Entanglement in Quantum Field Theory

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I. MEASURES OF ENTANGLEMENT: VON NEUMANN ENTROPY

Let us consider a pure quantum state $\psi\rangle$ of a system that we divide into two parts A and B. The total Hilbert space of the system $\mathcal{H}$ is the tensor product of the Hilbert spaces of the parts, i.e. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The state ψ is said to be not entangled, respect to the parts A and B, if it can be written as

$$|\psi\rangle = |\psi_1\rangle_A \otimes |\psi_2\rangle_B, \qquad |\psi\rangle_{A,B} \in \mathcal{H}_{A,B} \tag{1}$$

The problem of how to know that a given state is not entangled can be solved in a neat mathematical manner. Let us denote by $|e_i\rangle$ $(i = 1, \ldots, n_A)$ an orthonormal basis of $\mathcal{H}_A$ and by $|f_j\rangle$ $(j = 1, \ldots, n_B)$ an orthonormal basis of $\mathcal{H}_B$. A generic state $\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} \psi_{ij}\, |e_i\rangle_A\, |f_j\rangle_B \tag{2}$$

where ψ_{ij} is a $n_A \times n_B$ complex matrices normalized as

$$\langle\psi|\psi\rangle = 1 \Longrightarrow \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} |\psi_{ij}|^2 = 1$$

An important theorem in linear algebra is that a generic $m \times n$ complex matrix M can be written as

$$M = U\, D\, V^t, \qquad U\, U^\dagger = \mathbf{I}, \quad V\, V^\dagger = \mathbf{I}$$

where U and V are unitary $m \times m$ and $n \times n$ unitary matrices respectively and D is a diagonal $n \times m$ matrix whose entries are non negative numbers

$$D = \begin{pmatrix} d_1 & 0 & \ldots \\ 0 & d_2 & \ldots \\ 0 & 0 & \ldots \end{pmatrix}, \qquad d_1 \geq d_2 \geq d_3 \cdots \geq 0$$

This result is known as the singular value decomposition (SVD) of the matrix M. Applying this result to a $n_A \times n_B$ matrix Ψ , whose components are ψ_{ij}, one gets

$$\Psi = U\, D\, V^t \Longrightarrow \psi_{ij} = \sum_a U_{ia}\, d_a\, V_{ja} \tag{3}$$

Plugging this result into the equation for $|\psi\rangle$ one finds

$$\begin{aligned} |\psi\rangle &= \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} \sum_a U_{ia}\, d_a\, V_{ja}\, |e_i\rangle_A\, |f_j\rangle_B \\ &= \sum_a d_a \left(\sum_{i=1}^{n_A} U_{ia}\, |e_i\rangle_A \right) \left(\sum_{j=1}^{n_B} V_{ja} |f_j\rangle_B \right) \end{aligned}$$

If $n_A = n_B$, the terms in paranthesis define new orthonormal basis of $\mathcal{H}_{A,B}$

$$\widetilde{e}_a = \sum_{i=1}^{n_A} U_{ia} \; |e_i\rangle_A, \qquad \widetilde{f}_a = \sum_{j=1}^{n_B} V_{ja} |f_j\rangle_B$$

If $n_A \neq n_B$ the vectors defined by these equations can be supplemented with additional vectors in order to construct new orthonormal basis of $\mathcal{H}_{A,B}$. After this change of basis the state $|\psi\rangle$ takes the simple form

$$|\psi\rangle = \sum_{a=1}^{\chi} d_a \; |\widetilde{e}_a\rangle_A |\widetilde{f}_a\rangle_B \tag{4}$$

which is known as the Schmidt decomposition. χ gives the number of non vanishing d_a and it is called the Schmidt number, which is bounded by the minimum of n_A and n_B. The normalization of ψ implies

$$\sum_{a=1}^{\chi} d_a^2 = 1, \qquad \chi \leq \min\left(\dim \mathcal{H}_A, \dim \mathcal{H}_B\right)$$

Recalling the definition of a not entangled state we see that it coincides with a state with Schmidt number $\chi = 1$, that is, there is only one term in its Schmidt decomposition. Whenever $\chi > 1$ the state will be entangled, i.e.

$$\psi \text{ is entangled} \Longleftrightarrow \chi > 1$$

The EPR and Bell states of two spin 1/2 particle correspond to

$$\text{EPR and Bell states} \rightarrow \chi = 2, d_1 = d_2 = \frac{1}{\sqrt{2}}$$

A. Reduced density matrix

Let us consider a pure state ψ of a quantum system made of two disjoint parts A and B. Tracing over each part one gets two operators

$$\rho_A = \mathrm{Tr}_B |\psi\rangle\,\langle\psi|, \qquad \rho_B = \mathrm{Tr}_A |\psi\rangle\,\langle\psi|$$

which satisfy the standard properties of a density matrix

$$\rho^\dagger = \rho, \qquad \mathrm{Tr}\,\rho = 1, \qquad \rho^2 \leq \rho$$

and for these reason they are called reduced density matrices. Each of these matrices allow one to compute the expectation value of an observable defined on the corresponding portion, i.e.

$$\langle \mathcal{O}_A \rangle_\psi = \langle\psi|\mathcal{O}_A|\psi\rangle = \mathrm{Tr}_{A\cup B}\left(\mathcal{O}_A\,|\psi\rangle\langle\psi|\right) = \mathrm{Tr}_A\left(\mathcal{O}_A\,\mathrm{Tr}_B|\psi\rangle\langle\psi|\right) = \mathrm{Tr}\left(\mathcal{O}_A\,\rho_A\right)$$

The reduced density matrices take a particular simple form using the Schmidt decomposition (4),

$$\rho_A = \sum_{a=1}^{\chi} d_a^2 \; |\widetilde{e}_a\rangle_{A\;A}\langle\widetilde{e}_a|, \qquad \rho_B = \sum_{a=1}^{\chi} d_a^2 \; |\widetilde{f}_a\rangle_{B\;B}\langle\widetilde{f}_a| \tag{5}$$

They are diagonal in the corresponding basis and their eigenvalues coincide with the square of the Schmidt coefficients. An alternative way to compute these coefficients is as follows. Starting from the general equation (2), the two density matrices are

$$\rho_A \;=\; \sum_{i,i'=1}^{n_A} \rho^A_{ii'}\;|e_i\rangle_A\,{}_A\langle e_{i'}|, \qquad \rho^A_{ii'} = \sum_{j=1}^{n_B} \psi_{ij}\psi^*_{i'j} \tag{6}$$

$$\rho_B \;=\; \sum_{j,j'=1}^{n_B} \rho^B_{jj'}\;|f_j\rangle_B\,{}_B\langle f_{j'}|, \qquad \rho^B_{jj'} = \sum_{i=1}^{n_A} \psi_{ij}\psi^*_{ij'} \tag{7}$$

Using the SVD of Ψ one can write the matrices $\rho^{A,B}$ as

$$\rho^A \;=\; \Psi\,\Psi^\dagger = U\,D^2\,U^\dagger$$
$$\rho^B \;=\; \Psi^t\,\Psi^* = V\,D^2\,V^\dagger$$

This shows that the U and V matrices are nothing but the unitary transformations needed to diagonalize the reduced density matrices in the corresponding subsystems. This way of obtaining d_a^2 is the standard one in the so called Density Matrix Renormalization Group Method (DMRG).

B. von Neumann entropy

The von Neumann entropy of a density matrix ρ, is defined as

$$S = -\mathrm{Tr}\,\rho\,\log\rho$$

This quantity is non negative and vanishes if and only if ρ corresponds to a pure state, i.e. $\rho = |\psi\rangle\langle\psi|$. Using this quantity one can associate an entropy to the reduced density matrices of a pure quantum system. Indeed, one defines the entropy of entanglement S_A as the von Neumann entropy of the reduced density matrix ρ_A. Since ρ_A and ρ_B have the same eigenvalues one derives that $S_A = S_B$, hence

$$S_A = -\mathrm{Tr}\,\rho_A\,\log\rho_A = -\sum_a d_a^2\log d_a^2$$

It is comfortable to see that $S_A = 0$ if and only if the state ψ is not entangled. On the other hand, for a given Schmidt number χ the state with highest entropy of entanglement is given by

$$d_a = \frac{1}{\sqrt{\chi}} \Longrightarrow S_A = \log\chi$$

This means in particular that the EPR and Bell states have the highest entropy of entanglement. For a two qubit system the density matrix of the subsystems is two dimensional. hence their two eigenvalues can be taken as $\cos^2\theta$ and $\sin^2\theta$. So a single parameter serves to characterize the entanglement of a two qubit system, where any pure state can be written in the form

$$|\psi\rangle = \cos\theta\,|0\rangle_A|0\rangle_B + \sin\theta\,|1\rangle_A|1\rangle_B$$

yielding the reduced density matrix

$$\rho_A = \cos^2\theta\,|0\rangle_A\,{}_A\langle 0| + \sin^2\theta\,|1\rangle_A\,{}_A\langle 1|$$

whose entropy is

$$S_A = H(x), \qquad x = \cos^2\theta$$

where $H(x)$ is the binary entropy

$$H(x) = -x\log x - (1-x)\log(1-x) \tag{8}$$

See fig 1 for a plot of $H(x)$.

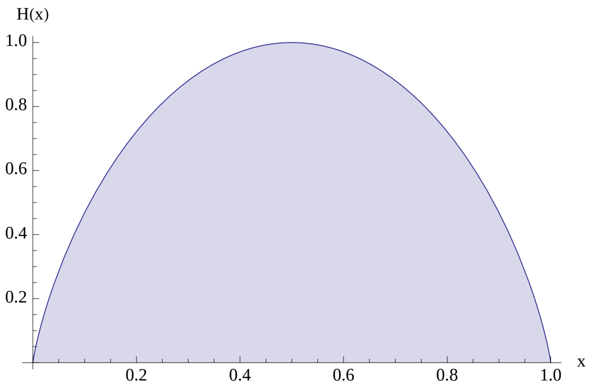

FIG. 1: Plot of the entropy $H(x)$ defined in eq. (8) using log in base two

II. BLACK HOLES AND ENTROPY

In 1973 Bekenstein proposed that a black hole (BH) has an entropy S proportional to the area A of its horizon.

$$S = \gamma A \tag{9}$$

He also argueed that the coefficient in (9) is of order one and equal to $\gamma = \log 2/8\pi\hbar$, in units $G = c = 1$. This proposal appeared somehow in previous papers as in[2], and it was based on analogies between formulas in thermodynamics and BH physics. In particular the expression

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ \tag{10}$$

where M, κ, Ω, J are respectively the mass, surface gravity, angular velocity and angular momenta of the BH, was put in correspondence with the thermodynamic relation

$$dU = TdS + pdV. \tag{11}$$

In his famous paper in 1975 on black hole evaporation[3], Hawking fixed the proportionality value in (9)

$$S_{BH} = \frac{kA}{4\ell_P^2}, \qquad \ell_P = \sqrt{\frac{G\hbar}{c^3}} \tag{12}$$

This formula became known as the Bekenstein-Hawing law of the black-hole entropy (curiously enough the initials BH refers either to these authors or to the black-hole itself!). The factor $\log 2$ in the Bekenstein formula came from considerations of information theory, since it represents the minimal information associated to a bit in Shannon's theory. Together with the thermodynamic analogy between eqs.(10) and (11), information theory played a major role in Bekenstein formulation of the area law, which is stated explicitely as (page 2336 in **??**):

It is then natural to introduce the concept of black-hole entropy as the measure of the inaccesibility of informattion (to an exterior observer) as to which particular internal configuration of the black hole is actually realized in a given case

To strength the information content of this concept Bekenstein continues saying:

At the outset it should be clear that the black hole entropy we are speaking of is not the thermal entropy inside the black hole. In fact, our black hole entropy refers to the equivalence class of all black holes which have the same mass, charge, and angular momentum, not to one particular black hole.

