The Haldane-Shastry model of spin chains

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Introduction

The Hubbard model has been the subject of many research in the last years mainly related to its relevance to high Tc superconductors, heavy fermion materials, etc. These are systems of fermions with short range repulsive on site interactions. The Hamiltonian is

$$H = -t \sum_{\langle i,j \rangle, \sigma} (c^\dagger_{i,s}\, c_{j,s} + c^\dagger_{j,s}\, c_{i,s}) + U \sum_i n_{i,+} n_{i,-}$$

where $c_{i,s}$, $c^\dagger_{i,s}$ are fermionic creation and annihilation operators at the site i and spin $s = +$ (up), $s = -$ (down). The parameter t is the hopping amplitude and $U > 0$ is the on-site Coulomb repulsion.

This model was solved exactly in 1D by Lieb and Wu, using the Bethe ansatz. The elementary excitations were shown to consist of exotic objects called spinons which carry the spin degrees of freedom and holons which carry the charge degrees of freedom. Hence the elementary excitations are not quasiparticles, that is renormalized electrons with an effective mass $m_{\rm eff}$, charge $-e$ and spin $1/2$ as described by the Landau Liquid Theory. In 1D the electron breaks into

$$\text{electron} \rightarrow \text{spinon} + \text{holon}$$

This phenomena is called the spin-charge separation. These particles have relativistic dispersion relations, and they are described by the Luttinger Liquid Theory (LLT), which is the Cond-Mat version Conformal Field Theory. The CFTs used in connection to the LLT are gaussians models with a boson ($c = 1$), and eventually Wess-Zumino-Witten models, specially $SU(2)$ at level $k = 1$ which also has $c = 1$.

In 2D the model has not been solved exactly but has been the subject of many investigations. In particular it is not clear the existence of exotic particles or the spin-charge separation as in 1D.

From the numerical viewpoint the Hubbard model is difficult to study numerically. One of the reasons is that the Hilbert space of each site is four dimensional:

$$\text{Hubbard site}: \qquad |0\rangle, \qquad c^\dagger_{i,+}|0\rangle, \qquad c^\dagger_{i,-}|0\rangle, \qquad c^\dagger_{i,+} c^\dagger_{i,-}|0\rangle$$

where $|0\rangle$ is the vacuum states. In the limit where $U >> t$, there will be a huge energy cost for two electrons, with spin up and down, to occupy the same site. It is then possible to obtain an effective model where the doubly occupied sites have been "integrated out" leaving and effective Hamiltonian

$$H_{\rm t-J} = -t \sum_{\langle i,j\rangle,\sigma} (\widetilde{c}^{\dagger}_{i,s}\ \widetilde{c}_{j,s} + \widetilde{c}^{\dagger}_{j,s}\ \widetilde{c}_{i,s}) + J\sum_{\langle i,j\rangle}(\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j + \frac{1}{4} n_i n_j) + \text{three body term}$$

where $\widetilde{c}_{j,s}$ are the projection of the fermionic operators to the on site Hilbert space

$$\text{t} - \text{J site}: \qquad |0\rangle, \qquad \widetilde{c}^{\dagger}_{i,+}|0\rangle, \qquad \widetilde{c}^{\dagger}_{i,-}|0\rangle,$$

$\mathbf{S}_i$ is the spin 1/2 operator at the site $i = 1, \ldots, N$ built from the fermionic ops as

$$\vec{\mathbf{S}}_i = \frac{1}{2} \widetilde{c}^{\dagger}_{i,s}\ \vec{\sigma}_{s,s'}\ \widetilde{c}_{i,s},$$

n_i is the number of electrons at site i

$$n_i = n_{i,+} + n_{i,-}, \qquad n_{i,s} = c^{\dagger}_{i,s} c_{i,s}, \qquad 0 \leq \langle n_i \rangle \leq 1$$

and J is a coupling constant related to U and t as

$$J = \frac{t^2}{4U}$$

This model has also been the subject of many investigations specially in connection to high-Tc superconductors, where it has become the standard model. In 1D it is exactly solvable provided $J = 2t$ and commutes with the generators of the superalgebra $SU(1|2)$.

In the Hubbard and tJ model an important quantity is the so called filling fraction defined as

$$x = \frac{N_e}{N}, \qquad \begin{cases} 0 \leq x \leq 2 & \text{Hubbard model} \\ 0 \leq x \leq 1 & \text{tJ model} \end{cases}$$

where N_e is the total number of electrons which is a conserved quantity:

$$N_e = \sum_i n_i,$$

At $x = 1$ in the tJ model there is an electron per site. Electrons cannot move since doubly occupied sites are forbidden, so the only available degree of freedom is the spin one. This case is refereed to as half filling, and the Hamiltonian is reduces to

$$H_H = J \sum_{\langle i,j\rangle} \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j$$

up to a constant. This is the famous spin 1/2 antiferromagnetic Heisenberg model which describes the dynamics of the spin degrees of freedom. The charge degrees of feedom have been frozen. The Hilbert space on one site is then given

$$\text{Heisenberg site}: \qquad |\uparrow\rangle = c^{\dagger}_{i,+}|0\rangle, \qquad |\downarrow\rangle = c^{\dagger}_{i,-}|0\rangle,$$

We have the chain of models

$$\text{Hubbard model} \stackrel{U>>t}{\Longrightarrow} \text{tJ model} \stackrel{x=1}{\Longrightarrow} \text{Heisenberg model}$$

with the corresponding reduction of degrees of freedom of the total Hilbert space

$$4^N \Longrightarrow 3^N \Longrightarrow 2^N$$

One can also go directly from the Hubbard to the Heisenberg model :

$$\text{Hubbard model} \overset{U/t\to\infty, x=1}{\Longrightarrow} = \text{Heisenberg model}$$

In particular the ground state (GS) of both models will be related by

$$\lim_{U/t\to\infty, x=1} |GS, \text{Hubbard}\rangle = |GS, \text{Heisenberg}\rangle$$

This relation is valid in any dimension. In 2D the cuprate superconductors are characterized by a dopping factor x which is related to the filling fraction defined above. The undoped compounds, which are called parent materials, correspond to filling fraction $x = 1$. These compounds are fairly well described by the 2D AF Heisenberg model (AFH) , which exhibits AF long rangle order and whose excitations are magnons with spin ± 1. This 2D model has been analized using the spin wave theory (which is a mean field theory), the O(3) non linear sigma model, numerical methods, MonteCarlo, etc.

The spin 1/2 AFH model in 1D was solved by Bethe in 1991 using the famous Bethe ansatz, which marked the beginning of the many body exactly solvable models, together the Onsager solutions of the 2D Ising model in 1944.

A. The antiferromagnetic Heisenberg model in 1D

The Hamiltonian of Heisenberg model for a chain with N spins with periodic boundary conditions is ($\vec{\mathbf{S}}_{N+1} = \vec{\mathbf{S}}_1$)

$$H = J \sum_{i=1}^{N} \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1}$$

where J is called the exchange coupling constant which is positive for antiferromagnetic interactions which favor that the nearest neighbour (NN) spins are antiparallel, while negative values correspond to ferromagnetic interactions wich favors parallel NN spins. The ferromagnetic case is much more easier to analyze than the AF case.

