, thus a generic quantum state of the 1 qubit system cannot be copied.

The consequences of this simple but profound theorem are far reaching:

- It yields a sufficient (but not necessary) condition for the impossibility of superluminal communication via quantum entanglement. To show this let us suppose that Alice and Bob are located at a very large space-time distance and that they shared a EPR pair. Alice wants to send a classical bit to Bob according the the following protocol. To send a 0, Alice will do nothing to her qubit.

XI. TELEPORTATION

In 1993 Bennet et al proposed a protocol to teletransport an unknown quantum state from Alice to Bob using a classical communication channel and a EPR pair[32]. The name teleportation is inspired in Science fiction movies. At the beginning, Alice has two particles, denoted 1 and 2, and Bob has one particle denoted 3. Particle 1 is in the unknown quantum state $|\phi\rangle_1$, while particles 2 and 3 are in entangled EPR state $|\psi^{(-)}\rangle_{23}$. All the one particle states are two level systems, hence one can write

$$|\phi\rangle_1 = a\,|\uparrow\rangle_1 + b\,|\downarrow\rangle_1, \qquad |a|^2 + |b|^2 = 1 \tag{48}$$

It is convenient to choose the so called Bell basis for the 4 dimensional Hilbert space of the particles 2 and 3,

$$\begin{aligned} |\psi^{(\pm)}\rangle_{23} &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_2|\downarrow\rangle_3 \pm |\downarrow\rangle_2|\uparrow\rangle_3\right) \\ |\phi^{(\pm)}\rangle_{23} &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_2|\uparrow\rangle_3 \pm |\downarrow\rangle_2|\downarrow\rangle_3\right) \end{aligned} \tag{49}$$

whose inverse is given by

$$\begin{aligned} |\uparrow\rangle_2|\downarrow\rangle_3 &= \frac{1}{\sqrt{2}}\left(|\psi^{(+)}\rangle_{23} + |\psi^{(-)}\rangle_{23}\right) \\ |\downarrow\rangle_2|\uparrow\rangle_3 &= \frac{1}{\sqrt{2}}\left(|\psi^{(+)}\rangle_{23} - |\psi^{(-)}\rangle_{23}\right) \\ |\uparrow\rangle_2|\uparrow\rangle_3 &= \frac{1}{\sqrt{2}}\left(|\phi^{(+)}\rangle_{23} + |\phi^{(-)}\rangle_{23}\right) \\ |\downarrow\rangle_2|\downarrow\rangle_3 &= \frac{1}{\sqrt{2}}\left(|\phi^{(+)}\rangle_{23} - |\phi^{(-)}\rangle_{23}\right) \end{aligned} \tag{50}$$

The initial state of the three particles is given by

$$\begin{aligned} |\phi\rangle_1\,|\psi^{(-)}\rangle_{23} &= \frac{a}{\sqrt{2}}\left(|\uparrow\rangle_1|\uparrow\rangle_2|\downarrow\rangle_3 - |\uparrow\rangle_1|\downarrow\rangle_2|\uparrow\rangle_3\right) \\ &+ \frac{b}{\sqrt{2}}\left(|\downarrow\rangle_1|\uparrow\rangle_2|\downarrow\rangle_3 - |\downarrow\rangle_1|\downarrow\rangle_2|\uparrow\rangle_3\right) \end{aligned}$$

which using the relations (50) becomes

$$\begin{aligned} |\phi\rangle_1\,|\psi^{(-)}\rangle_{23} &= \frac{a}{2}\left(|\phi^{(+)}\rangle_{12} + |\phi^{(-)}\rangle_{12}\right)|\downarrow\rangle_3 - \frac{a}{2}\left(|\psi^{(+)}\rangle_{12} + |\psi^{(-)}\rangle_{12}\right)|\uparrow\rangle_3 \\ &+ \frac{b}{2}\left(|\psi^{(+)}\rangle_{12} - |\psi^{(-)}\rangle_{12}\right)|\downarrow\rangle_3 - \frac{b}{2}\left(|\phi^{(+)}\rangle_{12} - |\phi^{(-)}\rangle_{12}\right)|\uparrow\rangle_3 \end{aligned}$$