III. QUANTUM SOURCE OF ENTROPY FOR BLACK HOLES (BOMBELLI ET AL 1986)

Inspired by the Bekenstein-Hawking formula, Bombelli et al[4]) computed the entropy of the reduced density matrix of a real scalar field satisfying the Klein-Gordon equation on a fixed background. In[4] the ground state of this field theory was traced over the inner degrees of freedom of a region of the space finding a dependence of the area. Bombelli et al also obtained a general expression for the entropy of any real Gaussian density matrix.

A. Two harmonic oscillators

As a simple example of the general idea both Bombelli et al compute the reduced density matrix of two harmonic oscillators coupled through the Hamiltonian (we introduce a simplified notation which makes more clear the results):

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\sum_{a,b=1,2} V_{ab}(\chi)\, q_a\, q_b \tag{13}$$

where the potential is given by the positive symmetric matrix

$$V(\chi) = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}, \qquad -\infty < \chi < \infty \tag{14}$$

with eigenvalues $e^{\pm\chi}$ (notice that $\det V(\chi) = 1$ and $\mathrm{Tr}V(\chi) = 2\cosh\chi$). The normalized ground state wave function is given by the gaussian

$$\psi(q_1, q_2) = \frac{1}{\sqrt{\pi}}\exp\left[-\frac{1}{2}\sum_{a,b} V_{ab}\left(\frac{\chi}{2}\right) q_a q_b\right] = \frac{1}{\sqrt{\pi}}\exp\left[-\frac{1}{2}\cosh\frac{\chi}{2}\,(q_1^2 + q_2^2) - \sinh\frac{\chi}{2}\, q_1 q_2\right] \tag{15}$$

which has an energy

$$E_0 = \frac{1}{2}\mathrm{Tr}\, V\left(\frac{\chi}{2}\right) = \cosh\frac{\chi}{2}$$

If $\chi = 0$ the two oscillators are completely decoupled. Let us now introduce the creation and annihilation operators for each oscillator

$$a_j = \frac{1}{\sqrt{2}}(p_j - iq_j), \quad a_j^\dagger = \frac{1}{\sqrt{2}}(p_j + iq_j), \quad [a_j, a_k] = \delta_{jk} \quad (j, k = 1, 2) \tag{16}$$

and the corresponding vacuum states

$$a_j\, |0\rangle_j = 0, \quad (j = 1, 2) \tag{17}$$

If $\chi = 0$, the GS of (13) is the product state

$$\chi = 0 \Longrightarrow H\, |0\rangle_1 \otimes |0\rangle_2 = |0\rangle_1 \otimes |0\rangle_2$$

One can easily prove that (15) satisfies the eqs.

$$a_1\, \psi = \gamma\, a_2^\dagger\, \psi, \quad a_2\, \psi = \gamma\, a_1^\dagger\, \psi, \qquad \gamma = \tanh\frac{\chi}{4} \tag{18}$$

so that ψ can be written as

$$|\psi\rangle = C e^{\gamma a_1^\dagger a_2^\dagger} |0\rangle_1 \otimes |0\rangle_2 = C \sum_{n=0}^{\infty} \gamma^n |n\rangle_1 \otimes |n\rangle_2 \qquad C = \sqrt{1-\gamma^2} \tag{19}$$

where C is a normalization constant and $|n\rangle_j$ are the basis of the Hilbert spaces $\mathcal{H}_j$ $(j = 1, 2)$ of the harmonic oscillators,

$$|n\rangle_j = \frac{(a_j^\dagger)^n}{\sqrt{n!}} |0\rangle_j, \quad n = 0, \ldots, \infty, \quad j = 1, 2 \tag{20}$$

The expression of ψ, in the basis $|n\rangle_j$, yields the Schmidt decomposition of that state (see later). The density matrix ρ in this operator language reads

$$\rho = |\psi\rangle\langle\psi| = C^2 \sum_{n,m\geq 0} \gamma^n \gamma^m \left(|n\rangle\langle m|\right)_1 \otimes \left(|n\rangle\langle m|\right)_2 \tag{21}$$

and the reduced density matrices for the oscillators 1 and 2 are obtained by tracing over the degrees of freedom of the other oscillator, namely

$$\begin{aligned} \rho^{(1)} &= \mathrm{Tr}_{\mathcal{H}_2} \rho = \sum_{n=0}^{\infty} {}_2\langle n|\rho|n\rangle_2 = C^2 \sum_{n\geq 0} \gamma^{2n} \left(|n\rangle\langle n|\right)_1 , \\ \rho^{(2)} &= \mathrm{Tr}_{\mathcal{H}_1} \rho = \sum_{n=0}^{\infty} {}_1\langle n|\rho|n\rangle_1 = C^2 \sum_{n\geq 0} \gamma^{2n} \left(|n\rangle\langle n|\right)_2 \end{aligned} \tag{22}$$

The von Neumann entropies of these reduced density matrices coincide and are given by

$$S^{(1)} = S^{(2)} = -\mathrm{Tr}_{\mathcal{H}_1} \rho^{(1)} = -\sum_{n=0}^{\infty} C^2 \gamma^{2n} \log(C^2 \gamma^{2n}) = -\frac{(1-\gamma^2)\log(1-\gamma^2) + \gamma^2 \log \gamma^2}{1-\gamma^2} \tag{23}$$

See fig. 2 for a plot of $S^{1)}$ as a function of γ^2. When $\gamma \to 1$, the entropy diverges as $-\log(1-\gamma^2)$, which corresponds to a very strong coupling between the oscillators. The fact that $S^{(1)} = S^{(2)}$ implies that the entropy of the reduced density matrices describes a property common to both subsystems and it is a general feature of pure states of bipartite systems.

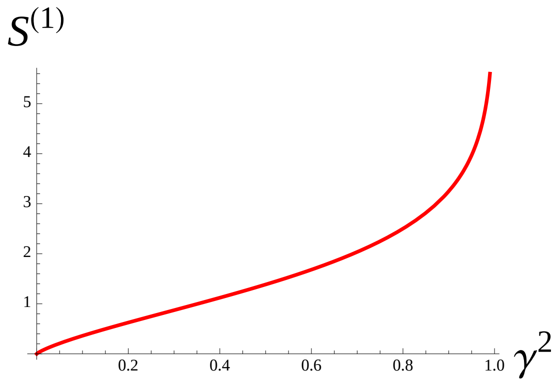

FIG. 2: Entropy $S^{(1)}$ for the two coupled harmonic oscillators as a function of $\gamma^2 \in (0, 1)$.

Let us now compute the density matrix ρ in the coordinate basis. First we write (15) as

$$\psi(q_1,q_2) = \frac{1}{\sqrt{\pi}} \exp\left[-\frac{1}{2}A(q_1^2+q_2^2) - Bq_1q_2\right] \tag{24}$$

where

$$A = \cosh\frac{\chi}{2} = \frac{1+\gamma^2}{1-\gamma^2}, \quad B = \sinh\frac{\chi}{2} = \frac{2\gamma}{1-\gamma^2} \tag{25}$$

The density matrix ρ is given by

$$\rho(q_1q_2, q_1'q_2') = \langle q_1q_2|\rho|q_1'q_2'\rangle = \psi(q_1,q_2)\psi^*(q_1',q_2') = \frac{1}{\pi}e^{-\frac{1}{2}A(q_1^2+q_2^2+q_1'^2+q_2'^2)-B(q_1q_2+q_1'q_2')} \tag{26}$$

and the reduced density matrix by

$$\begin{aligned}
\rho^{(1)}(q_1,q_1') &= \int dq_2\,\rho(q_1q_2,q_1'q_2) = \frac{1}{\pi}e^{-\frac{1}{2}A(q_1^2+q_1'^2)}\int dq_2\,e^{-Aq_2^2-B(q_1+q_1')q_2} \\
&= \frac{1}{\pi}e^{-\frac{1}{2}A(q_1^2+q_1'^2)}\int dq_2\,e^{-A\left(q_2+\frac{B}{2A}(q_1+q_1')\right)^2+\frac{B^2}{4A}(q_1+q_1')^2} \\
&= \frac{1}{\sqrt{\pi A}}e^{-\frac{1}{2}A(q_1^2+q_1'^2)+\frac{B^2}{4A}(q_1+q_1')^2}
\end{aligned}$$

where we have completed the squares. Bombelli et al write this expression as

$$\rho^{(1)}(q,q') = \sqrt{\frac{M}{\pi}}e^{-\frac{1}{2}M(q^2+q'^2)-\frac{N}{4}(q-q')^2} \tag{27}$$

where

$$M = \frac{1}{A} = \frac{1-\mu}{1+\mu}, \qquad N = \frac{B^2}{A} = \frac{4\mu}{1-\mu^2}, \qquad \mu = \gamma^2 \tag{28}$$

Notice that the entropy (23) can be written as

$$S^{(1)} = H(\mu) \equiv -\frac{\mu\log\mu + (1-\mu)\log(1-\mu)}{1-\mu} \tag{29}$$

The parameters M and N have dimensions of length^{-2}, but the entropy $S^{(1)}$ is dimensionaless, so it can only depend on the ratio

$$\lambda = \frac{N}{M} \tag{30}$$

which is related to μ by the following relations

$$\mu = 1 + \frac{2M}{N} - 2\sqrt{\frac{M}{N}\left(1+\frac{M}{N}\right)} = 1 + 2\lambda^{-1} - 2\sqrt{\lambda^{-1}(1+\lambda^{-1})} \tag{31}$$

Making the transformation

$$q \to \frac{q}{\sqrt{M}}$$

the density matrix (27) takes the simple form

$$\rho^{(1)}(q,q') = \sqrt{\pi}e^{-\frac{1}{2}(q^2+q'^2)-\frac{\lambda}{4}(q-q')^2} \tag{32}$$

B. General collection of harmonic oscillators

The previous example can be easily generalized to a general collection of harmonic oscillators whose Lagrangian is given by

$$L = \frac{1}{2} G_{MN}\, \dot{q}^M \dot{q}^N - \frac{1}{2} V_{MN}\, q^M q^N \tag{33}$$

where $q^M, \dot{q}^M$ ($M = 1, \ldots, d$) are the coordinates and velocities of d oscillators. G_{MN} and V_{MN} are positive and symmetric matrices. The matrix G_{MN} plays the role of a metric and its inverse

$$G^{MN} G_{NP} = \delta^M_P$$

is used to raise indices. The canonical momenta P_M is given by

$$P_M = \frac{\partial L}{d\dot{q}^M} = G_{MN}\, \dot{q}^N$$

in terms of which the Hamiltonian reads

$$H = \frac{1}{2} G^{MN}\, P_M P_N + \frac{1}{2} V_{MN}\, q^M q^N \tag{34}$$

Defining the symmetric matrix

$$W_{MA}\, W^A_N = V_{MN} \tag{35}$$

one can write (34) as

$$H = \frac{1}{2} G^{MN} \left(P_M - i W_{MA} q^A \right)^\dagger \left(P_N - i W_{NB} q^B \right) + \frac{1}{2} \mathrm{Tr}\, W \tag{36}$$

where one has used the canonical commutation relations

$$[q^M, P_N] = i\, \delta^M_N, \qquad P_N = -i \frac{\partial}{\partial q^N}$$

The ground state of (36) is found by solving the equations

$$\left(P_N - i W_{NB} q^B \right) |\psi\rangle = 0, \qquad \forall N$$

which in the coordinate basis reads

$$\left(\frac{\partial}{\partial q^N} + W_{NB} q^B \right) \psi(\{q^A\}) = 0, \qquad \forall N$$

and whose solution is

$$\psi(\{q^A\}) = \left(\det \frac{W}{\pi} \right)^{1/4} \exp\left(-\frac{1}{2} W_{AB}\, q^A q^B \right) \tag{37}$$