The Bethe ansatz (BA) is a choice of the eigenstates of the Hamiltonian H which consist in the linear superposition of plane waves of magnons. The starting point of the BA is the fully ferromagnetic state with all the spins up:

$$|F\rangle = |+, +, \ldots, +\rangle$$

which is an eigenstate of H with energy

$$E_0^F = \frac{JN}{4}$$

This is the GS of the ferromagnetic Heisenberg model ($J < 0$) and has third component of the spin $S^z = N/2$. In the sector with $S^z = N/2 - 1$ there are N states. The eigenstates of H can be build from the plane waves

$$|k\rangle = \sum_{x=1}^{N} e^{ikx} |x\rangle, \qquad |x\rangle = S_x^- |F\rangle$$

$|x\rangle$ is the state with all spins up expect at the site x where the spin is down, due to the application of the lowering operator $S_x^- = S_x^1 - iS_x^2$. Writting H as

$$H = J \sum_{i=1}^{N} \left(S_i^z \; S_{i+1}^z + \frac{1}{2}(S_i^+ \; S_{i+1}^- + S_i^- \; S_{i+1}^+) \right)$$

one easily finds

$$H|x\rangle = (E_0 - J)|x\rangle + \frac{J}{2}(|x+1\rangle + |x-1\rangle)$$

and

$$H|k\rangle = (E_0 + \varepsilon_{\rm mag}(k))|k\rangle$$

where

$$\varepsilon_{\rm mag}(k) = \frac{J}{2}(\cos k - 1) = -J\sin^2\frac{k}{2},$$

is the magnon dispersion relation for the momentum k whose quantization follows from the periodicity of the magnon wave function

$$e^{ikN} = 1 \rightarrow \qquad k = \frac{2\pi n}{N}, \qquad 0 \leq k < 2\pi$$

In the F case the magnon dispersion energy are positive and in the vecinity of $k = 0,\ \pi$ behaves has the non relativistic form $\varepsilon_{mag} \sim Jk^2/4$. In the AF case the energy of the magnons is negative meaning that the F state is a false vacuum. The two body magnon states are given by

$$|k_1, k_2\rangle = \left(A(k_1,k_2)e^{i(k_1x_1+k_2x_2)} + A(k_2,k_1)e^{i(k_2x_1+k_1x_2)}\right)|x_1,x_2\rangle$$

where the amplitudes A are related by the so called S matrix scattering amplitude

$$S(k_1,k_2) = \frac{A(k_2,k_1)}{A(k_1,k_2)} = -\frac{1-2e^{ik_2}+e^{i(k_1+k_2)}}{1-2e^{ik_1}+e^{i(k_1+k_2)}}$$

The positions of the magnons are ordered as $x_1 < x_2$. A state with M magnons is a linear superposition of plane waves with momenta $k_1, \ldots, k_M$:

$$|k_1,\ldots,k_M\rangle = \sum_P A(k_{P1},\ldots,k_{PM})e^{i(k_{P1}x_1+\cdots+k_{PM}x_M)}|x_1,\ldots,x_M\rangle$$

where one sums over the M! permutations P of the coordinates of the magnons and

$$|x_1,\ldots,x_M\rangle = S^-_{x_1}\ldots S^-_{x_M}|F\rangle, \qquad x_1 < \cdots < x_M.$$

The amplitudes A_P can be factorized into the product of two body S-matrices, e.g.

$$\begin{aligned} A(k_3,k_2,k_1) &= S(k_1,k_2)A(k_3,k_1,k_2) = S(k_1,k_2)S(k_1,k_3)A(k_1,k_3,k_2) \\ &= S(k_1,k_2)S(k_1,k_3)S(k_2,k_3)A(k_1,k_2,k_3) \end{aligned}$$

The momenta which correspond to eigenstates of H satisfy the Bethe ansatz equation

$$e^{ik_i} = \prod_{j(\neq i)} S(k_i,k_j), \qquad i = 1,\ldots,M$$

which means that the total phase shift of a mangon after crossing all the other ones is equal to one, for the wave function to be single valued. This is a complicate set of non linear differential eqs. that can be solved analytically in the thermodynamic limit $N >> 1$. The main results that one obtains for the AF case are

- The GS has total spin equal to zero $S_{tot} = 0$ for a chain with an even number of sites and $S_{tot} = 1/2$ for a chain with an odd number of sites. These states are constructed with $M = N/2$ (N even) magnons and $M = (N-1)/2$ (N odd) magnons respectively.

- The elementary excitations are spin 1/2 particles with dispersion relation

$$\varepsilon_{sp}(k) = v_s \sin k, \qquad v_s = \frac{\pi J}{2}, \qquad 0 < k < \pi$$

which near $k \sim 0$ and π can be linearized, i.e. $\varepsilon_{\rm sp} \sim v_s k$ or $\varepsilon_{sp} \sim v_s |k - \pi|$ which corresponds to a relativistic dispersion relation. v_s is the spinon velocity which plays the role of the speed of light in a Quantum Field Theory. Notice that the range of the momenta of the spinons is half of the Brillouin zone $(0, 2\pi)$. For a long time it was stated in the physical literature, that spin waves of the antiferromagnetic chain of spin 1/2 magnets has spin 1. Indeed, spin wave, i.e. magnon, corresponds to a turn of one spin, amounting to spin $1/2 + 1/2 = 1$. However, Faddeev and Taktajan showed that turn of a spin corresponds to 2 spinons, so that the momentum of this state runs through the whole Brillouin zone $0 \leq k \leq 2\pi$. This is a non perturbative phenomena:

$$magnon\ (S^z = 1) = spinon\ (S^z = 1/2) + spinon\ (S^z = 1/2)$$

- The spin susceptibility at zero temperature and zero magnetic field is

$$\chi_0 = \frac{1}{J\pi^2}$$

The product of χ_0 and the spinon velocity v_s is a dimensionaless quantity which plays the role of Wilson ratio

$$v_s\ \chi_0 = \frac{1}{2\pi}$$

- The low energy spectrum can be described by a CFT given by a $SU(2)$ Wess-Zumino-Witten model at level 1. The Virasoro central charge of this model is $c = 1$, which coincides with that of a free massless boson. Indeed this CFT can be constructed from a massless boson.

Guzwiller proyection

The connection between the Hubbard and Heisenberg models implied that

$$\lim_{U/t \to \infty, x=1} |GS, \text{Hubbard}\rangle = |GS, \text{Heisenberg}\rangle$$

Let us define the Guztwiller proyector as the operator that projects out the states where a site is double occupied

$$\begin{aligned} P_G &= \prod_{i=1}^{N} (1 - n_{i,+}\, n_{i,-}) \\ P_G |0\rangle &= |0\rangle, \qquad P_G\, c^\dagger_{i,s}|0\rangle = c^\dagger_{i,s}|0\rangle, \qquad P_G\, c^\dagger_{i,+} c^\dagger_{i,-}|0\rangle = 0 \end{aligned}$$

The limit $U/t \to \infty$ enforces that the operator P_G is effectively realized on the Hubbard GS and in particular

$$P_G \lim_{U/t \to \infty, x=1} |GS, \text{Hubbard}\rangle = \lim_{U/t \to \infty, x=1} P_G\ |GS, \text{Hubbard}\rangle = |GS, \text{Heisenberg}\rangle$$

One may wonder which are the properties of the state obtained by Gutzwiller projection, not of the GS at $U/t = \infty$, but the GS at $U/t = 0$, namely the non interacting Fermi sea.

$$|\psi_G\rangle = P_G\,|FS\rangle, \qquad |FS\rangle = \prod_{|k|<k_F} c^\dagger_{k,+}\, c^\dagger_{k,-}|0\rangle$$

where the Fermi momenta is choosen to guarantee that there is an electron per site, so that the state has only spin degrees of freedom.

$$k_F = \frac{\pi}{2} \to \langle\psi_G|n_i|\psi_G\rangle = 1, \qquad \forall i$$

In the state $|\psi_G\rangle$, the electrons are free to move along the lattice but they are forced not to occupy the same state, which is in a way equivalent to the $U/t = \infty$ limit.

Gebhard and Vollhardt computed in 1987 the exact expression of the spin-spin correlation function[14]

$$\langle S^z_i\; S^z_{i+j}\rangle \equiv \frac{\langle\psi_G|S^z_i\; S^z_{i+j}|\psi_G\rangle}{\langle\psi_G|\psi_G\rangle} = \frac{Si(\pi j)}{\pi}\,\frac{(-1)^j}{j}, \qquad j > 0$$

where $Si(x)$ is the sine integral

$$Si(x) = \int_0^x dy\;\frac{\sin y}{y}$$

The alternating sign, $(-1)^j$, and the algebraic decay of the spin-spin correlations, $1/j$, are similar to the behaviour of this correlator in the GS of the nearest neighbour AFH model. A comparison of the exact NN and NNN of these correlators between the two models shows also their proximity:

$$\langle S^z_i\; S^z_{i+j}\rangle = \begin{cases} j & \psi_G & \psi_{AFH} \\ 1 & -0.589490 & -0.5908623 \\ 2 & 0.225706 & 0.242719 \end{cases}$$

These results suggested that the state ψ_G have properties rather close to those of the AFH model, so a natural question was: what is the Hamiltonian for which ψ_G is the exact GS. The answer was given independently by Haldane and Shastry in 1988.