Collecting terms one finally gets

$$|\phi\rangle_1\,|\psi^{(-)}\rangle_{23} = \frac{1}{2}\Big[|\psi^{(-)}\rangle_{12}\left(-a|\uparrow\rangle_3 - b|\downarrow\rangle_3\right) + |\psi^{(+)}\rangle_{12}\left(-a|\uparrow\rangle_3 + b|\downarrow\rangle_3\right) \tag{51}$$

$$+\, |\phi^{(-)}\rangle_{12}\left(a|\downarrow\rangle_3 + b|\uparrow\rangle_3\right) + |\phi^{(+)}\rangle_{12}\left(a|\downarrow\rangle_3 - b|\uparrow\rangle_3\right)\Big] \tag{52}$$

So far, one has simply written the initial Alice-Bob state in a convenient basis. Observe that each of the Bell states, of the particles 1 and 2, appear with the same probability, i.e. 1/4. In the next step, Alice measures with a device that distinguishes which of these four Bell states is obtained. Let us suppose that Alice obtains the state $|\psi^{(-)}\rangle_{12}$ in her aparatus, hence the collapse of the wave function yields the state

$$|\phi\rangle_1\,|\psi^{(-)}\rangle_{23} \to |\psi^{(-)}\rangle_{12}\left(-a|\uparrow\rangle_3 - b|\downarrow\rangle_3\right) = -|\psi^{(-)}\rangle_{12}\,|\phi\rangle_3$$

leaving Bob with a state $|\ phi\rangle_3$, which is identical to the original Alice' s state, except for a unobservable minus sign. The other possible outcomes of Alice' s measurements yield the following states after the collapse

$$\begin{aligned} |\phi\rangle_1 \, |\psi^{(-)}\rangle_{23} &\to |\psi^{(+)}\rangle_{12} \, (-a|\uparrow\rangle_3 + b|\downarrow\rangle_3) = -|\psi^{(-)}\rangle_{12} \, \sigma_3^z |\phi\rangle_3 \\ &\to |\phi^{(-)}\rangle_{12} \, (a|\downarrow\rangle_3 + b|\uparrow\rangle_3) = |\psi^{(-)}\rangle_{12} \, \sigma_3^x \, |\phi\rangle_3 \\ &\to |\phi^{(+)}\rangle_{12} \, (a|\downarrow\rangle_3 - b|\uparrow\rangle_3) = i|\psi^{(-)}\rangle_{12} \, \sigma_3^y |\phi\rangle_3 \end{aligned}$$

Now Bob' s state is not identical to Alice' s state but he can recover it by applying a unitary transformations $\sigma^{x,y,z}$ to the state of particle 3 after the collapse. To do so Alice tells Bob which state she has obtained in her measurement through a classical communication channel, involving two classical bits, which can be taken as $00, 01, 10, 11$. Notice that Alice loose any trace of the original quantum state. Also the EPR state is destroyed after the measurement. A space-time diagram of the teleportation process is shown in fig. **??**.

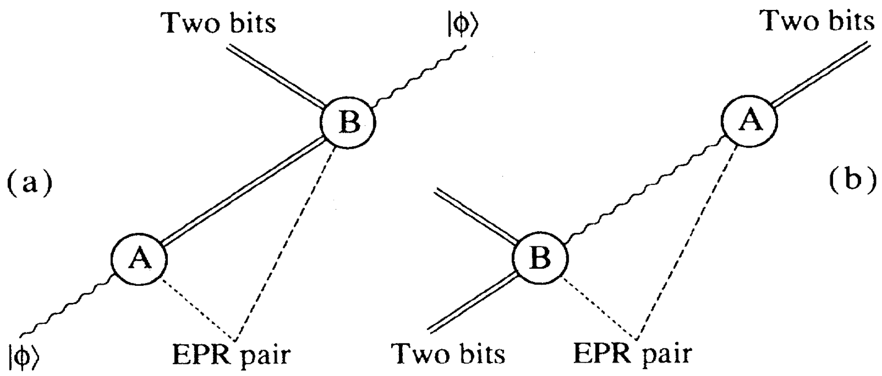

FIG. 24: Space-time representation of (a) teleportation (left) and (b) 4-way coding. Fig. taken from citeTele93

The generalization to quantum states of dimension N is straightforward. Let $|j\rangle$ $(j = 1, 2, \ldots, N$ be a local basis for single particle states. A generalized EPR state is given by

$$|\psi\rangle_{12} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle_1 \, |j\rangle_2$$

while a basis for Alice' s measurements is

$$|\psi_{n,m}\rangle_{12} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi i j n/N} |j\rangle_1 \, |\, (j+m) \bmod N\rangle_2$$