The density matrix is given by

$$\rho(\{q^A\},\{q'^B\}) = \psi(\{q^A\})\psi^*(\{q^B\}) = \left(\det\frac{W}{\pi}\right)^{1/2} \exp\left[-\frac{1}{2}W_{AB}\,(q^A q^B + q'^A q'^B)\right] \tag{38}$$

Next, one splits the d oscillators in two sets of d_1 oscillators and d_2 oscillators ($d = d_1 + d_2$),

$$A = (a, \alpha), \qquad a = 1, \ldots, d_1, \quad \alpha = d_1 + 1, \ldots, d_1 + d_2$$

and correspondingly the matrix W_{AB} and its inverse W^{AB} split as

$$W_{AB} = \begin{pmatrix} W_{ab} & W_{a\beta} \\ W_{\alpha b} & W_{\alpha\beta} \end{pmatrix}, \qquad W^{AB} = \begin{pmatrix} W^{ab} & W^{a\beta} \\ W^{\alpha b} & W^{\alpha\beta} \end{pmatrix} \tag{39}$$

The matrix W^{AB} is not obtained from W_{AB} raising the indices with G^{AB}. The reduced density matrix for the oscillators q^a is defined as

$$\begin{aligned} \rho^{(1)}(\{q^a\},\{q'^b\}) &= \int \prod_\alpha dq^\alpha\, \rho(\{q^a, q^\alpha\},\{q'^b, q^\alpha\}) \\ = \left(\det\frac{W_{AB}}{\pi}\right)^{1/2} &\times\ \exp\left[-\frac{1}{2}W_{ab}\,(q^a q^b + q'^a q'^b)\right] \int \prod_\alpha dq^\alpha \exp\left[-W_{\alpha\beta}\, q^\alpha q^\beta - W_{a\alpha}(q^a + q'^a) q^\alpha)\right] \end{aligned} \tag{40}$$

To perform the gaussian integral it is convenient to introduce the inverse of some of tha matrices in (39), namely

$$W_{ab}\,\widetilde{W}^{bc} = \delta^c_a, \quad W^{ab}\,\widetilde{W}_{bc} = \delta^a_c, \quad W_{\alpha\beta}\,\widetilde{W}^{\beta\gamma} = \delta^\gamma_\alpha, \quad W^{\alpha\beta}\,\widetilde{W}_{\beta\gamma} = \delta^\alpha_\gamma \tag{41}$$

Then completing squares in (40) one finds

$$\rho^{(1)}(\{q^a\},\{q'^b\}) = \left(\det\frac{\widetilde{W}_{ab}}{\pi}\right)^{1/2} \exp\left[-\frac{1}{2}W_{ab}\,(q^a q^b + q'^a q'^b)\right] \exp\left[\frac{1}{4}\widetilde{W}^{\alpha\beta}\, W_{\alpha a}\, W_{\beta b}(q+q')^a (q+q')^b\right] \tag{42}$$

where one has used the identity

$$\det W_{AB} = \det \widetilde{W}_{ab}\ \det W_{\alpha\beta}$$

that follows from the relation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{pmatrix} \Longrightarrow \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\ \det D$$

and

$$\widetilde{W}_{ab} = W_{ab} - W_{a\alpha}\,\widetilde{W}^{\alpha\beta}\, W_{\beta b}$$

This eq. can be proved using the definitions (41)

$$\begin{aligned} W^{ca}(W_{ab} - W_{a\alpha}\,\widetilde{W}^{\alpha\beta}\, W_{\beta b}) &= W^{cA}W_{Ab} - W^{c\gamma}W_{\gamma b} - (W^{cA}W_{A\alpha} - W^{c\gamma}W_{\gamma\alpha})\widetilde{W}^{\alpha\beta}W_{\beta b} \\ &= \delta^c_b - W^{c\gamma}W_{\gamma b} + W^{c\gamma}\delta^\beta_\gamma W_{\beta b} = \delta^c_b \end{aligned}$$

Finally, one defines

$$M_{ab} = \widetilde{W}_{ab}, \qquad N_{ab} = W_{a\alpha}\,\widetilde{W}^{\alpha\beta}\,W_{\beta b} \tag{43}$$

which brings (42) into the form

$$\rho^{(1)}(\{q^a\},\{q'^b\}) \;=\; \left(\det\frac{M_{ab}}{\pi}\right)^{1/2}\exp\left[-\frac{1}{2}M_{ab}\,(q^aq^b+q'^aq'^b)-\frac{1}{4}N_{ab}(q-q')^a(q-q')^b\right] \tag{44}$$

The W matrices corresponding to the two harmonic oscillators studied previously correspond to (recall eq. (14)

$$W_{AB} = \begin{pmatrix} \cosh\frac{\chi}{2} & \sinh\frac{\chi}{2} \\ \sinh\frac{\chi}{2} & \cosh\frac{\chi}{2} \end{pmatrix}, \qquad W^{AB} = \begin{pmatrix} \cosh\frac{\chi}{2} & -\sinh\frac{\chi}{2} \\ -\sinh\frac{\chi}{2} & \cosh\frac{\chi}{2} \end{pmatrix} \tag{45}$$

and the entries M_{11} and N_{11} coincide with M and N defined in (28). To diagonalize (44) one first perform an orthogonal transformation U which diagonalizes the symmetric matrix M_{ab},

$$q \to Uq, \qquad M = U\begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}U^T$$

This brings $\rho^{(1)}$ to

$$\rho^{(1)}(\{q^a\},\{q'^b\}) \;=\; \prod_n\left(\frac{M_n}{\pi}\right)^{1/2}\exp\left[-\frac{1}{2}M_n\,(q^nq^n+q'^nq'^n)-\frac{1}{4}N'_{nm}(q-q')^n(q-q')^m\right]$$

where $N' = U^TNU$. As we did for two harmonic oscillators, we perform the scaling

$$q_n \to \frac{q_n}{\sqrt{M_n}}$$

so that (up to a rescaling of the overall factor to keep the right normalization)

$$\rho^{(1)}(\{q^a\},\{q'^b\}) \;=\; \prod_n\pi^{-1/2}\exp\left[-\frac{1}{2}\,(q^nq^n+q'^nq'^n)-\frac{1}{4}M_n^{-1/2}\,N'_{nm}\,M_m^{-1/2}(q-q')^n(q-q')^m\right]$$

Finally one diagonalizes the matrix $M_n^{-1/2}\,N'_{nm}\,M_m^{-1/2}$, or alternatively the matrix

$$\Lambda^a_b = (M^{-1})^{ac}\,N_{cb} \tag{46}$$

which leads to

$$\rho^{(1)}(\{q^a\},\{q'^b\}) \;=\; \prod_n\{\pi^{-1/2}\exp\left[-\frac{1}{2}\,(q^nq^n+q'^nq'^n)-\frac{1}{4}\lambda_n(q-q')^n(q-q')^n\right]\} \tag{47}$$

This expression shows that the reduce density matrix is the collection of the density matrices associated to the eigenvalues λ_n of decoupled oscillators,

$$\rho^{(1)} = \otimes_n\rho(\lambda_n) \tag{48}$$

and consequently the entropy is given by

$$S^{(1)} = \sum_n S[\rho(\lambda_n)] = \sum_n H(\mu_n) \tag{49}$$

where $H(\mu)$ was defined in (29) and μ_n is related to λ by eq.(31), i.e.

$$\lambda_n = \frac{4\mu_n}{(1-\mu_n)^2}, \qquad \mu_n = 1 + 2\lambda_n^{-1} - 2\sqrt{\lambda_n^{-1}(1+\lambda_n^{-1})} \tag{50}$$

Notice that (49) is a generalization of eq.(29). Using identities involves the W matrices, Bombelli et al show that the matrix λ positive semidefinite and can be written as

$$\Lambda^a_b = -W^{a\alpha}\, W_{\alpha b} \tag{51}$$

C. Entropy of a free scalar field

The latter general results are next used by Bombelli et al to study the case of a free scalar field in D space dimensions $D = 3$ in reference[4]), making the correspondence

$$\frac{1}{2} V_{AB}\, q^A q^B \to \frac{1}{2}\langle\phi|\nabla^2 + m^2|\phi\rangle = \frac{1}{2}\int d^D x[(\nabla\phi)^2 + m^2\phi^2]$$

In the continuum limit, the oscillator label A represents the position $x \in \mathbb{R}^D$, of the scalar field $\phi(x)$, so that the matrix V_{AB} corresponds to the function

$$V(x,y) = \int \frac{d^D k}{(2\pi)^D} (k^2 + m^2)\, e^{ik\cdot(x-y)} \tag{52}$$

and the matrices W_{AB} and W^{AB} to

$$W(x,y) = \int \frac{d^D k}{(2\pi)^D} (k^2+m^2)^{1/2}\, e^{ik\cdot(x-y)}, \qquad W^{-1}(x,y) = \int \frac{d^D k}{(2\pi)^D} (k^2+m^2)^{-1/2}\, e^{ik\cdot(x-y)} \tag{53}$$

Calling $\Omega \subset \mathbb{R}^D$ the region of the space where one takes the trace, the matrix Λ becomes

$$\begin{aligned} \Lambda(x,y) &= -\int_\Omega d^D z\, W^{-1}(x,z)\, W(z,y) \qquad (54) \\ &= -\int_\Omega d^D z \int \frac{d^D k}{(2\pi)^D} (k^2+m^2)^{-1/2}\, e^{ik\cdot(x-z)} \int \frac{d^D p}{(2\pi)^D} (p^2+m^2)^{1/2}\, e^{ik\cdot(z-y)} \end{aligned}$$

The problem is then to find the eigenvalues of this matrix

$$\int_{\Omega^c} d^D y\, \Lambda(x,y)\, f(y) = \lambda\, f(x), \tag{55}$$

where Ω^c is the complement of Ω in $\mathbb{R}^D$. Bombelli et al show that the entropy S_Ω that results in the later computation is infinite even with a finite mass. This entropy is a dimensionless quantity, which can only depend on the linear size of Ω, say R, and the mass m in the form mR. If $m = 0$, the entropy can only be ∞ or 0, but as a calculation shows it is actually infinite. The addition of the mass does not give a finite result, because taking the limit $R \to 0$, amounts to taking the limit $m \to 0$, and therefore one gets infinity again. The divergency of the entropy has an ultraviolet origin and therefore a short distance cutoff is needed in order to get a finite answer. The paper discuss three possible regularizations: 1) lattice cutoff, 2) momentum cutoff and 3) distance cutoff separating Ω from the rest of the system. The choice 3) is the one implemented in[4]. To show the area law of S_Ω, Bombelli et al use a half-space geometry, i.e

$$\Omega = \{x \,|\, -\infty < x^D < -\epsilon\}, \qquad \Omega^c = \{x \,|\, 0 < x^D < \infty\} \tag{56}$$

where $\epsilon > 0$ is the short distance cutoff. This choice implies that there is a thin hyperplane of width ϵ separating Ω and Ω^c. Any D dimensional vector, as the momenta, is decomposed into a perpendicular component $v_\perp$ and $D-1$ parallel components $v_\parallel$ to the hyperplane separating Ω for the other half (the parallel component is absent for $D=1$). The W and W^{-1} functions then decomposed as

$$\begin{aligned} W(x,y) &= \int \frac{dk_\perp}{2\pi} \int \frac{d^{D-1}k_\parallel}{(2\pi)^{D-1}} \, (k_\perp^2 + k_\parallel^2 + m^2)^{1/2} \, e^{ik_\perp(x_\perp - y_\perp)} e^{ik_\parallel \cdot (x_\parallel - y_\parallel)}, \\ W^{-1}(x,y) &= \int \frac{dk_\perp}{2\pi} \int \frac{d^{D-1}k_\parallel}{(2\pi)^{D-1}} \, (k_\perp^2 + k_\parallel^2 + m^2)^{-1/2} \, e^{ik_\perp(x_\perp - y_\perp)} e^{ik_\parallel \cdot (x_\parallel - y_\parallel)}, \end{aligned}$$

where the integration of the momenta run over the whole momentum space. Consequently the matrix Λ decomposes as

$$\begin{aligned} \Lambda(x,y) &= -\int_{-\infty}^{-\epsilon} dz_\perp \int_{\mathbb{R}^{D-1}} d^{D-1}z_\parallel \int \frac{dk_\perp}{2\pi} \int \frac{d^{D-1}k_\parallel}{(2\pi)^{D-1}} \, (k_\perp^2 + k_\parallel^2 + m^2)^{-1/2} e^{ik_\perp(x_\perp - z_\perp)} e^{ik_\parallel \cdot (x_\parallel - z_\parallel)} \\ &\times \int \frac{dp_\perp}{2\pi} \int \frac{d^{D-1}p_\parallel}{(2\pi)^{D-1}} \, (p_\perp^2 + p_\parallel^2 + m^2)^{1/2} \, e^{ip_\perp(z_\perp - y_\perp)} e^{ip_\parallel \cdot (z_\parallel - y_\parallel)} \end{aligned} \tag{57}$$