Haldane-Shastry model

The Guzwiller projected state in the spin basis

Let us first define the Klein operators

$$a_{n,+} = c_{n,+}, \qquad a_{n,-} = c_{n,-}e^{i\pi N_+}, \qquad N_+ = \sum_i n_{i,+}$$

which commute among themselves

$$[a_{n,+}, a_{m,-}] = 0, \qquad \forall n, m$$

while they satisfy canonical anticommutations relations between operators with the same spin

$$\{a_{n,s}, a^\dagger_{m,s}\} = \delta_{n,m}, \qquad \{a_{n,s}, a_{m,s}\} = 0, \qquad \forall n, m$$

Now let us consider the state

$$|\phi\rangle = \prod_{k\in K} a^\dagger_{k,+} \prod_{q\in Q} a^\dagger_{q,-}|0\rangle$$

where Q is the set of momenta of the M spins down and K the set of the momenta of the $N-M$ spins up (so that $S^z = N/2 - M$). Now we perform a particle-hole transformation on the up spins:

$$|0\rangle \leftrightarrow a^\dagger_n\,|0\rangle$$

in terms of the unitary transformation $a^\dagger_m + a_m$ which satisfies

$$(a^\dagger_{m,+} + a_{m,+})n_{m,+} = (1 - n_{m,+})(a^\dagger_{m,+} + a_{m,+})$$

The operator that performs this transformation on all sites is

$$U = (a^\dagger_{N,+} + a_{N,+})\dots(a^\dagger_{1,+} + a_{1,+})$$

One considers the transform state

$$|\overline{\psi}\rangle = UP_G|\phi\rangle = \overline{P}_G\,U|\phi\rangle, \qquad \overline{P}_G = UP_GU^\dagger = \prod_{m=1}^{N}(1 - n_{m,-} + n_{m,+}n_{m,-})$$

The action of U can be simply computed finding

$$|\overline{\psi}\rangle = \overline{P}_G \prod_{k\in -K} a^\dagger_{k,+} \prod_{q\in Q} a^\dagger_{q,-}|0\rangle$$

where $-K$ is the set complementary of K. Expanding the momentum operator in terms of the position operators

$$a^\dagger_{k,s} = \frac{1}{\sqrt{N}}\sum_n e^{ikn}a^\dagger_{n,s}$$

The number of momenta in the sets Q and $-K$ is equal to M (recall that K contains $N-M$ momenta). Introducing the above eq in $\overline{\psi}$ one finds (up to an overall constant)

$$\begin{aligned}
|\overline{\psi}\rangle &= \overline{P}_G\, a^\dagger_{p_1,+}\dots a^\dagger_{p_M,+}a^\dagger_{q_1,-}\dots a^\dagger_{q_M,+}|0\rangle \\
&= \sum_{n_1,\dots,n_M}\sum_{n'_1,\dots,n'_M} e^{i(p_1n_1+\dots+p_Mn_M)}e^{i(q_1n'_1+\dots+q_Mn'_M)}\overline{P}_G\, a^\dagger_{n_1,+}\dots a^\dagger_{n_M,+}a^\dagger_{n'_1,-}\dots a^\dagger_{n'_M,-}|0\rangle \\
&= \sum_{n_1,\dots,n_M} \det(e^{ip_in_j})\ \det(e^{iq_in_j})\, b^\dagger_{n_1}\dots b^\dagger_{n_M}|0\rangle
\end{aligned}$$

where p_j $(j = 1,\dots,M)$ label the momenta in $-K$ and q_j $(j = 1,\dots,M)$ the momenta in Q. The antisymmetric character of the fermion operators leads to the product of the two determinants, while the proyector $\overline{P}_G$ forces that every time there is a spin down fermion there is also an spin up, and they appear in the same amount. $b^\dagger_n$ is the hard core boson operator

$$b^\dagger_n = a^\dagger_{n,+}\,a^\dagger_{n,-}$$

This operator gives rise to a spin 1/2 representation at the nth- site

$$S_n^+ = b_n, \qquad S_n^- = b_n^\dagger, \qquad S_n^z = \frac{1}{2} - b_n^\dagger b_n$$

so that the state $\overline{\psi}$ is equivalent to the spin state

$$|\psi\rangle \;=\; \sum_{n_1,\dots,n_M} \det(e^{ip_i n_j})\; \det(e^{iq_i n_j})\; S_{n_1}^- \dots S_{n_M}^- |F\rangle$$

where $|F\rangle$ is the ferromagnetic state used in the Bethe ansatz construction.

Let us come back to the Gutzwiller state at half filling $M = N/2$. Choosing for simplicity antiperiodic BCs for the fermions, the momenta of the particles are ($N/2$ an even number)

$$k_n = \frac{2\pi}{N}(n + \frac{1}{2}), \qquad n = -\frac{N}{4}, \dots, \frac{3N}{4} - 1$$

The momenta corresponding to the FS (i.e. set Q) and its complementary (i.e. set $-K$) correspond to the values

$$q_n : \; n = -\frac{N}{4}, \dots, \frac{N}{4} - 1, \qquad p_n = \pi - q_n \Longrightarrow e^{ip_i n_j} = e^{i\pi n_i}\; e^{-iq_i n_j}$$

Hence the two determinants became the same, up to a phase, and the wave function of the spin deviations is

$$\psi(n_1, \dots, n_M) = e^{i\pi \sum_i n_i} |\det(e^{iq_i n_j})|^2$$

where we recall that the coordinates $n_1, \dots, n_M$ denotes the position of the hard core bosons or equivalent, the down spins. Defining the phase

$$z = e^{2\pi i/N} \longrightarrow e^{iq_j n_l} = z^{(j+1/2)n_l}$$

the determinant becomes

$$\det(e^{iq_j n_l}) = z^{\sum_l n_l/2} \det(z^{j n_l})$$

Since we are interested in the absolute value of this determinant, the overall phase does not matter and the range of j can be taken from 0 to $M-1$. Using the variables $z_j = z^{n_j}$, one obtains a Vardemonde determinant for the $M \times M$ matrix

$$\det(z^{j n_l}) = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_M \\ z_1^2 & z_2^2 & \dots & z_M^2 \\ \dots & \dots & \dots & \dots \\ z_1^{M-1} & z_2^{M-1} & \dots & z_M^{M-1} \end{pmatrix} = (-1)^{M(M-1)/2} \prod_{i<j} (z_i - z_j)$$

and since

$$|z_i - z_j| = |e^{2\pi i n_i} - e^{2\pi i n_j}| = 2|\sin\left(\frac{\pi(n_i - n_j)}{N}\right)|$$

one finally finds the wave function

$$\psi(n_1, \dots, n_M) \propto e^{i\pi \sum_i n_i} \prod_{i<j} \sin^2\left(\frac{\pi(n_i - n_j)}{N}\right)$$

Comments:

- The wave function has the typical Jastrow form

$$\psi(r_1, \ldots, r_N) = \prod_{i<j} f(r_i - r_j)$$

- The term $e^{i\pi \sum_i n_i}$ satisfies the sign Marshall rule:

$$e^{i\pi \sum_i n_i} = \begin{cases} 1 & \sum_i n_i : even \\ -1 & \sum_i n_i : odd \end{cases}$$

so that the sign of the wave function is 1 if the number of down spins in the odd sites is even, and -1 it that number is odd. This rule also applies to the GS of the AFH Hamiltonian. Its origin is the Perron-Frobenius theorem, which states that the eigenvector with highest eigenvalue of an irreducible symmetric matrix $A_{i,j}$ with non-negative entries, can be choosen with all its entries non negative real numbers:

$$A_{i,j} \geq 0, \ \ \forall i, j, \qquad \sum_j A_{i,j} v_j = \lambda_{max} \, v_j, \Longrightarrow v_i \geq 0$$

Consider now the operator

$$V = \prod_i \sigma^z_{2i+1}$$

where σ^z_i is the Pauli matrix that gives 1 (-1) if the site i has a spin up (down). Acting on the wave function ψ one obtains

$$V|\psi\rangle = \sum_{n_1, \ldots, n_M} e^{i\pi \sum_i n_i} \, \psi(n_1, \ldots, n_M) \, S^-_{n_1} \ldots S^-_{n_M} |F\rangle$$

so that the new wave function has all its entries positive as a Perron-Frobenius vector. In the AFH model, the GS satisfies the Marshall rule because the operator V generates a unitary transformation of the AFH Hamiltonian

$$V H_H V^\dagger = J \sum_i \left(S^z_i S^z_{i+1} - \frac{1}{2}(S^+_i S^-_{i+1} + S^-_i S^+_{i+1}) \right)$$

which, up to an overall constant, is a matrix with negative entries. Hence the vector with lowest eigenvalue, i.e. the GS, can be choosen with all its components positive.