Bob' s unitary transformation is given by the two integers (n, m)

$$U_{nm} = \sum_{k=1}^{N} e^{2\pi i k n/N} |k\rangle_1 \, {}_2\langle \, (k+m) \bmod N|$$

Is it necessary to use a EPR state for teleportation? Supppose Alice and Bob share the product state $|\phi\rangle_2 \, |\psi\rangle_3$. One can cleary see that manipulation of particle 2 does not yield any effect on particle 3. More general one can show that teleportation can be achieve if the entangled state has the generic form

$$|\Upsilon\rangle_{23} = \frac{1}{\sqrt{2}} \left(|u\rangle_2 \, |p\rangle_3 + |v\rangle_2 \, |q\rangle_3 \right)$$

where $|u, v\rangle$ is an orthonormal basis for the particle 2 and $|p, q\rangle$ an orthornormal basis for particle 3. By means of local unitary operations one can bring this state into one of the Bell state basis, $|\psi^{(\pm)}\rangle_{23}, |\phi^{(\pm)}\rangle_{23}$.

States with less entanglement reduce the *fidelity* of the transformation of the teleportation or the range of states that can be teleported. This implies that

Maximal entanglement is a necessary and sufficient condition for faithful teleportation

One cannnot use teleportation to transmit signals above the speed of light. Both' s recovery of Alice' s state requires classical communication which satisfies special relativity. On the other hand Alice does not need to know Bob' s position to teleport her state. She can just broadcast the result of her measurement to the region where Bob is supposed to be.

XII. FOUR WAY CODING

Prior to the famous teleportation paper, Bennet and Wiesner wrote showed that a EPR state can be used to encode 2 classical bits[33]. This result is also discussed in the teleportation paper[32].

Consider the usual EPR state shared by Alice and Bob

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Now let Bob apply a unitary transformation

$$\begin{aligned}
\sigma_B^z|\psi\rangle &= \frac{1}{\sqrt{2}}\,(-|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\
\sigma_B^x|\psi\rangle &= \frac{1}{\sqrt{2}}\,(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) \\
\sigma_B^y|\psi\rangle &= \frac{-i}{\sqrt{2}}\,(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)
\end{aligned} \tag{53}$$

The four possible choices of Bob, i.e. $1, \sigma^x, \sigma^y, \sigma^z$ amounts to two classical bits, which are encoded into the EPR state via the aforementioned transformations. After this is done, Bob' s send his particle to Alice who can measure the Bell states of the two particles, for where she can reconstruct Bob's transformation. This shows that 1 qbit is equivalent to 2 classical bits.

XIII. MEASURES OF ENTANGLEMENT: VON NEUMANN ENTROPY

Let us consider a pure quantum state $\psi\rangle$ of a system that we divide into two parts A and B. The total Hilbert space of the system $\mathcal{H}$ is the tensor product of the Hilbert spaces of the parts, i.e. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The state ψ is said to be not entangled, respect to the parts A and B, if it can be written as

$$|\psi\rangle = |\psi_1\rangle_A \otimes |\psi_2\rangle_B, \qquad |\psi\rangle_{A,B} \in \mathcal{H}_{A,B} \tag{54}$$

The problem of how to know that a given state is not entangled can be solved in a neat mathematical manner. Let us denote by $|e_i\rangle$ $(i = 1, \ldots, n_A)$ an orthonormal basis of $\mathcal{H}_A$ and by $|f_j\rangle$ $(j = 1, \ldots, n_B)$ an orthonormal basis of $\mathcal{H}_B$. A generic state $\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} \psi_{ij}\, |e_i\rangle_A\, |f_j\rangle_B \tag{55}$$

where ψ_{ij} is a $n_A \times n_B$ complex matrices normalized as

$$\langle\psi|\psi\rangle = 1 \Longrightarrow \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} |\psi_{ij}|^2 = 1$$