Translational invariance on the parallel plane suggests the following ansatz for the eigenfunctions of Λ,

$$f(x) = e^{i\nu_\parallel \cdot x_\parallel} \, f(x_\perp) \tag{58}$$

which converts the eigenvalue eq.(55) into

$$\begin{aligned} \lambda \, f(x_\perp) &= -\int_0^\infty dy_\perp \int_{-\infty}^{-\epsilon} dz_\perp \int_{-\infty}^{\infty} \frac{dk_\perp}{2\pi} \, (k_\perp^2 + \nu_\parallel^2 + m^2)^{-1/2} e^{ik_\perp(x_\perp - z_\perp)} \\ &\times \int \frac{dp_\perp}{2\pi} \, (p_\perp^2 + \nu_\parallel^2 + m^2)^{1/2} \, e^{ip_\perp(z_\perp - y_\perp)} \, f(y_\perp) \end{aligned} \tag{59}$$

For each momenta $\nu_\parallel$, eq.(59) is a one-dimensional problem of a boson with effective mass

$$m_e = (\nu_\parallel^2 + m^2)^{1/2}$$

which will give rise to a spectrum of eigenvalues $\lambda_n(m_e\epsilon)$ $n \in \mathbb{Z}$, each of then given a contribution to the entropy

$$H_n(m_e\epsilon) = -\frac{\mu_n \log \mu_n + (1-\mu_n) \, \log(1-\mu_n)}{1-\mu_n}$$

where μ_n is related to λ_n by eq.(50). The total entropy associated to the effective mass m_e will be given by

$$H(m_e\epsilon) = \sum_{n=0}^{\infty} H_n(m_e\epsilon)$$

and the total entropy is obtained integrating over all momenta $\nu_\parallel$. Using the fact that this entropy is proportional to the area A one finds (for $D=2$)

$$S \;=\; \frac{A}{(2\pi)^2} \int d^2\nu_{\parallel}\, H(m_e\epsilon) = \frac{A}{2\pi\epsilon^2} \int_{\epsilon\sqrt{R^{-2}+m^2}}^{\sqrt{1+\epsilon^2 m^2}} d\xi\, \xi\, H(\xi) \tag{60}$$

where we also introduce an infrared cutoff R associated to the finite area A. Bombelli et al then show that the integral over ξ is finite and in this manner they find the area law

$$S \sim C \frac{A}{\epsilon^2} \tag{61}$$

for some constants C. They also discuss the case where the domain Ω is a sphere but they do not compute the final result which they expect to follow also an area law.

IV. ENTROPY AND AREA (SREDNICKI (1993))

In 1993 Srednicki did a computation similar to that of Bombelli et al[5] (at the end of the paper Srednicki acknowledges reference[4] pointing out some difference). Srednicki realized that the entropy of a region Ω does not depend on wether one traces over the interior or the exterior, i.e.

$$S = -\mathrm{Tr}\, \rho_{\rm in}\, \log \rho_{\rm in} = -\mathrm{Tr}\, \rho_{\rm out}\, \log \rho_{\rm out} \tag{62}$$

and therefore S could only depend on a common property shared by Ω and its complement Ω_c, namely the area that separates the two domains. For a field theory Srednicki arguee that S must depend on an ultraviolet cutoff M, which for a crystal would be the inverse of the atomic spacing and an infrared cutoff μ, which would be the inverse of the linear size. However Srednicki arguees that if in the ground state the correlations fall off fast enough with the distance from the boundary, then S should be independent on μ and therefore S is expected to satisfy

$$S = \kappa M^2\, A \tag{63}$$

where κ is a numerical factor. This law is strikingly similar to the BH law for a black hole

$$S_{\rm BH} = \frac{1}{4} M_{\rm Pl}^2\, A^2 \tag{64}$$

which is observed as a rather mysterious fact. Srednicki then suggest that the area law is a much more general formula that has been realized so far, and not particularly tied to black holes, since in particular the region Ω is enterily imaginary. As in Bombelli et al paper, Srednicki first computed the entropy of two coupled harmonic oscillators and later on he generalizes the result to a more collection of oscillators, which is then applied to a free scalar field with Hamiltonian

$$H = \frac{1}{2} \int d^3x\, \left[\pi^2(x) + (\nabla\phi(x))^2\right] \tag{65}$$

which is discretized on a sphere of radius R using partial waves $\phi_{lm,j}$ and $\pi_{lm,j}$, where l is the total angular momenta, m its third component, and $j = 1, \ldots, N$ are the positions. These fields satisfy the canonical commutation relations

$$[\phi_{lm,j}, \pi_{l'm',j'}] = i\, \delta_{ll'}\, \delta_{mm'}\, \delta_{jj'}$$

The total Hamiltonian being is given by

$$H = \sum_{lm} H_{lm} = \sum_{lm} \frac{1}{2a} \sum_{j=1}^{N} \left[\pi_{lm,j}^2 + (j+\frac{1}{2})^2 \left(\frac{\phi_{lm,j}}{j} - \frac{\phi_{lm,j+1}}{j+1} \right)^2 + \frac{l(l+1)}{j^2} \phi_{lm,j}^2 \right] \tag{66}$$

where a is the lattice spacing. Srednicki then computed numerically the entropy of the first $1 \leq n \leq 30$ sites for a system with $N = 60$, finding

$$S = 0.30\, M^2 R^2, \qquad (D = 3) \tag{67}$$

where $M = a^{-1}$ and $R = (n + 1/2)a$ is the radius of the region Ω. This result was shown to be independent of the size N of the system. In $D = 2$ and $D = 1$ Srednicki finds

$$\begin{aligned} S &= \kappa\, M\, R \qquad\qquad\quad D = 2 \\ S &= \kappa_1 \log(M\, R) + \kappa_2 \log(\mu\, R) \qquad D = 1 \end{aligned} \tag{68}$$

which supports the general area law

$$S = \kappa\, M^{D-1}\, A, \qquad D > 1 \tag{69}$$

except for $D = 1$ which exhibits a logarithmic dependence with the size of the system.

V. GEOMETRIC ADN RENORMALIZED ENTROPY IN CONFORMAL FIELD THEORY (HOLZHEY ET AL 1994)

This work is motivated by the area law in black hole physics and the prolem of moving mirrors. It is the first work where the entropy in a 2D CFT is computed and applied to the latter problems.

The goal is to compute the entropy

$$S = -\text{Tr}\, \rho\, \log \rho$$

that *describes the correlations between the subsystem and the rest of the universe. Roughly speaking, it is the logarithm of the number of states of the inacessible part of the universe that are consistent with all the measurements restricted to the accesible part, together with a apriori knowledge that the universe as a whole is in a pure state*[8].

The universe U is splitted into an inner and out parts. In a quantum field theory the variables are local and described by a complete set of commuting observables $\widehat{\xi}_{in}, \widehat{\xi}_{out}$. The density matrix of the universe can be written as

$$\rho_U = \rho_U(\xi^1_{in}, \xi^1_{out}; \xi^2_{in}, \xi^2_{out}) \tag{70}$$

and the density matrix form the inner part is then

$$\rho_{in}(\xi^1_{in}; \xi^2_{in}) = \sum_{\xi_{out}} \rho_U(\xi^1_{in}, \xi_{out}; \xi^2_{in}, \xi_{out}) \tag{71}$$

The problem in a QFT is to obtain a finite result for observables, due specially to UV divergences. Concerning the geometric entropy Holzhey et al solved this problem using a method similar to that of Bombelli et at[4]. That is to separate the inner and outer regions by a short distance cutoff.

Let us consider a system $\mathcal{S}$ of length L with periodic boundary conditions. To describe it we introduce the complex variable $\zeta = \sigma + i\tau$, where $0 \leq \sigma \leq L$ is the spatial coordinate and τ the time coordinate. The system $\mathcal{S}$ is splitted into two subsystems which we take as $A = (0, \ell)$ and $B = (\ell, L)$. If one computes the entropy of a given state by tracing over the degrees of freedom in B the result will be infinite. The reason being that localized excitations arbitrary close to the boundary of A will correlate it with the subsystem B. This problem is resolved by introducing a short distance cutoff ϵ[4,8]. The system S is now regarded as the union

$$\begin{aligned} \mathcal{S} &= A \cup B \\ A = (\epsilon, \ell - \epsilon), &\qquad B = (\ell + \epsilon, L - \epsilon), \end{aligned} \tag{72}$$

where $\epsilon << \ell < L$, so that A and B are separated by two regions of sizes 2ϵ. The world sheet of the past before time $\tau = 0$, is a cylinder with two semidisks of radii ϵ cut out (see fig 3). Let us denote by C and D the boundaries of these disks:

$$C = \{\epsilon\, e^{-i\varphi},\ \ 0 \leq \varphi \leq \pi\}, \qquad D = \{\ell - \epsilon\, e^{i\varphi},\ \ 0 \leq \varphi \leq \pi\}$$

Hence the complete boundary of the world sheet of fig refmaps consists in the union $A \cup C \cup B \cup D$. To simplify this geometry one first performs the conformal transformation $\zeta \to w$:

$$w = -\frac{\sin\left(\frac{\pi(\zeta-\ell)}{L}\right)}{\sin\left(\frac{\pi\zeta}{L}\right)}, \qquad \zeta = \frac{L}{2\pi i} \log\left(\frac{w + e^{i\pi\ell/L}}{w + e^{-i\pi\ell/L}}\right) \tag{73}$$

The real ζ-axis is mapped into the real w-axis, in particular the boundaries of the intervals $A = (A_1, A_2),\ B = (B_1, B_2)$ are mapped into

$$w_{A_1} = \frac{1}{w_{A_2}} = -\frac{1}{w_{B_1}} = -w_{B_1} = \frac{L}{\pi\epsilon} \sin\left(\frac{\pi\ell}{L}\right) \equiv R$$

in the limit where $\epsilon << \ell < L$. If we further take $\ell << L$, then the length of the system L disappears in the previous equations since they only depend on $\pm\ell/\epsilon$. Moreover, the point in the infinite past $\zeta \to -i\infty$ is mapped into

$$w_\infty = e^{i\pi(L-\ell)/L}$$

Using eq.(73) one can show that the boundary C is mapped into the small semicircle $w = 1/R\, e^{i\varphi}$ $(0 \leq \varphi \leq \pi)$ in the upper-half $w-$plane, while the boundary D is mapped into the large semicircle $w = R\, e^{i\varphi}$ $(0 \leq \varphi \leq \pi)$. Hence the cylinder of fig.1a has been mapped into the half annulus of the upper-half plane of radii R and $1/R$. The boundaries of this half annulus of course corresponds to the original boundaries A, B, C, D.