- Using $z_i = z^{n_i}$, the wave function can also be written as

$$\psi(n_1, \ldots, n_M) \propto \prod_i z_{n_i} \prod_{i<j} (z_{n_i} - z_{n_j})^2, \qquad z_n = e^{2\pi i n/N}$$

- Consider the following wave function for a spin system with N spin variables, $s_n = \pm 1$ ($n = 1, \ldots, N$), and total spin zero, $\sum_n s_n = 0$:

$$|\Phi\rangle = \sum_{s_1, \ldots, s_N} \Phi(s_1, \ldots, s_N) \, |s_1, \ldots, s_N\rangle$$

where

$$\Phi(s_1,\ldots,s_N) = e^{i\pi/2\sum_{n:odd}(s_n-1)} \prod_{n>m}^{N} \left[\sin\left(\frac{\pi(n-m)}{N}\right)\right]^{s_n s_m/2}$$

The integers n denotes the position of the spins, regardeless they are up or down. This wave function is proportional to the HS wave function shown above. To show this connection, let us define the variable a_n as

$$s_n = 1 - 2a_n \Longrightarrow a_n = 0\ (1),\quad s_n = 1\ (-1)$$

Plugging this into the previous eq. one finds, up to overall constants

$$\Phi(s_1,\ldots,s_N) \propto e^{i\pi\sum_{n:odd} a_n} \prod_{n<m}^{N} \left[\sin\left(\frac{\pi(n-m)}{N}\right)\right]^{2a_n a_m - a_n - a_m}$$

The term proportional to $a_n + a_m$ gives an overall constant using translational invariance and the fact that $\sum_n a_n = N/2$. The last equation is a consequence of the eq. $\sum s_n = 0$. One is left with

$$\Phi(s_1,\ldots,s_N) \propto e^{i\pi\sum_{n:odd} a_n} \prod_{n<m}^{N} \left[\sin\left(\frac{\pi(n-m)}{N}\right)\right]^{2a_n a_m}$$

Finally, the sites n where the spins are up, contribute with 1, so the wave function only depends on the positions n_i os the spins down, i.e. $a_{n_i} = 1$, i.e.

$$\Phi(s_1,\ldots,s_N) \propto e^{i\pi\sum_i n_i} \prod_{i<j}^{N/2} \left[\sin\left(\frac{\pi(n_i-n_j)}{N}\right)\right]^{2}$$

which coincides with the HS wave function. The wave function expressed in terms of the spin variables s_n will be later used to relate the HS model with the WZW model.

The Haldane-Shastry Hamiltonian

Haldane and Shastry found in 1988[1,2] that the Gutwiller wave function is the exact eigenstate of the Hamiltonian with long range exchange couplings

$$H = \frac{J\pi^2}{N^2} \sum_{n<m} \frac{\vec{\mathbf{S}}_n \cdot \vec{\mathbf{S}}_m}{\sin^2(\pi(n-m)/N)}$$

Their work was motivated by previous results of Sutherland for a continuum bose gas model that we shall review later on. The coupling constant between two spins has an interesting geometrical meaning. It is inversely proportional to the square of the distance between two arbitrary sites on the ring,

$$d(z_n, z_m) = |z_n - z_m| = 2|\sin\left(\frac{\pi(n-m)}{N}\right)|, \qquad n, m = 1,\ldots,N$$

Following Shastry we perform the unitary V transformation to the Hamiltonian which becomes

$$H = J_0 \sum_{n \neq m} \frac{1}{\sin^2(\pi(n-m)/N)} \left[S_n^z S_m^z + \frac{(-1)^{n+m}}{2} (S_n^+ S_m^- + S_n^- S_m^+) \right],$$

where $J_0 = J\pi^2/(2N^2)$. Next we express the spin operators in terms of the hard core boson operators in the subspace of M bosons (i.e. $M = \sum_n \langle b_n^\dagger b_n \rangle$):

$$H = -\frac{1}{2} \sum_{n \neq m} J_{n-m}^z (b_n^\dagger b_m + H.C.) + \sum_{n \neq m} J_{n-m}^\perp b_n^\dagger b_n b_m^\dagger b_m + E_0'(N,M)$$

where

$$J_n^z = \frac{J_0}{\sin^2(\pi n/N)}, \qquad J_n^\perp = (-1)^{n+1} J_n^z$$

and

$$E_0'(N,M) = (\frac{N}{4} - M) \sum_{n=1}^{N-1} J_n^z = J_0 \frac{N^2-1}{3} (\frac{N}{4} - M)$$

In the last expression we have used

$$\sum_{n=1}^{N} \frac{1}{\sin^2(\pi n/N)} = \frac{1}{3}(N^2-1)$$

which is a particular case of the general identity found by Haldane[1]

$$S_{00}(J) = \sum_{n=1}^{N-1} \frac{z^{Jn}}{(1-z^n)(1-z^{-n})} = \frac{1}{12}(N^2-1) - \frac{1}{2}J(N-J), \qquad z = e^{2\pi i/N}, \qquad 0 \leq J \leq N$$

The ferromagnetic state, $M = 0$, has an energy equal to

$$E_0^F \equiv E_0(N,0) = E_0'(N,0) = J_0 \frac{N^2-1}{3} \frac{N}{4} = \frac{J\pi^2}{24}(N - \frac{1}{N})$$

which grows linearly with the size of the systems and has a $1/N$ correction.
The one magnon state with momentum k is given by

$$|k\rangle = \sum_n e^{ikn} \, S_n^- |F\rangle = \sum_n e^{ikn} \, b_n^\dagger |0\rangle$$

It is not difficult to show that this is an eigenstate of H

$$H \, |k\rangle = (E_0^F + \varepsilon_{mag}(k))|k\rangle$$

where the magnon dispersion relation is

$$\varepsilon_{mag}(k) = \frac{J}{4}(k^2 - \pi^2), \qquad |k| \leq \pi$$

In the AF situation, $J > 0$, this quantity is negative as in the AFH model. However the dependence of the momenta is quadratic while in the latter model is $\cos k - 1$. In Haldane' s derivation the dispersion relation is $K(K - 2\pi)/4$.

The difference, as compared with Shastry result is that in the latter the Hamiltonian has been $V-$transformed which adds a momentum π to all the momenta, i.e. $K = \pi + k$.

The HS state with two magnon, $M = 2$, is given by

$$|\psi\rangle = \sum_{n_1,n_2} \psi(n_1, n_2) b^\dagger_{n_1} b^\dagger_{n_2} |0\rangle, \qquad \psi(n_1, n_2) = \sin^2(\pi(n_1 - n_2)/N)$$

Employing Haladane' s identity for $S_{00}(J)$ one can show that this is an eigenstate state of H with eigenvalue

$$E_0(N, 1) = E_0^F + \frac{J\pi^2}{2}(-1 + \frac{4}{N^2})$$

The case with $M = 3$ is already quite involved to prove. The basic identity which is needed is[1]

$$\cot(\theta_1 - \theta_2)\cot(\theta_2 - \theta_3) + (\text{cyclic permutation of } 1, 2, 3) = 1$$

The GS of the HS for general values of $M \leq N/2$ is

$$E_0(N, M) = E_0^F + J\pi^2 M(\frac{M^2 - 1}{3N^2} - \frac{1}{4})$$

which reproduces the previous results for $M = 1$ with $k = 0$ and $M = 2$. The minimun value of $E_0(N, M)$ in the interval $0 \leq M \leq N/2$, corresponds to half filling

$$M = \frac{N}{2} \rightarrow E_0^{HS}(N) \equiv E_0(N, \frac{N}{2}) = -\frac{J\pi^2}{24}(N + \frac{5}{N})$$

Spin susceptibility

Let us add to the HS Hamiltonian a magnetic field h the the z-direction

$$H = H_{HS} + h\, S^z_{tot}$$

The energy of a state with M magnons will be

$$E(N, M, h) = E_0(N, M) + h(\frac{N}{2} - M)$$

Minimizing this energy respect to M, gives the magnetic field $h = h(m)$ as a function of the magnetization $m = M/N$

$$\frac{dE(N, M, h)}{dM} = 0 \Longrightarrow h(m) = \frac{dE_0(N, M)}{dM} = J\pi^2 \left(m^2 - \frac{1}{4}\right)$$

The spin susceptibility at zero temperature is defined as

$$\chi(h) = \frac{dm}{dh} = \left(\frac{dh}{dm}\right)^{-1} = \frac{1}{2J\pi^2 m}$$

which at zero field $h = 0$, i.e. $m = 1/2$ is

$$\chi(0) = \frac{1}{J\pi^2}$$

This value coincides with the spin susceptibility of the AFH chain.