An important theorem in linear algebra is that a generic $m \times n$ complex matrix M can be written as

$$M = U\,D\,V^t, \qquad U\,U^\dagger = \mathbf{I}, \quad V\,V^\dagger = \mathbf{I}$$

where U and V are unitary $m \times m$ and $n \times n$ unitary matrices respectively and D is a diagonal $n \times m$ matrix whose entries are non negative numbers

$$D = \begin{pmatrix} d_1 & 0 & \dots \\ 0 & d_2 & \dots \\ 0 & 0 & \dots \end{pmatrix}, \qquad d_1 \geq d_2 \geq d_3 \cdots \geq 0$$

This result is known as the singular value decomposition (SVD) of the matrix M. Applying this result to a $n_A \times n_B$ matrix Ψ , whose components are ψ_{ij}, one gets

$$\Psi = U\,D\,V^t \Longrightarrow \psi_{ij} = \sum_a U_{ia}\,d_a\,V_{ja} \tag{56}$$

Plugging this result into the equation for $|\psi\rangle$ one finds

$$\begin{aligned} |\psi\rangle &= \sum_{i=1}^{n_A}\sum_{j=1}^{n_B}\sum_a U_{ia}\,d_a\,V_{ja}\,|e_i\rangle_A\,|f_j\rangle_B \\ &= \sum_a d_a \left(\sum_{i=1}^{n_A} U_{ia}\ |e_i\rangle_A\right)\left(\sum_{j=1}^{n_B} V_{ja}|f_j\rangle_B\right) \end{aligned}$$

If $n_A = n_B$, the terms in paranthesis define new orthonormal basis of $\mathcal{H}_{A,B}$

$$\widetilde{e}_a = \sum_{i=1}^{n_A} U_{ia}\ |e_i\rangle_A, \qquad \widetilde{f}_a = \sum_{j=1}^{n_B} V_{ja}|f_j\rangle_B$$

If $n_A \neq n_B$ the vectors defined by these equations can be supplemented with additional vectors in order to construct new orthonormal basis of $\mathcal{H}_{A,B}$. After this change of basis the state $|\psi\rangle$ takes the simple form

$$|\psi\rangle = \sum_{a=1}^{\chi} d_a\ |\widetilde{e}_a\rangle_A|\widetilde{f}_a\rangle_B \tag{57}$$

which is known as the Schmidt decomposition. χ gives the number of non vanishing d_a and it is called the Schmidt number, which is bounded by the minimum of n_A and n_B. The normalization of ψ implies

$$\sum_{a=1}^{\chi} d_a^2 = 1, \qquad \chi \leq \min\left(\dim \mathcal{H}_A, \dim \mathcal{H}_B\right)$$

Recalling the definition of a not entangled state we see that it coincides with a state with Schmidt number $\chi = 1$, that is, there is only one term in its Schmidt decomposition. Whenever $\chi > 1$ the state will be entangled, i.e.

$$\psi \text{ is entangled} \Longleftrightarrow \chi > 1$$

The EPR and Bell states of two spin 1/2 particle correspond to

$$\text{EPR and Bell states} \rightarrow \chi = 2, d_1 = d_2 = \frac{1}{\sqrt{2}}$$

A. Reduced density matrix

Let us consider a pure state ψ of a quantum system made of two disjoint parts A and B. Tracing over each part one gets two operators

$$\rho_A = \mathrm{Tr}_B|\psi\rangle\,\langle\psi|, \qquad \rho_B = \mathrm{Tr}_A|\psi\rangle\,\langle\psi|$$

which satisfy the standard properties of a density matrix

$$\rho^\dagger = \rho, \qquad \mathrm{Tr}\,\rho = 1, \qquad \rho^2 \le \rho$$

and for these reason they are called reduced density matrices. Each of these matrices allow one to compute the expectation value of an observable defined on the corresponding portion, i.e.

$$\langle \mathcal{O}_A\rangle_\psi = \langle\psi|\mathcal{O}_A|\psi\rangle = \mathrm{Tr}_{A\cup B}\,(\mathcal{O}_A\,|\psi\rangle\langle\psi|) = \mathrm{Tr}_A\,(\mathcal{O}_A\,\mathrm{Tr}_B|\psi\rangle\langle\psi|) = \mathrm{Tr}\,(\mathcal{O}_A\,\rho_A)$$

The reduced density matrices take a particular simple form using the Schmidt decomposition (57),

$$\rho_A = \sum_{a=1}^{\chi} d_a^2\; |\widetilde{e}_a\rangle_{A\;A}\langle\widetilde{e}_a|, \qquad \rho_B = \sum_{a=1}^{\chi} d_a^2\; |\widetilde{f}_a\rangle_{B\;B}\langle\widetilde{f}_a| \tag{58}$$