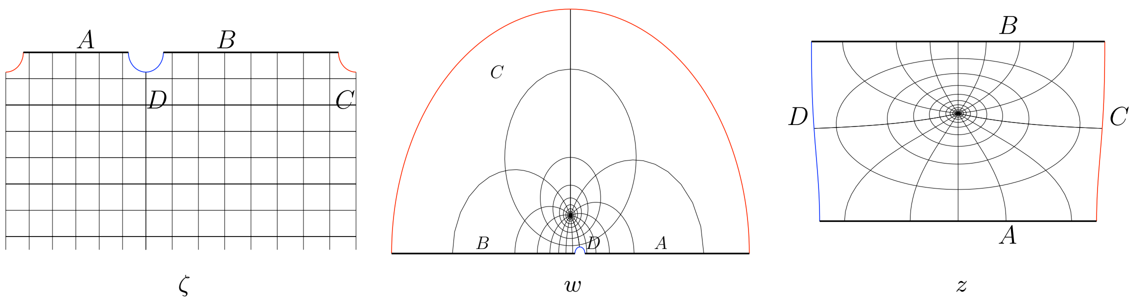

FIG. 3: Riemann surfaces describing the past events in ζ and z. The distinguished point in z is the infinite past $\zeta_\infty = -i\infty$ (taken from[9]).

Next one makes the conformal transformation

$$z = \log\, w$$

which maps the annulus into a strip of width π and length d (see fig 3)

$$z_{A_1} = \frac{d}{2}, \quad z_{A_2} = -\frac{d}{2}, \quad z_{B_1} = i\pi - \frac{d}{2}, \quad z_{B_2} = i\pi + \frac{d}{2}, \qquad d = 2\log\left[\frac{L}{\pi\epsilon}\sin\left(\frac{\pi\ell}{L}\right)\right]$$

The point at infinity $\tau = -\infty$ is mapped into

$$z_\infty = i\pi\left(1 - \frac{\ell}{L}\right)$$

The interval A is now represented by the lower half of the strip, while the interval B is represented by the upper half. the intervals C and D corresponds to the other two sides of the strip and one imposes periodic boundary conditions, which amounts to matching conditions of the fields at the edges of the original subsystems.

We are interested in finding an expression for the entanglement entropy for primary states in a CFT including the vacuum $|0\rangle$ (see references[9–11]). The latter can be obtained by acting on the vacuum state $|0\rangle$ with an primary field $\phi(z, \overline{z})$

$$|\phi\rangle = \lim_{z,\overline{z}\to 0} \phi(z, \overline{z})\,|0\rangle \tag{74}$$

with conformal weights $h, \overline{h}$. Eq. (74) describes an incoming state. The outgoing state can be defined similarly

$$\langle\phi| = \lim_{z,\overline{z}\to 0} \overline{z}^{-2h} z^{-2\overline{h}}\,\langle 0|\,\phi(\frac{1}{\overline{z}}, \frac{1}{z}) \tag{75}$$

so that the scalar product of these two states is one.

$$\langle\phi|\phi\rangle = \lim_{z,\overline{z}\to\infty} \overline{z}^{-2h} z^{-2\overline{h}}\,\langle 0|\,\phi(\frac{1}{\overline{z}}, \frac{1}{z})\,\phi(0,0)|0\rangle = 1 \tag{76}$$

The wave function associated to the vaccum (i.e the ground state) is given by the path integral

$$\Psi_{XY}(GS) \propto \int \mathcal{D}\phi\; e^{-S(\phi)} \tag{77}$$

and the one associated to the primary state is

$$\Psi_{XY}(\Upsilon) \propto \int \mathcal{D}\phi\; \Upsilon[\phi(z_\infty)]\; e^{-S(\phi)} \tag{78}$$

where ϕ denotes the local field whose action is $S(\phi)$. The field Υ will be a functional of ϕ, and it will be evaluated at the infinite past z_∞, in agreement with eq. (74). In the path integral X and Y denote the values of the field ϕ in the subsystems A and B respectively. The density matrix $\rho \equiv \rho_A$ for the subsystem A is obtained by tracing over the variables in B

$$\rho_{XX'}(\Upsilon) = \int \mathcal{D}Y\; \Psi_{XY}(\Upsilon)\, \Psi^*_{YX'}(\Upsilon) \tag{79}$$

Plugging (77) and (78) into (79) one finds for the GS and the excited states

$$\rho_{XX'}(GS) = \frac{1}{Z(1)} \int \mathcal{D}\phi\; e^{-S(\phi)} \tag{80}$$

and

$$\rho_{XX'}(\Upsilon) = \frac{1}{Z(1)\langle\Upsilon(z_\infty)\,\Upsilon^\dagger(z'_\infty)\rangle} \int \mathcal{D}\phi\; \Upsilon[\phi(z_\infty)]\; \Upsilon^*[\phi(z'_\infty)]\; e^{-S(\phi)} \tag{81}$$

where

$$z'_\infty = i\pi\left(1 + \frac{\ell}{L}\right)$$

represents to position of the infinite future where the outgoing fields are created.

The functional integral is over a strip of height 2π and boundary conditions $\phi = X$ on the lower side and $\phi = X'$ on the upper side. The normalization factor is determined by the condition $\mathrm{tr}\,\rho = 1$, which implies that $Z(1)$ is the function integral with no operator insertion and the top and bottom edges of the strip identified (i.e. a torus partition function), and $\langle \Upsilon\, \Upsilon^\dagger \rangle$ is the two point correlator on the same torus. To compute the entanglement entropy one uses the replica trick. For the GS one first takes the n^{th} power of (80)

$$\rho^n_{XX'}(GS) = \frac{1}{Z(1)^n} \int \mathcal{D}\phi\; e^{-S(\phi)} \tag{82}$$

where the path integral is over a strip of height $2\pi n$. Then one identifies X with X' obtaining

$$\mathrm{Tr}\,\rho^n(GS) = \frac{Z(n)}{Z(1)^n} \tag{83}$$

where $Z(n)$ denotes the partition function on a torus of lengths $2\pi n$ and d, along the b and a cycles respectively, so that the moduli parameter is given by $\tau = 2\pi i n/d$. The entropy is finally obtained taking the limit

$$S = -\frac{d}{dn} \mathrm{Tr}\,\rho^n|_{n=1} \tag{84}$$

which using eq.(83) yields

$$S_{GS} = (1 - n\frac{d}{dn}) \log Z(n)|_{n=1} \tag{85}$$

An alternative formula used to compute this quantity is in terms of the so called Renyi entropies

$$S^{(n)} = \frac{1}{1-n} \log\, \mathrm{Tr}\,\rho^n \tag{86}$$

whose $n \to 1$ limit gives the von Neumann entropy,

$$S^{(1)} = \lim_{n\to 1} S^{(n)} = -\mathrm{Tr}\,\rho\, \log\, \rho \tag{87}$$

Similarly for the excited states one finds[9–11]

$$\mathrm{tr}\ \rho^n_\Upsilon = \frac{Z(n)}{Z(1)^n} \frac{\prod_{k=0}^{n-1} \langle \Upsilon(z_\infty + 2i\pi k)\, \Upsilon^\dagger(z'_\infty + 2i\pi k)\rangle_{\tau_n}}{\langle \Upsilon(z_\infty)\, \Upsilon^\dagger(z'_\infty)\rangle_{\tau_1}{}^n} \tag{88}$$

where $\tau_n = 2\pi i n/d$ denotes the moduli of the corresponding torii. Notice that the $2n$ point correlator of fields Υ, $\Upsilon^\dagger$ depends on the ratio ℓ/L and the moduli parameter τ.

An example: a free boson

The partition function of a massless free boson is given by

$$Z(\tau) = \frac{1}{(\mathrm{Im}\,\tau)^{1/2}\, |\eta(\tau)|^2} \tag{89}$$

where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1-q^k), \qquad q = e^{2i\pi\tau} \tag{90}$$

which transforms under the modular transformations as follows

$$\eta(\tau+1) = e^{i\pi/12}\,\eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau) \tag{91}$$

This eqs. imply that $Z(\tau)$ is modular invariant

$$Z(\tau+1) = Z(\tau), \qquad Z(-1/\tau) = Z(\tau) \tag{92}$$

In the previous example the moduli is given by $\tau_n = 2\pi i n/d$, whose modulus goes to zero as $d >> 1$. Hence the gnome q approaches 1, and all the terms in the product defining $\eta(\tau)$ contribute. However the transformed moduli $\widetilde{\tau}_n = id/2\pi n$ grows with d and correspondingly the gnome $\widetilde{q} = e^{2\pi i \widetilde{\tau}_n} = e^{-d/n}$ goes to zero, in which case $\eta(\widetilde{q})$ can be easily computed,

$$Z(\tau_n) = \frac{1}{|\tau_n|^{1/2}\,|\eta(\tau_n)|^2} = \frac{|\tau_n|^{1/2}}{|\eta(-1/\tau_n)|^2} \to \left(\frac{2\pi n}{d}\right)^{1/2} e^{\frac{d}{12n}} \Longrightarrow \text{Tr}\,\rho_{GS}^n = e^{\frac{d}{12}(\frac{1}{n}-n)}$$

Keeping the leading terms one finds

$$\text{Tr}\,\rho_{GS}^n = e^{\frac{d}{12}(\frac{1}{n}-n)} \Longrightarrow S^{(n)} = \frac{n+1}{12n} d = \frac{n+1}{6n} \log\left[\frac{L}{\pi\epsilon}\log\frac{\pi\ell}{L}\right] \tag{93}$$

which implies for the von Neumann entropy

$$S = \frac{1}{3}\log\left[\frac{L}{\pi\epsilon}\log\frac{\pi\ell}{L}\right] \tag{94}$$

In fig. 4 plot we plot $S^{(n)}$. Notice the symmetry

$$S^{(n)}(\ell) = S^{(n)}(L-\ell), \quad \forall n$$

which reflects the fact that eigenvalues of the density matrix does not depend on tracing on a subsystem or its complement. In a general CFT with central charge c, the partition function $Z(n)$ of the theory is

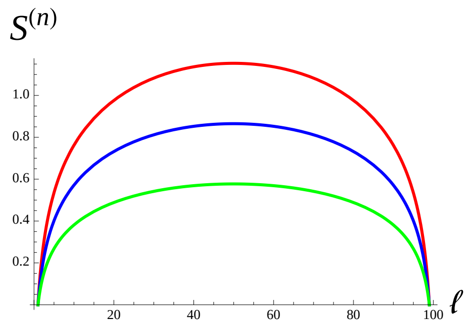

FIG. 4: Renyi entropies $S^{(n)}$ for a free boson given by eq.(93) for $n = 1$ (red), 2 (blue), ∞ (green). The length of the system is $L = 100$ and we take $\epsilon = 1$.

$$Z(n) = Z(\tau,\overline{\tau}) = \text{Tr}\; q^{L_0-\frac{c}{24}}\;\overline{q}^{\overline{L}_0-\frac{c}{24}}$$

In the limit $d >> 1$, using again the modular transformation $\tau \to -1/\tau$ one finds

$$\text{tr}\;\rho^n_{GS} = \frac{Z(n)}{Z(1)^n} \sim e^{\frac{c}{12}(\frac{1}{n}-n)\,d} = \left[\frac{L}{\pi\epsilon}\sin\left(\frac{\pi\ell}{L}\right)\right]^{\frac{c}{6}(\frac{1}{n}-n)}$$

and hence the Renyi entropies and von Neumann entropy are given by

$$S^{(n)} = \frac{c(n+1)}{6n}\log\left[\frac{L}{\pi\epsilon}\log\frac{\pi\ell}{L}\right], \qquad S = \frac{c}{3}\log\left[\frac{L}{\pi\epsilon}\log\frac{\pi\ell}{L}\right] \tag{95}$$