Pair correlation function

The pair correlation function of the state

$$|\psi\rangle = \sum_{n_1,\dots,n_M} \psi(n_1,\dots,n_M) b^\dagger_{n_1}\dots b^\dagger_{n_M}\ |0\rangle$$

is defined as

$$\langle b^\dagger_n\ b_m\rangle = \frac{\langle\psi|b^\dagger_n\ b_m|\psi\rangle}{\langle\psi|\psi\rangle}$$

and it coincides with the spin correlator $\langle S^-_n\ S^+_m\rangle$, and its expression is given by

$$\langle b^\dagger_n\ b_m\rangle = M(M-1)C^{-1/2} \sum_{n_2,\dots,n_M} \psi(n,n_2,\dots,n_M)\psi(m,n_2,\dots,n_M)$$

where C is the normalization of ψ. The sum over the integers can be quite complicated to do directly however one can replace these sums by integrals and the use the results obtained by Sutherland of the pair correlator in the continuous bose model. To explain why this happens, let us consider the determinant form of the wave function that we obtained earlier:

$$\psi(n_1,\dots,n_M) = e^{i\pi\sum_i n_i} |\det(e^{ik_i n_j})|^2$$

where

$$k_n = \frac{2\pi}{N}(n+\frac{1}{2}), \qquad n = -\frac{N}{4},\dots,\frac{N}{4}-1, \qquad |k_n| < \frac{\pi}{2}$$

Expanding the determinants one has

$$\det(e^{ik_i n_j}) = \sum_{P\in\mathcal{S}_M} \epsilon_P\ e^{i(k_{P1}n_1+\dots+k_{PM}n_M)}$$

where P are the permutations of the symmetric group $\mathcal{S}_M$ and ε_P the their sign. The wave function is then

$$\psi(n_1,\dots,n_M) = e^{i\pi\sum_i n_i} \sum_{P,Q\in\mathcal{S}_M} \epsilon_P\ \epsilon_Q\ e^{i((k_{P1}-k_{Q1})n_1+\dots+(k_{PM}-k_{QM})n_M)}$$

and

$$\psi(n,n_2,\dots,n_M)\psi(m,n_2,\dots,n_M) = e^{i\pi(n+m)} \sum\nolimits_{P,Q,P',Q'\in\mathcal{S}_M} \epsilon_P\ \epsilon_Q\ \epsilon_{P'}\ \epsilon_{Q'}$$
$$\times e^{i[(k_{P1}-k_{Q1})n+(k_{P'1}-k_{Q'1})m+(k_{P2}-k_{Q2}+k_{P'2}-k_{Q'2})n_2+\dots+(k_{PM}-k_{QM}+k_{P'M}-k_{Q'M})n_M]}$$

The pair correlator involves $M-1$ sums over the integers $n_2,\dots,n_M$ of the form

$$\sum_{n=1}^{N} e^{iKn}, \qquad K = k_a - k_b + k_c - k_d = \frac{2\pi\times\text{integer}}{N}, \qquad |K| < 2\pi,$$

where the bound on K follows from the condition $|k| < \pi/2$. It is then a simple matter to check that under these conditions

$$\sum_{n=1}^{N} e^{iKn} = \begin{cases} N, & \text{if } K = 0 \\ 0, & \text{if } K \neq 0 \end{cases}$$

where the zero result follows from the eq. $e^{iKN} = 1$. On the other hand, treating n as a continous variable between $(0, N)$ one has

$$\int_0^N dn\, e^{iKn} = \begin{cases} N, & \text{if } K = 0 \\ 0, & \text{if } K \neq 0 \end{cases}$$

Hence, to compute the pair correlator one can make the replacement

$$\sum_{n=1}^{N} \rightarrow \int_0^N dn$$

so that the real-space (N point) correlations of the lattice model are identical to those of the equivalent state of the continuum Bose gas model.

Haldane considered a more general form of the wave function

$$\psi(n_1, \ldots, n_N) \propto e^{2\pi i J \sum_i n_i/N} \prod_{i<j}^{N/2} \left[\sin\left(\frac{\pi(n_i - n_j)}{N} \right) \right]^2$$

where J belongs to the range

$$M - 1 \leq J \leq N - M + 1$$

The case considered by Shastry corresponds to $J = N/2$. In the thermodynamic limit with

$$\frac{M}{N} \rightarrow m, \qquad \frac{J}{N} \rightarrow j, \qquad m < j < 1 - m$$

the longitudinal and transverse spin correlations

$$\begin{aligned} C_{\parallel}(x) &= \langle S^z_n\, S^z_{n'} \rangle - \langle S^z_n \rangle \langle\, S^z_{n'} \rangle \\ C_{\perp}(x) &= \langle S^x_n\, S^x_{n'} \rangle = \langle S^y_n\, S^y_{n'} \rangle \end{aligned}$$

with $x = 2\pi(n - n')$, are given by

$$\begin{aligned} C_{\parallel}(x) &= \frac{m}{x} Si(mx) \cos(mx) - \frac{1}{x^2} [Si(mx) + \sin(mx)] \sin(mx) \\ C_{\perp}(x) &= \frac{m}{x} Si(mx) \cos(jx) \end{aligned}$$

where $Si(x)$ is the sine integral. The dominant decay is algebraic with an exponent $\eta = 1$, which is independent of m and without logaritmic corrections. These two features are different from the AFH model where the exponent η is renormalized for $m \neq 1/2$. The absence of log corrections indicates that there is no a backscattering process (spin umkalpp), which are associated to a marginal irrelevant operator of the fixed point WZW model. In this regard it is interesting to observe that a Hamiltonian with NN and NNN couplings where the backscattering term is also absent is

$$H_{J_1, J_2} = \sum_i (J_1 \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1} + J_2 \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+2}), \qquad \frac{J_2}{J_1} \sim .241$$

In the HS model the ratio of NN and NNN couplings is rather close to that value

$$\frac{J(2)}{J(1)} = \left(\frac{\sin(\pi/N)}{\sin(2\pi N)}\right)^2 = \frac{1}{4\cos^2(\pi N)} \to \frac{1}{4}$$

Following Haldane, the states with $M \leq J \leq N-M$ are states at the top of their multiplets, i.e. $S^z = S = N/2-M$. If one takes $m = j = 1/2$ the correlations become isotropic

$$\langle S_n^a\, S_0^b\rangle = \delta_{a,b}\frac{(-1)^n}{4\pi n} Si(\pi n)$$

and coincide with the result obtained by Gebhard and Vollhardt[14].

The trigonometric Sutherland model

The Haldane-Shastry model is closely related to the Sutherland model of M particles moving on a circle of length N and Hamiltonian

$$H_{Suth} = -\sum_{i=1}^{M} \frac{\partial^2}{\partial x_i^2} + 2\left(\frac{\pi}{N}\right)^2 \lambda(\lambda-1)\sum_{i<j} \frac{1}{\sin^2(\pi(x_i - x_j)/N}$$

where $x_1, \ldots, x_M$ are the positions of the coordinates of the particles.
The GS of this Hamiltonian is

$$\psi(x_1, \ldots, x_M) = \prod_{i<j}^{M}\left[\sin\left(\frac{\pi(x_i - x_j)}{N}\right)\right]^\lambda$$

which for $\lambda = 2$ is a continuous version of the HS wave function, except for the Marshall sign which is absent in the later wave function. To prove this result let us first consider the derivative

$$\frac{\partial}{\partial x_n}\psi = \frac{\lambda\pi}{N}\sum_{m(\neq n)} \cot\frac{\pi x_{n,m}}{N}\;\psi$$

where $x_{n,m} = x_n - x_m$. Taking another derivative and summing over n one finds

$$\begin{aligned}\frac{\Delta\psi}{\psi} &= -\lambda\left(\frac{\pi}{N}\right)^2 \sum_{n\neq m}\frac{1}{\sin^2(\pi x_{n,m}/N)} \\ &\quad + \left(\frac{\lambda\pi}{N}\right)^2 \sum_{n}\sum_{m(\neq n)}\sum_{l(\neq n)} \cot\frac{\pi x_{n,m}}{N}\;\cot\frac{\pi x_{n,l}}{N}\end{aligned}$$