They are diagonal in the corresponding basis and their eigenvalues coincide with the square of the Schmidt coefficients. An alternative way to compute these coefficients is as follows. Starting from the general equation (55), the two density matrices are

$$\rho_A = \sum_{i,i'=1}^{n_A} \rho^A_{ii'}\; |e_i\rangle_{A\;A}\langle e_{i'}|, \qquad \rho^A_{ii'} = \sum_{j=1}^{n_B}\psi_{ij}\psi^*_{i'j} \tag{59}$$

$$\rho_B = \sum_{j,j'=1}^{n_B} \rho^B_{jj'}\; |f_j\rangle_{B\;B}\langle f_{j'}|, \qquad \rho^B_{jj'} = \sum_{i=1}^{n_A}\psi_{ij}\psi^*_{ij'} \tag{60}$$

Using the SVD of Ψ one can write the matrices $\rho^{A,B}$ as

$$\begin{aligned}\rho^A &= \Psi\,\Psi^\dagger = U\,D^2\,U^\dagger \\ \rho^B &= \Psi^t\,\Psi^* = V\,D^2\,V^\dagger\end{aligned}$$

This shows that the U and V matrices are nothing but the unitary transformations needed to diagonalize the reduced density matrices in the corresponding subsystems. This way of obtaining d_a^2 is the standard one in the so called Density Matrix Renormalization Group Method (DMRG).

B. von Neumann entropy

The von Neumann entropy of a density matrix ρ, is defined as

$$S = -\mathrm{Tr}\,\rho\,\log\rho$$

This quantity is non negative and vanishes if and only if ρ corresponds to a pure state, i.e. $\rho = |\psi\rangle\langle\psi|$. Using this quantity one can associate an entropy to the reduced density matrices of a pure quantum system. Indeed, one defines the entropy of entanglement S_A as the von Neumann entropy of the reduced density matrix ρ_A. Since ρ_A and ρ_B have the same eigenvalues one derives that $S_A = S_B$, hence

$$S_A = -\text{Tr}\,\rho_A\,\log\rho_A = -\sum_a d_a^2 \log d_a^2$$

It is comfortable to see that $S_A = 0$ if and only if the state ψ is not entangled. On the other hand, for a given Schmidt number χ the state with highest entropy of entanglement is given by

$$d_a = \frac{1}{\sqrt{\chi}} \Longrightarrow S_A = \log\chi$$

This means in particular that the EPR and Bell states have the highest entropy of entanglement. For a two qubit system the density matrix of the subsystems is two dimensional. hence their two eigenvalues can be taken as $\cos^2\theta$ and $\sin^2\theta$. So a single parameter serves to characterize the entanglement of a two qubit system, where any pure state can be written in the form

$$|\psi\rangle = \cos\theta\,|0\rangle_A|0\rangle_B + \sin\theta\,|1\rangle_A|1\rangle_B$$

yielding the reduced density matrix

$$\rho_A = \cos^2\theta\,|0\rangle_{A\;A}\langle 0| + \sin^2\theta\,|1\rangle_{A\;A}\langle 1|$$

whose entropy is

$$S_A = H(x), \qquad x = \cos^2\theta$$

where $H(x)$ is the binary entropy

$$H(x) = -x\log x - (1-x)\log(1-x) \tag{61}$$

See fig 25 for a plot of $H(x)$.

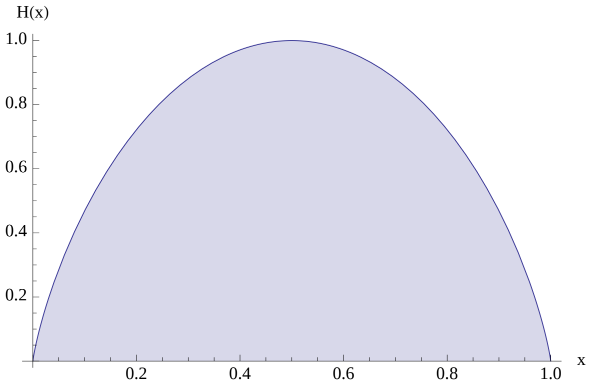

FIG. 25: Plot of the entropy $H(x)$ defined in eq. (61) using log in base two

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