VI. ENTANGLEMENT OF LOW-ENERGY EXCITATIONS IN CONFORMAL FIELD THEORY (ALCARAZ ET AL, 2011)

Let us now consider the formula (88) for the primary states (see references[9–11]). The remaining terms of this expression only depend on the correlators of the fields $\Upsilon\,\Upsilon^\dagger$ on the cylinder. The two-point correlator on the denominator is computed on a cylinder of radius 2π, while the $2n$-point correlator in the numerator is computed on a cylinder of radius $2\pi n$. It is thus convenient to rescale the fields in the later correlator to have a common radius of 2π. Assuming that the field Υ has conformal dimensions $h, \overline{h}$ we can write (88) as

$$F^{(n)}_{\Upsilon} \equiv \frac{\text{tr}\;\rho^n_{\Upsilon}}{\text{tr}\;\rho^n_{\Upsilon_0}} \;=\; n^{-2n(h+\overline{h})}\;\frac{\prod_{k=0}^{n-1}\langle\Upsilon(\frac{z_\infty}{n}+\frac{2i\pi k}{n})\,\Upsilon^\dagger(\frac{z'_\infty}{n}+\frac{2i\pi k}{n})\rangle_{\tau_n}}{\langle\Upsilon(z_\infty)\,\Upsilon^\dagger(z'_\infty)\rangle^n_{\text{cyl}}} \tag{96}$$

The coordinates of the fields Υ and $\Upsilon^\dagger$ are

$$\Upsilon:\quad z_j = \frac{i\pi}{n}(2j-1-x), \qquad \Upsilon^\dagger:\quad z_j = \frac{i\pi}{n}(2j-1+x), \qquad j = 1,2,\ldots,n$$

where $x = \ell/L$. Doing the shift $z_j \to z_j - \frac{i\pi}{n}(1-x)$, and exchanging the σ and τ variables one can write these coordinates as

$$\Upsilon:\quad z_j = \frac{2\pi j}{n}, \qquad \Upsilon^\dagger:\quad z_j = \frac{2\pi}{n}(j+x), \qquad j = 0,1,\ldots,n-1$$

so that $F^{(n)}_{\Upsilon}$ becomes

$$F^{(n)}_{\Upsilon}(x) \equiv \frac{\text{tr}\;\rho^n}{\text{tr}\;\rho^n_{GS}} \;=\; n^{-2n(h+\overline{h})}\;\frac{\langle\prod_{j=0}^{n-1}\Upsilon(\frac{2\pi j}{n})\,\Upsilon^\dagger(\frac{2\pi}{n}(j+x))\rangle_{\text{cyl}}}{\langle\Upsilon(0)\,\Upsilon^\dagger(2\pi x)\rangle^n_{\text{cyl}}} \tag{97}$$

For $n = 2$ this is

$$F^{(2)}_{\Upsilon}(x) \;=\; 2^{-4(h+\overline{h})}\;\frac{\langle\Upsilon(0)\,\Upsilon^\dagger(\pi x)\Upsilon(\pi)\,\Upsilon^\dagger(\pi(1+x))\rangle_{\text{cyl}}}{\langle\Upsilon(0)\,\Upsilon^\dagger(2\pi x)\rangle^2_{\text{cyl}}} \tag{98}$$

The entanglement entropy for the excited state Υ can then be computed using the replica trick

$$S_A(\Upsilon) = -\frac{\partial}{\partial n}\text{tr}\;\rho^n|_{n=1} = S_A(GS) - \frac{\partial F^{(n)}_{\Upsilon}}{\partial n}|_{n=1} \tag{99}$$

Example 1.- Consider a chiral scalar field $\Upsilon = \Upsilon^\dagger = i\partial\phi(z)$. Using the correlator on the cylinder ($z_{12} = z_1 - z_2$)

$$\langle \Upsilon(z_1)\, \Upsilon(z_2)\rangle_{\rm cyl} = \frac{1}{\left(\sin\frac{z_{12}}{2}\right)^2} \tag{100}$$

and the Wick theorem one obtains

$$\begin{aligned} F_\Upsilon^{(2)}(x) &= 2^{-4}\,(\sin\pi x)^4 \left[\frac{1}{\sin^4\frac{\pi x}{2}} + \frac{1}{\cos^4\frac{\pi x}{2}} + 1\right] = \frac{1}{64}\,(7+\cos 2\pi x)^2 \\ &= 1 - 2\sin^2\frac{\pi x}{2} + 3\sin^4\frac{\pi x}{2} - 2\sin^6\frac{\pi x}{2} + \sin^8\frac{\pi x}{2} \end{aligned} \tag{101}$$

In fig. 5 we plot the value of $F_\Upsilon^{(n)}$ for $n = 2, 3$ together with the numerical values for the free fermion model. We omit the analytic expression for $n = 3$ which is more lengthly. The small x expansion of $F_\Upsilon^{(n)}$ for any n can be found using the OPE formula

$$\Upsilon(z_1)\,\Upsilon(z_2) = \frac{1}{\sin^2\frac{z_{12}}{2}} + :\Upsilon(z_1)\,\Upsilon(z_2):, \qquad \Upsilon(z) = i\partial\phi(z) \tag{102}$$

where $:\cdot:$ denotes normal ordering. In particular one gets

$$\Upsilon(\frac{2\pi j}{n})\,\Upsilon^\dagger(\frac{2\pi}{n}(j+x)) \sim \frac{1}{\left(\sin\frac{\pi x}{n}\right)^2} + :\Upsilon^2(\frac{2\pi j}{n}): \;\sim \left(\frac{n}{\pi x}\right)^2\left(1 + \frac{(\pi x)^2}{3n^2} + \frac{(\pi x)^2}{n^2} :\Upsilon^2(\frac{2\pi j}{n}):\right)$$

Plugging this eq. into (97) one finally gets

$$F_\Upsilon^{(n)}(x) \sim 1 + \frac{(\pi x)^2}{3}\left(\frac{1}{n} - n\right), \qquad x << 1$$

For $n = 2$ this eq. agrees with the expansion of (101). Hence from eq.(99) one finds

$$S_A(\Upsilon) - S_A(\Upsilon_0) \sim \frac{2\pi^2 x^2}{3} \tag{103}$$

which agrees with the numerical results obtained for the free fermion model.

Generalization to all n
Let us compute $F_\Upsilon^{(n)}$ for generic values of n. Consider the correlator

$$\langle \prod_{j=1}^{M} \Upsilon(z_j)\rangle_{\rm cyl}$$

Making the conformal transformation to the plane $w = e^{iz}$ we get

$$i^M e^{i\sum_j z_j}\, \langle \prod_{j=1}^{M} \Upsilon(e^{iz_j})\rangle_{\rm plane}$$

Now using

$$\langle \Upsilon(w_1)\Upsilon(w_2)\rangle = \frac{1}{(w_1-w_2)^2}$$

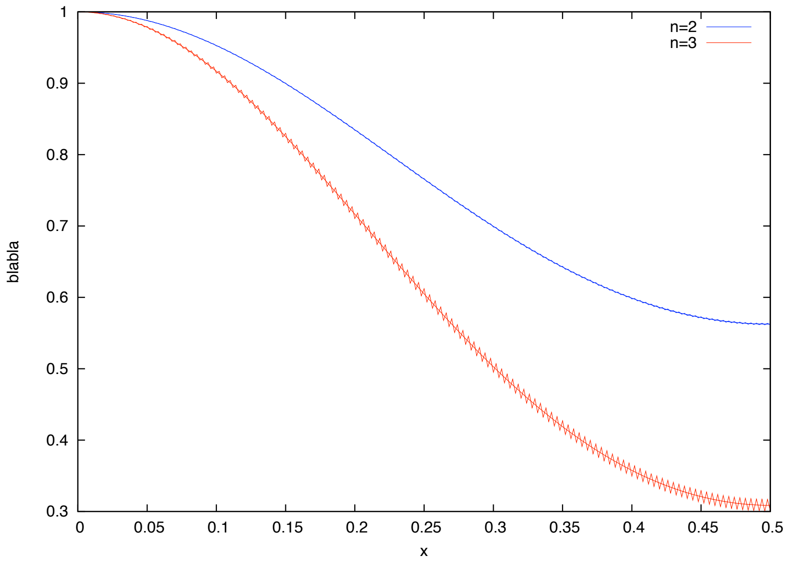

FIG. 5: Plot of $F^{(n)}_{\partial\phi}$ for $n = 2, 3$. The analytic formula for $n = 2$ is given by eq. (101). The number of sites is $L = 500$.

and the Wick theorem

$$\langle \prod_{j=1}^{M} \Upsilon(e^{iz_j}) \rangle_{\rm plane} = {\rm Hf} \frac{1}{(e^{iz_j} - e^{iz_k})^2}$$

where Hf is the Haffnian of a $2n \times 2n$ symmetric matrix which is defined as

$$\mathrm{Hf}(A) = \frac{1}{2^n\, n!} \sum_{\sigma \in \mathcal{S}_{2n}} \prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)}$$

The Haffnian that appears can actually be computed as a determinant, i.e.

$$\mathrm{Hf}\, \frac{1}{(w_j - w_k)^2} = \det \frac{1}{w_j - w_k}$$

Collecting terms one gets:

$$\langle \prod_{j=1}^{M} \Upsilon(z_j) \rangle_{\rm cyl} = i^M\, e^{i \sum z_j}\, \det \frac{1}{e^{iz_j} - e^{iz_k}}$$

In particular we want to compute

$$\langle \prod_{j=0}^{n-1} \Upsilon(\frac{2\pi j}{n})\, \Upsilon^{\dagger}(\frac{2\pi}{n}(j + x)) \rangle_{\rm cyl} = (-1)^n e^{2\pi i x}\, \det \frac{1}{e^{iz_j} - e^{iz_k}}$$

which finally yields

$$F^{(n)}_{\Upsilon} = (-1)^n \left(\frac{2}{n} \sin(\pi x) \right)^{2n} e^{2\pi i x}\, \det \frac{1}{e^{iz_j} - e^{iz_k}}$$

where the coordinates z_j are

$$z_j = \frac{2\pi j}{n}, \quad \frac{2\pi}{n}(j+x), \qquad j = 0, 1, \dots, n-1$$

In the determinant the diagonal terms are omitted. One can similary write this equation as

$$F_{\Upsilon}^{(n)} = (-1)^n \left(\frac{2}{n}\sin \pi x\right)^{2n} \det \frac{1}{e^{iz_j} - e^{iz_k}} \tag{104}$$

where

$$z_j = \frac{\pi}{n}(2j - x), \quad \frac{\pi}{n}(2j+x), \qquad j = 0, 1, \dots, n-1 \tag{105}$$

Example 2.- Consider a chiral vertex operator $\Upsilon = e^{i\alpha\phi(z)}$ ($\Upsilon^\dagger = e^{-i\alpha\phi(z)}$) with conformal weight $h = \alpha^2/2$. The correlator of a product of vertex operators is

$$\langle \prod_i e^{i\alpha_i \phi(z_i)} \rangle_{\rm cyl} = \prod_{i>j} \left(\sin \frac{z_{ij}}{2}\right)^{\alpha_i \alpha_j}$$

One can prove that

$$F_Y^{(n)} = 1, \qquad \forall n$$

This eq. implies that in the fermionic model all the primary fields have the same entanglement entropy and Renyi entropy as the ground state.