The last term we split as

$$\sum_{n}\sum_{m(\neq n)}\sum_{l(\neq n)} = \sum_{n\neq m}\delta_{l,m} + \sum_{n\neq m\neq l\neq n}$$

obtaining

$$\begin{aligned}\frac{\Delta\psi}{\psi} &= -\lambda\left(\frac{\pi}{N}\right)^2 \sum_{n\neq m}\frac{1}{\sin^2(\pi x_{n,m}/N)} + \left(\frac{\lambda\pi}{N}\right)^2 \sum_{n\neq m}\cot^2\frac{\pi x_{n,m}}{N} \\ &\quad + \left(\frac{\lambda\pi}{N}\right)^2 \sum_{n\neq m\neq l\neq n} \cot\frac{\pi x_{n,m}}{N}\;\cot\frac{\pi x_{n,l}}{N}\end{aligned}$$

Using the identity ($\theta_{i,j} = \theta_i - \theta_j$)

$$\cot\theta_{1,2}\ \cot\theta_{2,3} + \cot\theta_{2,3}\ \cot\theta_{3,1} + \cot\theta_{3,1}\ \cot\theta_{1,2} = 1$$

one gets

$$\sum_{n\neq m\neq l\neq n} \cot\frac{\pi x_{n,m}}{N}\ \cot\frac{\pi x_{n,l}}{N} = -\frac{M(M-1)(M-2)}{3}$$

while

$$\sum_{n\neq m} \cot^2\frac{\pi x_{n,m}}{N} = \sum_{n\neq m} \frac{1}{\sin^2(\pi x_{n,m}/N)} - M(M-1)$$

Hence

$$\frac{\Delta\psi}{\psi} = -\lambda(\lambda-1)\left(\frac{\pi}{N}\right)^2 \sum_{n\neq m} \frac{1}{\sin^2(\pi x_{n,m}/N)} - \left(\frac{\lambda\pi}{N}\right)^2 \frac{M(M^2-1)}{3}$$

Which shows that

$$H_{Suth}\ \psi = E_0(N,M,\lambda)\ \psi$$

with

$$E_0(N,M,\lambda) = \left(\frac{\lambda\pi}{N}\right)^2 \frac{M(M^2-1)}{3}$$

Excitations

To find the excitations of the Sutherland Hamiltonian one make the ansatz

$$\psi(x_1,\ldots,x_M) = \Phi(x_1,\ldots,x_M)\ \psi_0(x_1,\ldots,x_M)$$

where ψ_0 is the GS wave function derived above and Φ a trial wave function which one wants to find. Let us first make the change of variables

$$\theta_j = \frac{2\pi x_j}{N}, \qquad 0 \leq \theta_j \leq 2\pi$$

The Hamiltonian can be written in this variables as

$$H = \left(\frac{2\pi}{N}\right)^2 \left[-\sum_{i=1}^{M} \frac{\partial^2}{\partial\theta_i^2} + \frac{\lambda(\lambda-1)}{2} \sum_{i<j} \frac{1}{\sin^2(\theta_i-\theta_j)/2} \right]$$

The Schroedinger eq.

$$H\psi = E\ \psi$$

after replacing the ansatz $\psi = \Phi\,\psi_0$ and using that ψ_0 is an eigenstate of H, becomes

$$H'\Phi = \epsilon\,\Phi, \qquad \epsilon = \left(\frac{N}{2\pi}\right)^2 (E - E_0)$$

where

$$H' = -\sum_{i=1}^{M} \frac{\partial^2}{\partial\theta_i^2} - 2\sum_{i=1}^{M} \frac{\partial \log\psi_0}{\partial\theta_i}\frac{\partial}{\partial\theta_i}$$

which can be written as

$$\begin{aligned} H' &= H_1 + H_2 \\ H_1 &= -\sum_{i=1}^{M} \frac{\partial^2}{\partial\theta_i^2} \\ H_2 &= -i\lambda \sum_{j>l} \frac{e^{i\theta_j} + e^{i\theta_l}}{e^{i\theta_j} - e^{i\theta_l}} \left(\frac{\partial}{\partial\theta_j} - \frac{\partial}{\partial\theta_l}\right) \end{aligned}$$

The operator H' in general will not be hermitean. To solve this eq. one first defines a basis of plane waves characterized by a collection of integers $\{n_j\}$

$$\{n_j\} = \{n_1, n_2, \dots, n_M\}, \qquad n_1 \leq n_2 \cdots \leq n_M$$

$$\Phi\{n_j\} = \sum_{P\in\mathcal{S}_M} \prod_{j=1}^{M} e^{in_j\theta_{Pj}} = e^{i(n_1\theta_1+\cdots+n_M\theta_M)} + \text{Permutations } \theta_j$$

This functions is a symmetric polynomial in the variables $z_j = e^{i\theta_j}$ of degree deg $\Phi = \sum_j n_j$. The action of H_1 on this basis is diagonal

$$H_1\Phi\{n_j\} = \epsilon_1\,\Phi\{n_j\}, \qquad \epsilon_1 = \sum_{j=1}^{M} n_j^2$$

while the action of H_2 will be shown below to be upper triangular. Let' s act with H_2 on the wave function $\{n_1, \dots, m, \dots, n, \dots, n_M\}$ and consider the term

$$\begin{aligned} A &\equiv -i\lambda e^{i\sum' \theta_j n_{Pj}} \tfrac{e^{i\theta}+e^{i\phi}}{e^{i\theta}-e^{i\phi}} \left(\tfrac{\partial}{\partial\theta} - \tfrac{\partial}{\partial\phi}\right)\left(e^{i(n\theta+m\phi)} + e^{i(m\theta+n\phi)}\right) \\ &= \lambda(n-m) e^{i\sum' \theta_j n_{Pj}}\, e^{im(\theta+\phi)}\, \tfrac{e^{i\theta}+e^{i\phi}}{e^{i\theta}-e^{i\phi}} (e^{ik\theta} - e^{ik\phi}) \end{aligned}$$

where $k = n - m \geq 0$. The prime over the sum indicates the exclusion of the terms corresponding to θ and ϕ. Next one uses the fundamental identity

$$\begin{aligned} (e^{i\theta} + e^{i\phi})\ \frac{e^{ik\theta} - e^{ik\phi}}{e^{i\theta} - e^{i\phi}} &= (e^{i\theta} + e^{i\phi}) e^{i(k-1)\theta}\ \frac{1 - e^{ik(\phi-\theta)}}{1 - e^{i(\phi-\theta)}} \\ = \qquad & (e^{i\theta} + e^{i\phi}) e^{i(k-1)\theta}\ (1 + e^{i(\phi-\theta)} + \cdots + e^{i(k-1)(\phi-\theta)}) = e^{ik\theta} + e^{ik\phi} + 2\sum_{l=1}^{k-1} e^{i[(k-l)\theta + l\phi]} \end{aligned}$$

to get

$$A = \lambda(n-m)\, e^{i\sum' \theta_j n_{Pj}} \left[e^{i(n\theta+m\phi)} + e^{i(m\theta+n\phi)} + 2\sum_{l=1}^{n-m-1} e^{i[(n-l)\theta+(m+l)\phi]} \right]$$

The first two terms contribute to the energy of the state as

$$\epsilon_2 = \lambda \sum_{l>j} (n_l - n_j) = -\lambda \sum_{l=1}^{M} n_l\, (M+1-2l)$$

The third term corresponds to a sum of vectors labelled by different values of the integers $\{n_j\}$, where two of them, $m \leq n$, have been shifted as $m \to m+l,\ n \to n-l$. The factor of 2 accounts for the permutation of these integers. This motivates the definition of squeezing. A partition $\{n'_j\}$ is obtained by squeezing a partition $\{n_j\}$ if

$$\{n_1, \ldots, n_j, \ldots, n_k, \ldots, n_M\} \to \{n_1, \ldots, n_{j'}, \ldots, n_{k'}, \ldots, n_M\}$$

where the pair $n_j < n_k$ has been changed as

$$n_j \to n_{j'} = n_j + l, \qquad n_k \to n_{k'} = n_k - l, \qquad 0 \leq l \leq \frac{n_k - n_j}{2}$$

An example with $l = 1$ is

$$\begin{array}{ccccc} n_1 & n_2 & n_3 & n_4 & n_5 \\ 1 & 3 & 4 & 6 & 9 \\ & \uparrow +1 & & \uparrow -1 & \end{array} \quad \Longrightarrow \quad \begin{array}{ccccc} n_1 & n_2 & n_3 & n_4 & n_5 \\ 1 & 4 & 4 & 5 & 9 \end{array}$$

Notice that after que squeezing the integers may not be in proper order. Two partitions related as above imply that $\langle n'|H_2|n\rangle$ pick up a term $2\lambda(n_k - n_j)$. The squeezing operation implies

$$\{n_j\} \overset{\text{squeezing}}{\to} \{n'_j\} \Longrightarrow \langle n'|H_2|n\rangle \neq 0 \quad \text{and} \quad \langle n|H_2|n'\rangle = 0$$

The squeezing allows one to introduce an ordering of the basis

$$\{n_j\} > \{n'_j\} \quad \text{if} \quad \{n_j\} \overset{\text{squeezing}}{\to} \{n'_j\}$$

so that H' becomes upper triangular

$$H' = \begin{pmatrix} * & * & * & \ldots \\ 0 & * & * & \ldots \\ 0 & 0 & * & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

The diagonal elements give the eigenvalues of H', namely $\epsilon = \epsilon_1 + \epsilon_2$ with

$$\epsilon = \sum_{j=1}^{M} n_j^2 + \lambda \sum_{l>j} (n_l - n_j) = \sum_{j=1}^{M} (n_j^2 - \lambda n_j (M+1-2j)$$

Sutherland also gives an algorithm to find the eigenvectors of H'.

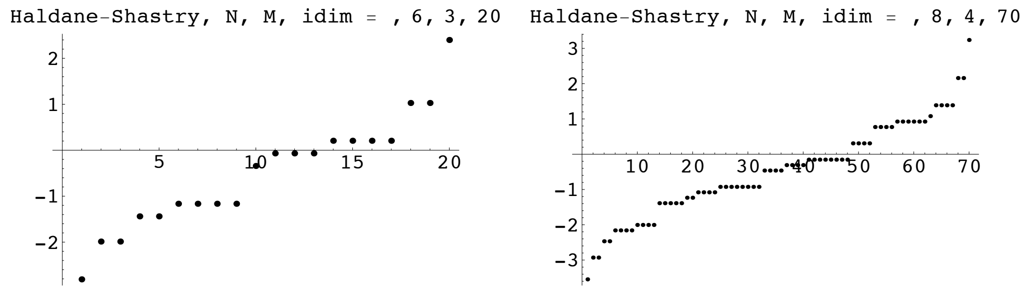

FIG. 1: Spectrum of the HS Hamiltonian for $(N, M) = (6, 3)$ and (8,4). N is the number of sites and M the number of down spins, idim is the dimension of the Hilber space.

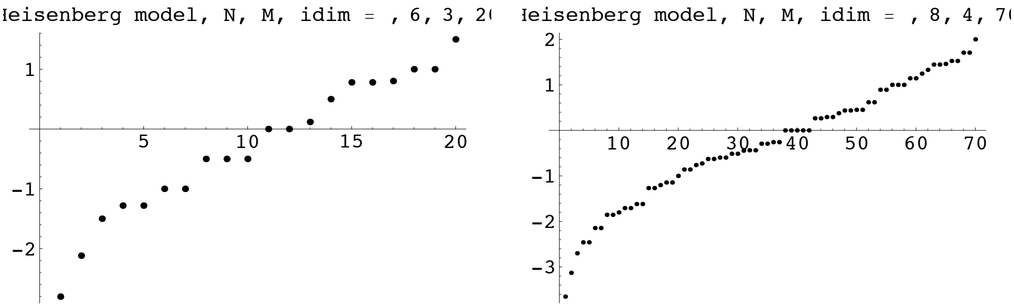

FIG. 2: Spectrum of the AFH model with NN interactions.

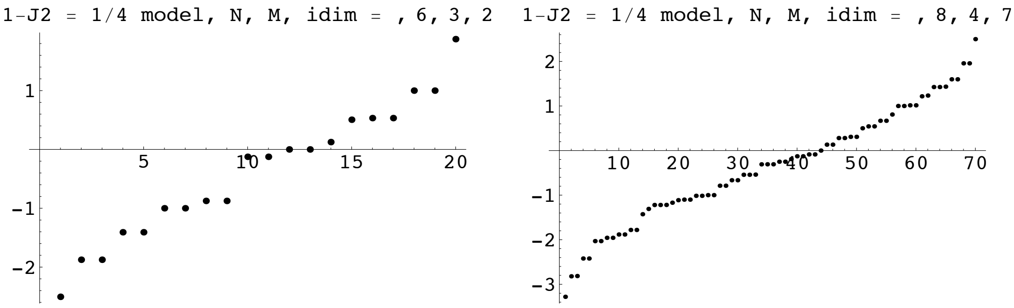

FIG. 3: Spectrum of the $J_1 - J_2$ AFH model with $J2/J1 = 1/4$.

Spectrum of the Heisenberg, HS and $J1 - J2$ models

Excitations of the HS model from the Sutherland model: from discrete to continuum

Let us write the Sutherland Hamiltonian H' using complex variables

$$z_j = e^{i\theta_j}, \qquad z_j \frac{\partial}{\partial z_j} = -i \frac{\partial}{\partial \theta_j}$$

We get $H' = H_1 + H_2$ with

$$
\begin{aligned}
H_1 &= \sum_{j=1}^{M} \left(z_j \frac{\partial}{\partial z_j} \right)^2 \\
H_2 &= \lambda \sum_{j>l} \frac{z_j + z_l}{z_j - z_l} \left(z_j \frac{\partial}{\partial z_j} - z_l \frac{\partial}{\partial z_l} \right)
\end{aligned}
$$

On the other hand, using

$$
\frac{1}{\sin^2(\pi(i-j)/N)} = -4 \frac{z_i z_j}{(z_i - z_j)^2}, \qquad z_j = e^{2\pi i j/N}
$$

one can write the HS Hamiltonian as

$$
H = \frac{J\pi^2}{2N^2} \sum_{i \neq j} \frac{\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j}{\sin^2(\pi(i-j)/N)} = -\frac{2J\pi^2}{N^2} \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j
$$

and we split it as

$$
\begin{aligned}
H &= \frac{2J\pi^2}{N^2} (\widetilde{H}_1 + \widetilde{H}_2) \\
\widetilde{H}_1 &= -\sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} S_i^z \, S_j^z \\
\widetilde{H}_2 &= -\sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} \frac{1}{2} \left(S_i^+ \, S_j^- + S_i^- \, S_j^+ \right)
\end{aligned}
$$

Now let us consider the wave function

$$
\psi = \phi \, \psi_0
$$

where ψ_0 is the GS of the HS Hamiltonian. We want to consider the action of these operators on the state ψ. First of all recall that the basis we are working with is

$$
|n_1, \ldots, n_M\rangle = S_{n_1}^- \ldots S_{n_M}^- \, |F\rangle
$$

where $|F\rangle$ is the ferromagnetic state $|F\rangle = |\uparrow, \ldots, \uparrow\rangle$. The operator $\widetilde{H}_1$ is diagonal in these basis, hence

$$
\langle n_1, \ldots, n_M | \widetilde{H}_1 | \phi \psi_0 \rangle = \phi(n_1, \ldots, n_M) \, \langle n_1, \ldots, n_M | \widetilde{H}_1 | \psi_0 \rangle
$$

To compute the action of $\widetilde{H}_2$ on ψ consider the term

$$
S_i^- \, S_{n_j}^+ |n_1, \ldots, n_M\rangle = S_{n_1}^- \ldots S_i^- \ldots S_{n_m}^- \, |F\rangle = |n_1, \ldots, n_{j-1}, i, n_{j+1}, \ldots n_M\rangle
$$

where $i \neq n_j$. This implies that

$$
\langle n_1, \ldots, n_M | S_i^+ \, S_{n_j}^- |\psi\rangle = \langle n_1, \ldots, n_{j-1}, i, n_{j+1}, \ldots n_M | \psi \rangle = \psi(n_1, \ldots, n_{j-1}, i, n_{j+1}, \ldots n_M)
$$

and hence

$$\langle n_1,\ldots,n_M|\widetilde{H}_2|\phi\psi_0\rangle =$$
$$= -\sum_{j=1}^M \sum_{i\neq n_j}^N \frac{z_i z_{n_j}}{(z_i - z_{n_j})^2}\ \phi(n_1,\ldots,n_{j-1},i,n_{j+1},\ldots n_M)\psi_0(n_1,\ldots,n_{j-1},i,n_{j+1},\ldots n_M)$$