A. Relation with the trace anomaly

Holzhey at al give an alternative derivation of the entropy in the case $\ell << L$

$$S = \frac{c}{3}\log\frac{\ell}{\epsilon} \tag{106}$$

using the trace anomaly following the Cardy review of CFT[12]. They want to show that

$$\frac{\partial S}{\partial \log\,\epsilon} = -\frac{c}{3} \tag{107}$$

To do this computation these authors consider the w-coordinates where the wave function $\Psi_{X,Y}$ corresponds to the upper half annulus of radii R, R^{-1} (with $R = \ell/\epsilon$). The positive axis corresponds to the subsystem A and the negative to the subsystem B. The density matrix $\rho_{X,X'}$ is then given by the path integral over the full disk with a cut along the positive axis where the field takes values X and X'. Thus $Z(n) \propto {\rm Tr}\,\rho^n$, is the partition function of a annulus covered n times. Suppose that one makes an analytic extension to n a number slightly less that one. Geometrically this is a cone whose vertex has an angle $2\pi n < 2\pi$. The problem is reduced to find the dependence of $Z(n)$ on ϵ.

The latter problem can be seen as a coarse grain procedure where $\epsilon \to (1+\alpha)\epsilon$ ($\alpha > 0$). Accordingly R and R^{-1} will decrease and increase respectively. To simplify the computation one can keep fixed the inner radius and change the outer radius twice. This amounts to a rescaling $x^\mu \to x'^\mu = (1-2\alpha)x^\mu$. The partition function of the annulus with radius R is defined by the functional integral

$$Z(R) = \int d\phi\, e^{-S(R)} \tag{108}$$

where $S(R)$ is the action of the system. Under a small changes of coordinates $R \to R'$ the partition function must remain invariant since this is just a RG transformation,

$$Z(R) = \int d\phi\, e^{-S(R)} = \int d\phi\, e^{-S(R') - \delta S} = Z(R') \langle e^{-\delta S} \rangle \sim Z(R') e^{-\langle \delta S \rangle} \tag{109}$$

which implies[12]

$$\delta \log Z = Z(R') - Z(R) = \delta S \tag{110}$$

The action changes as

$$\delta S = -\frac{1}{2\pi} \int d^2x\, T_{\mu\nu}(x) \frac{\partial x'^{\mu}}{\partial x^{\nu}} \tag{111}$$

where $T_{\mu\nu}(x)$ is the energy momentum tensor. Hence

$$\delta \log Z = -\frac{1}{2\pi} \int d^2x\, \langle T_{\mu\nu}(x) \rangle \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{\alpha}{\pi} \int d^2x\, \langle T^{\mu}_{\mu}(x) \rangle \tag{112}$$

we have assumed that the path integral measure $d\phi$ is invariant under the rescaling. In CFT the expectation value of the trace of the energy momentum tensor is given by the formula

$$\langle T^{\mu}_{\mu}(x) \rangle = -\frac{c}{12} R(x) \tag{113}$$

where $R(x)$ is the scalar curvature. This formula is known as the trace anomaly and it arises purely from quantum effects since in the classical theory the stress tensor is traceless. Plugging (113) into (112) yields

$$\delta \log Z = -\frac{\alpha c}{12\pi} \int d^2x\, R(x) \tag{114}$$

The curvature tensor $R(x)$ has a delta singularity at the vertex of the cone which explains why (114) does not vanish. To compute it, one returns to the expression (112) and writes (using the conservation law $\partial/\partial x^{\nu} T_{\mu\nu} = 0$ and the Stokes theorem)

$$\int d^2x\, \langle T^{\mu}_{\mu}(x) \rangle = \int d^2x\, \langle T_{\mu\nu}(x) \rangle \frac{\partial x^{\mu}}{\partial x^{\nu}} = \int d^2x\, \frac{\partial}{\partial x^{\nu}} \left(x^{\mu} \langle T_{\mu\nu}(x) \rangle \right) = \int dS^{\nu}\, x^{\mu} \langle T_{\mu\nu}(x) \rangle \tag{115}$$

where the integration is on the outer boundary of the annulus. In complex coordinates this eq. reads

$$\int dS^{\nu}\, x^{\mu} \langle T_{\mu\nu}(x) \rangle = -i \int dw\, w\, \langle T_{\rm cone}(w) \rangle + h.c. \tag{116}$$

The coordinate w describes a cone with angular circunference $2\pi n$, provided we identify $w = 1$ with $w = e^{2\pi n i}$. We can now make the conformal transformation to the complex plane $w \to z = w^{1/n}$, so that $w = 1 \to z = 1$ and $w = e^{2\pi n i} \to w = e^{2\pi i} = 1$. The energy momentum on the cone $T_{\rm cone}(w)$ and on the plane $T_{\rm plane}(z)$ are related by the eq.

$$T_{\rm cone}(w) = \left(\frac{dz}{dw} \right)^2 T_{\rm plane}(z(w)) + \frac{c}{12} \{z; w\} \tag{117}$$

where $\{z; w\}$ is the Schwarzian derivative

$$\{z; w\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2, \qquad z' = \frac{dz}{dw}, \quad \text{etc} \tag{118}$$

Using that $\langle T_{\rm plane}(z)\rangle = 0$ and $z = w^{1/n}$ one finds

$$\langle T_{\rm cone}(w)\rangle = \frac{c}{24\, w^2}\left(1 - \frac{1}{n^2}\right) \tag{119}$$

hence

$$\int d^2x\, \langle T^\mu_\mu(x)\rangle = \int dS^\nu\, x^\mu \langle T_{\mu\nu}(x)\rangle = -i\frac{c}{24}\left(1 - \frac{1}{n^2}\right)\int \frac{dw}{w} + h.c. = \frac{c}{12}\left(1 - \frac{1}{n^2}\right) 2\pi n \tag{120}$$

Notice that $\int dw/w = 2\pi i n$ captures the circular angle around the vertex of the cone. Plugging this expression in (112)

$$\delta \log Z = \frac{\alpha\, c}{6}\left(n - \frac{1}{n}\right) \tag{121}$$

Finally we can compute (106) as follows (use $d\epsilon/\epsilon = \alpha$).

$$\frac{\partial S}{\partial \log\, \epsilon} = \left(1 - n\frac{d}{dn}\right)\frac{\partial \log Z}{\partial \log\, \epsilon}|_{n=1} = \left(1 - n\frac{d}{dn}\right)\frac{\delta \log Z}{\alpha}|_{n=1} = -\frac{c}{3} \tag{122}$$

From this result Holzhey conclude that the divergence of the geometric entropy can be traced directly to the singular short-distance behaviour of QFT.

VII. ENTANGLEMENT ENTROPY AND QUANTUM FIELD THEORY (CALABRESE AND CARDY (2004)

In this paper Calabrese and Cardy (CC) generalized the work of Holzhey et al proposing a systematic approach to the computation of entanglement entropies in CFT and relativistic 1+1 Quantum Field Theories[13]. This technical tool is to reformulate the replica trick in terms of a euclidean field theory on a n-sheeted Riemann surface.

CC first reproduced the Holzhey et al result for the entanglement entropy of a interval A of length ℓ inside a system of length L with periodic boundary conditions

$$S_A = \frac{c}{3}\log \frac{L}{\pi a}\sin \frac{\pi \ell}{L} + c'_1 \tag{123}$$

where a is the short distance cutoff. For a system with open boundary conditions and A an interval obtained splitting the system in two parts they CC obtain

$$S_A = \frac{c}{6}\log \frac{L}{\pi a}\sin \frac{\pi \ell}{L} + 2g + c'_1 \tag{124}$$

where g is the boundary entropy of Affleck and Ludwig[14]. Another result is the entropy for the entropy of a thermal state of an infinite long strip at finite temperature

$$S_A = \frac{c}{3}\log \frac{\beta}{\pi a}\sinh \frac{\pi \ell}{\beta} + c'_1 \tag{125}$$

In these formulas c'_1 is a non universal constant. For a massive 1+1-dimensional relativistic QFT (which corresponds to an o?-critical quantum spin chain where the correlation length $\xi >> a$) the entanglement entropy for an infinite system divided into two semi-infinite pieces is

$$S_A = \frac{c}{6} \log \frac{\xi}{a} \tag{126}$$

If there are more intervals the corresponding entropy is given by (126) multiplied by the number of intervals. This is the 1D analogue of the area law. The result (126) is confirmed with the study of two exactly solvable models, the Ising model and the XXZ model. The main observation made by CC in this regard is that the density matrix ρ_A can be related to the Baxter' s corner transfer matrix whose eigenvalues are known for the latter models. This observation was made earlier by Nishino in connection with the Density Matrix Renormalization Group (DMRG)[15].

The general set up of the problem is a lattice QFT 1+1 with local commuting observables $\{\widehat{\phi}(x)\}$ with eigenvalues $\{\phi(x)\}$ and Hamiltonian $\widehat{H}$. The thermal state ρ at inverse temperature β has matrix elements

$$\langle \{\phi''(x'')\} | \rho | \{\phi'(x')\}\rangle = \frac{1}{Z(\beta)} \langle \{\phi''(x'')\} | e^{-\beta \widehat{H}} | \{\phi'(x')\}\rangle \tag{127}$$

where

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} \tag{128}$$

is the partition function. Eq.(127) can be expressed in terms of an euclidean path integral as

$$\langle \{\phi''(x'')\} | \rho | \{\phi'(x')\}\rangle = \frac{1}{Z(\beta)} \int [d\phi(x,\tau)] \prod_x \delta(\phi(x,0) - \phi'(x')) \prod_x \delta(\phi(x,\beta) - \phi''(x''))\, e^{-S_E} \tag{129}$$

where $S_E = \int_0^\beta d\tau\, L_E$ and L_E the euclidean Lagrangian. The normalization factor in (127) guarantees that $\mathrm{Tr}\, \rho = 1$. The partition function $Z(\beta)$ is obtained doing the path integral with the identification $\phi'(x) = \phi''(x)$ at $\tau = 0$ and $\tau = \beta$, and integrating over these variables. The geometry of the integration surface is a cylinder of length β.

Let us now take a system A made of a disjoint union of intervals $(u_1, v_1) \dots (u_N, v_N)$. One wants to compute the reduced density matrix ρ_A by tracing over the points not in A. This operation amounts to gluing those points in (129) and doing the integral over them. The effect is to leave cuts $(u_1, v_1) \dots (u_N, v_N)$ along the $\tau = 0$ line.

To compute $\mathrm{Tr}\, \rho_A^n$ one can use the replica trick. One first makes n copies of (129), labelled by $k = 1, \dots, n$ and gluing then together cyclically

$$\phi'_k(x) = \phi''_{k+1}(x), \quad (k = 1, \dots, n-1), \qquad \phi'_n(x) = \phi''_1(x), \qquad \forall\, x \in A \tag{130}$$

The path integral on this n-sheeted geometry is denoted as $Z_n(A)$ and hence

$$\mathrm{Tr}\, \rho_A^n = \frac{Z_n(A)}{Z^n} \tag{131}$$

CC then arguee that then LHS of (131) is analytic for all $\mathrm{Re}\, n > 1$ and that its derivative respect to n in the limit $n \to 1^+$ gives the entropy

$$S_A = -\lim_{n \to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n} \tag{132}$$

A final remark is that in 2D the log of a general partition function Z of a domain with total area $\mathcal{A}$ and boundaries with length $\mathcal{L}$ behaves according to Cardy and Peschel as[16]

$$\log Z = f_1 \frac{\mathcal{A}}{a^2} + f_2 \frac{\mathcal{L}}{a} + O(\log a) \tag{133}$$

where f_1 and f_2 are the non-universal bulk and boundary free energies and the term of order $\log a$ arises from points of non zero curvature and it is universal. Now, taking the log in (131) one sees that the area and boundary lengths of the n-sheet surface A and the n copies of it are the same, so they cancell and one is left with the result that $\mathrm{Tr}\, \rho_A^n$ only depends on the conical singularities at the brach points. This is also in agreement with the Holzhey at al results.

A. Entanglement entropy in 2D CFT

Consider a CFT with central charge c. CC first reproduce the Holzhey et al result for one interval, but the computation follows a different method.

A. Single interval

Here there is a single interval of length ℓ in an infinitely long 1D quantum system at zero temperature and no boundaries. Denote by w the coordinate of the system, so

$$A = (u, v), \qquad B = (-\infty, u) \cup (v, \infty), \qquad \ell = v - u \tag{134}$$

CC then performs the conformal transformation

$$w \to \zeta = \frac{w - u}{w - v} \tag{135}$$

so that the interval A is mapped into the negative real axis $\zeta < 0$ and the subsystem B into the positive one. In the construction of the density matrix ρ_A one joins the points along the subsystem B, hence the Riemann surface associated is the complex plane with a cut along the negative real axis $\zeta < 0$. Then one takes n copies of this cutted Riemann surface and joining them cyclically as in eq.(130). The result is a n-sheeted surface $\mathcal{R}_n$ whose coordinate is given by

$$z = \zeta^{1/n} = \left(\frac{w - u}{w - v} \right)^{1/n} \tag{136}$$

The various sheets of $\mathcal{R}_n$ correspond to

$$\begin{array}{lccc} 1-\mathrm{st\,sheet} & : & -\pi < \arg\zeta < \pi, & -\frac{\pi}{n} < \arg z < \frac{\pi}{n} \\ 2-\mathrm{nd\,sheet} & : & \pi < \arg\zeta < 3\pi, & \frac{\pi}{n} < \arg z < \frac{3\pi}{n} \\ \dots & & \dots & \\ \mathrm{n-th\,sheet} & : & (2n-3)\pi < \arg\zeta < (2n-1)\pi, & -\frac{3\pi}{n} < \arg z < -\frac{\pi}{n} \end{array} \tag{137}$$

Notice that in the z-coordinate the Riemann surface is just the complex plane $\mathbb{C}$. As in Holzhey et al we can related the energy momentum tensor $T(w)$ of $\mathcal{R}_n$ with the energy momentum tensor $T(z)$ on $\mathbb{C}$ by the eq. (recall (117))

$$T(w) = \left(\frac{dz}{dw} \right)^2 T(z) + \frac{c}{12}\{z; w\} \tag{138}$$

where $\{z; w\}$ is the Schwarzian derivative defined in eq.(118). Using that $\langle T(z) \rangle_{\mathbb{C}} = 0$ and the transformation (136) one finds

$$\langle T(w) \rangle_{\mathcal{R}_n} = \frac{c}{12}\{z; w\} = \frac{c(1 - n^{-2})}{24} \frac{(v - u)^2}{(w - u)^2 (w - v)^2} \tag{139}$$

This correlator is compared with that a three point correlator on $\mathbb{C}$ of $T(w)$ and two primary operators $\Phi_n(u)$ and $\Phi_{-n}(v)$ with conformal dimensions

$$\Delta_n = \overline{\Delta}_n = \frac{c(1 - n^{-2})}{24} \tag{140}$$

which by general CFT is

$$\langle T(w)\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}} = \frac{\Delta_n}{(w-u)^2(w-v)^2(v-u)^{2\Delta_n-2}(\overline{v}-\overline{u})^{2\Delta_n}} \tag{141}$$

with the normalization

$$\langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}} = |u-v|^{-2\Delta_n-2\overline{\Delta}_n} \tag{142}$$

In eq.(141) the coordinate w is assumed to be in the complex plane, not in $\mathcal{R}_n$. From the latter equations one finds

$$\langle T(w)\rangle_{\mathcal{R}_n} \equiv \frac{\int d\phi\, T(w)\, e^{-S_E(\mathcal{R}_n)}}{\int d\phi\, e^{-S_E(\mathcal{R}_n)}} = \frac{\langle T(w)\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}}}{\langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}}} \tag{143}$$

Now CC make an argument similar to the second derivation of the entropy formula by Holzhey et al (see previous section). The latter authors made a rescaling of the coordinates and study how the partition function of the cone changes. In the case of the n-sheeted geometry the argument goes as follows. First of all, one makes an infinitesimal change $w \to w' = w + \alpha(w)$ on $\mathbb{C}$, which acts identically on all the sheets of $\mathcal{R}_n$. Recalling eqs.(110, 111,115,116), the infinitesimal change induced on $Z_n(A)$ is

$$\delta \log Z_n(A) = n\left(\frac{1}{2\pi i}\int_C dw\, \alpha(w)\, \langle T(w)\rangle_{\mathcal{R}_n} - \frac{1}{2\pi i}\int_C d\overline{w}\, \overline{\alpha(w)}\, \langle\overline{T}(\overline{w})\rangle_{\mathcal{R}_n}\right) \tag{144}$$

where the contour C encircles the points u and v. The overall factor n arises because the transformation is done on all the sheets. On the other hand consider the correlator $\langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}}$. The Ward identities implies that under the conformal transformation $w \to w' = w + \alpha(w)$ the change is

$$\delta\langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}} = \frac{1}{2\pi i}\int_C dw\, \alpha(w)\, \langle T(w)\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}} - \frac{1}{2\pi i}\int_C d\overline{w}\, \overline{\alpha(w)}\, \langle\overline{T}(\overline{w})\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}} \tag{145}$$

so using eq.(143) one gets

$$\delta\langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}} = \langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}}\left(\frac{1}{2\pi i}\int_C dw\, \alpha(w)\, \langle T(w)\rangle_{\mathcal{R}_n} - \frac{1}{2\pi i}\int_C d\overline{w}\, \overline{\alpha(w)}\, \langle\overline{T}(\overline{w})\rangle_{\mathcal{R}_n}\right) \tag{146}$$

Hence, comparing (144) and (146) one derives

$$\delta \log Z_n(A) = n\, \log\delta\langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}} \Longrightarrow Z_n(A) \propto \left(\langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}}\right)^n \tag{147}$$

Namely, the ratio $Z_n(A)/Z_n$ behaves under conformal transformations as the n^{th} power of the two point correlator of the primary operator Φ_n with $\Delta_n = \overline{\Delta}_n = c/24(1-n^{-2})$. This implies in particular that

$$\mathrm{Tr}\,\rho_A^n = c_n\left(\langle\Phi(u)_n\Phi_{-n}(v)\rangle_{\mathbb{C}}\right)^n = \frac{c_n}{((v-u)/a)^{2n(\Delta_n+\overline{\Delta}_n)}} = c_n\left(\frac{v-u}{a}\right)^{-\frac{c}{6}(n-\frac{1}{n})} \tag{148}$$

The parameter a is the short distance cutoff which makes the expression (148) dimensionless. The constants c_n cannot be determined by this method, but normalization of the trace implies

$$\mathrm{Tr}\,\rho_A = 1 \Longrightarrow c_1 = 1 \tag{149}$$

Using now eq. (132) one recover the Holzhey et al formula

$$S_A = \frac{c}{3}\log\frac{\ell}{a} + c_1', \qquad \ell = v - u \tag{150}$$

The CC result (148) is generalized to more general situations:

$$\mathrm{Tr}\,\rho_A^n = c_n\,\left(\langle\Phi(u)_n\Phi_{-n}(v)\rangle\right)^n \tag{151}$$

where the correlator of the primary fields $\Phi_{\pm n}$ are computed on the corresponding geometry. Consider first the case of an infinite strip in the σ direction at finite inverse temperature β. The coordinate w describing the Riemann surface of integration in the path integral satisfies

$$w = \sigma + i\tau, \qquad -\infty < \sigma < \infty, \qquad 0 \leq \tau \leq \beta, \qquad w = w + i\,\beta \tag{152}$$

where the identification $w = w + i\beta$ implements the fact that one is computing a thermal state. To compute the correlator (152) in this case one maps the strip (152) into the complex plane by means of the map

$$z = e^{2\pi w/\beta}, \qquad w = \frac{\beta}{2\pi}\log z \tag{153}$$

which guarantees that $z(w) = z(w + i\beta)$. The correlator on the strip can then be found using the standard transformation law of primary fields,

$$\langle\Phi(w_1,\overline{w}_1)_n\Phi_{-n}(w_2,\overline{w}_2)\rangle \;=\; \left|\frac{dz_1}{dw_1}\right|^{2\Delta_n}\left|\frac{dz_2}{dw_2}\right|^{2\Delta_n}\langle\Phi(z_1,\overline{z}_1)_n\Phi_{-n}(z_2,\overline{z}_2)\rangle = \left|\frac{\pi a}{\beta}\frac{1}{\sinh\frac{\pi(w_1-w_2)}{\beta}}\right|^{4\Delta_n} \tag{154}$$

which yields

$$\mathrm{Tr}\,\rho_A^n = c_n\left(\frac{\beta}{\pi a}\sinh\frac{\pi\ell}{\beta}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \tag{155}$$

and

$$S_A(\beta) = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi\ell}{\beta}\right) + c_1' \tag{156}$$

In the low temperature regime compared to the size of the subsystem one finds

$$\ell << \beta \to S_A(\beta) \sim \frac{c}{3}\log\frac{\ell}{a} \tag{157}$$

as in the computation done before. In the high temperature regime one gets

$$\ell >> \beta \to S_A(\beta) \sim \frac{\pi c}{3}\frac{\ell}{\beta} \tag{158}$$

where the von Neumann entropy becomes an extensive quantity, i.e. $S_A \propto \ell$. It is also in agreement with the Gibbs entropy of an isolated system obtained in CFT. To show this let us recall the thermodynamic relation (in units of the Boltzmann constant $K_B = 1$)

$$S = -\frac{\partial F}{\partial T} = \beta^2\frac{\partial F}{\partial \beta}, \qquad \beta = T^{-1} \tag{159}$$

where F is the free energy, which in CFT is given by

$$\beta\,F = -\frac{\pi c}{6}\frac{\ell}{\beta} \tag{160}$$

so that (157) can be easily reproduced. The previous results can be easily generalized to the case of a subsystem of length ℓ in a circle of length L, which amounts basically to replace $\beta \to -iL$. The w- domain is given by

$$w = \sigma + i\tau, \qquad 0 \leq \sigma \leq L, \qquad -\infty < \tau < \infty, \qquad w = w + L \tag{161}$$

where the identification $w = w + L$ implements the periodicity of the system. The map to complex plane is given by

$$z = e^{2\pi i w/L}, \qquad w = -\frac{iL}{2\pi} \log z \tag{162}$$

which guarantees that $z(w) = z(w + L)$. The correlator on the strip is now

$$\langle \Phi(w_1, \overline{w}_1)_n \Phi_{-n}(w_2, \overline{w}_2) \rangle = \left| \frac{dz_1}{dw_1} \right|^{2\Delta_n} \left| \frac{dz_2}{dw_2} \right|^{2\Delta_n} \langle \Phi(z_1, \overline{z}_1)_n \Phi_{-n}(z_2, \overline{z}_2) \rangle = \left| \frac{\pi a}{L} \frac{1}{\sin \frac{\pi(w_1 - w_2)}{L}} \right|^{4\Delta_n} \tag{163}$$

which yields

$$\mathrm{Tr}\, \rho_A^n = c_n \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right)^{-\frac{c}{6}\left(n - \frac{1}{n}\right)} \tag{164}$$

and

$$S_A(\ell, L) = \frac{c}{3} \log \left(\frac{L}{\pi a} \sinh \frac{\pi \ell}{L} \right) + c_1' \tag{165}$$

which again reproduces the Holzhey et al result.

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