To go further one needs the following identity

$$\sum_{n(\neq m)}^N \frac{z_n z_m}{(z_n - z_m)^2}\ P(z_n) = \left\{-\frac{1}{2}(z_m\partial_{z_m})^2 + \frac{N}{2} z_m \partial_{z_m} - \frac{N^2-1}{12}\right\}\ P(z_m)$$

where $P(z_n)$ is a polynomial of degree $< N$ in every variable. This formula can be checked using the Haldane formula for S_{00} with $z_n = z^n, z = e^{2\pi i/N}$. Now, since

$$\psi_0 = \prod_{i<j}^M (z_{n_i} - z_{n_j})^2 \prod_{i=1}^M z_{n_i}$$

each variable z_{n_i} appears to a power which is less or equal to $2(M-1)+1$, so that the degree of ϕ must be less or equal to $N-2M$ to guarantee that deg $(\phi\psi_0) \leq N-1$, in which case we can apply the later formula, which leads to

$$\sum_{i(\neq n_j)}^N \frac{z_i z_{n_j}}{(z_i - z_{n_j})^2}\psi(z's) = \left\{-\frac{1}{2}(z_{n_j}\partial_{z_{n_j}})^2 + \frac{N}{2} z_{n_j}\partial_{z_{n_j}} - \frac{N^2-1}{12}\right\}\psi(z's)$$

and so

$$\langle n_1,\ldots,n_M|\widetilde{H}_2|\phi\psi_0\rangle = -\sum_{j=1}^M \left\{-\frac{1}{2}(z_{n_j}\partial_{z_{n_j}})^2 + \frac{N}{2} z_{n_j}\partial_{z_{n_j}} - \frac{N^2-1}{12}\right\}\phi\psi_0$$

Applying the Leibniz rule we get

$$\begin{aligned}\langle n_1,\ldots,n_M|\widetilde{H}_2|\phi\psi_0\rangle &= \phi \sum_{j=1}^M \left\{\frac{1}{2}(z_{n_j}\partial_{z_{n_j}})^2 - \frac{N}{2} z_{n_j}\partial_{z_{n_j}} + \frac{N^2-1}{12}\right\}\psi_0 \\ &+ \psi_0 \sum_{j=1}^M \left\{\frac{1}{2}(z_{n_j}\partial_{z_{n_j}})^2 - \frac{N}{2} z_{n_j}\partial_{z_{n_j}} + z_{n_j}^2 \frac{\partial \log\psi_0}{\partial z_{n_j}}\partial_{z_{n_j}}\right\}\phi\end{aligned}$$

After some algebra one finds

$$\langle n_1,\ldots,n_M|\widetilde{H}_2|\phi\psi_0\rangle = \langle n_1,\ldots,n_M|\widetilde{H}_2|\psi_0\rangle + \langle n_1,\ldots,n_M|\widetilde{H'_2}|\phi_0\rangle$$

where

$$\widetilde{H'_2} = \sum_{j=1}^M \frac{1}{2}(z_{n_j}\partial_{z_{n_j}})^2 + \sum_{j>l} \frac{z_{n_j}+z_{n_l}}{z_{n_j}-z_{n_l}}\left(z_{n_j}\frac{\partial}{\partial z_{n_j}} - z_{n_l}\frac{\partial}{\partial z_{n_l}}\right) + (M - \frac{N}{2})\sum_{j=1}^M z_{n_j}\partial_{z_{n_j}}$$

The state ψ_0 is an eigenstate of $H_1 + H_2$ with eigenvalue

$$(H_1 + H_2)|\psi_0\rangle = \epsilon_0(M)|\psi_0\rangle, \qquad \epsilon_0(M) = \frac{1}{6}M(4M^2-1) - \frac{1}{2}M^2 N$$

On the other hand using Sutherland solution for $\lambda = 2$ with ϕ a homogenous polynomial

$$\phi = \sum_{P\in\mathcal{S}_M} \prod_{i=1}^{M} z_{n_i}^{r_{P(i)}}, \qquad 0 \le r_1 \le r_2 \cdots \le N-2M$$

one finds that

$$\widetilde{H_2'}|\phi\rangle = \left[\sum_{j=1}^{M} \frac{1}{2}(r_j^2 - 2r_j(M+1-2j)) + (M-\frac{N}{2})r_j\right]|\phi\rangle$$

And the total energy of the state ψ is

$$E = \frac{2J\pi^2}{N^2}\left(\epsilon_0(M) + \sum_{j=1}^{M} \frac{1}{2}(r_j^2 - 2r_j(M+1-2j)) + (M-\frac{N}{2})r_j\right)$$

A more convenient form to write this expression is using the so called quasimomenta

$$m_j = r_j + 2j - 1, \qquad 0 < m_j < N$$

so that

$$E(\{m_j\}) = \frac{2J\pi^2}{N^2}\sum_{j=1}^{M}\frac{1}{2}m_j(m_j - N) = \frac{J\pi^2}{N^2}\sum_{j=1}^{M} m_j(m_j-N)$$

Notice that in units $J(\pi/N)^2$ the energies of the states are integers!! Other normalization is in units of $J(2\pi/N)^2$, where the energies goes as $\frac{1}{4}m_i(m_i-N)$. The previous energy is measured with the convention that the fully polarised state has zero energy. Otherwise we have to add the term

$$E_0^F = \frac{J\pi^2}{24N^2}N(N^2-1)$$

The integers m_j ($j = 1, \ldots, M$) satisfy the following constraint

$$m_{j+1} - m_j = r_{j+1} - r_j + 2 \Longrightarrow m_{j+1} \ge m_j + 2$$

In the GS , i.e. $M = N/2$ (N even), these numbers are

$$GS : (m_1, m_2, \ldots, m_M) = (1, 3, \ldots, N-1)$$

The momenta defined as

$$k_i = \frac{2\pi m_i}{N}, \qquad 0 < k_i < 2\pi$$

satisfy the following BAE

$$k_i N = 2\pi I_i + \pi\sum_{j=1}^{M}\text{sign}(k_i - k_j)$$

where I_i are a set of distinct quantum numbers that take the values in the range $I_0+1, I_0+2, \ldots, I_{M+1}-1$ where

$$I_0 = \frac{M-1}{2}, \qquad I_{M+1} = N - I_0$$

To show this, let us write the BAE as

$$2m_i = 2I_i + \sum_{j=1}^{M} \text{sign}(m_i - m_j)$$

Using the identity

$$\sum_{j=1}^{M} \text{sign}(m_i - m_j) = 2i - 1 - M$$

one gets

$$2m_i = 2I_i + 2i - 1 - M$$

The conditions of the $m_i's$ imply

$$\begin{aligned} m_1 \geq 1 &\implies I_1 \geq \frac{M+1}{2} = I_0 + 1 \\ m_M \leq N-1 &\implies I_M \leq N - \frac{M+1}{2} = I_{M+1} + 1 \\ m_{j+1} - m_j \geq 2 &\implies I_{j+1} - I_j \geq 1 \end{aligned}$$

The total number of possible $I's$ is given by

$$N_I = I_{M+1} - I_0 - 1 = N - M$$

If we define the number of spinons N_{sp} as

$$S^z = \frac{N}{2} - M = \frac{N_{sp}}{2}$$

we get

$$N_I = M + N_{sp}$$

Each excited state is specified by the choice of M values of the $I's$, hence the total number of excited states constructed above is given by

$$g(M, N_{sp}) = \frac{(M + N_{sp})!}{M!\, N_{sp}!}$$

In particular, for the GS with N even one has $g(M = N/2, 0) = 1$, so that the state is unique. However if N is odd, one must have at least one spinon such that $g(M, 1) = M + 1$. Notice that the number of one spinon states is a half of the number of sites. This implies that the spinons are *semions*, which are particles with a statistics between bosons and fermions.

An example: $N = 6$

The numerical diagonalization of the Hamiltonian gives the following spectrum of eigenvalues (the state $M = 0$ has zero energy). The number is parenthesis give the degeneracy (0 means that there is not such a state in the spectrum).

$$\begin{aligned} M = 3,\ E = &\ -19(1),\ -16(2),\ -14(2),\ -13(4),\ -10(1),\ -9(3),\ -8(4),\ -5(2),\ 0(1) \\ M = 2,\ E = &\ -19(0),\ -16(1),\ -14(2),\ -13(2),\ -10(1),\ -9(2),\ -8(4),\ -5(2),\ 0(1) \\ M = 1,\ E = &\ -19(0),\ -16(0),\ -14(0),\ -13(0),\ -10(0),\ -9(1),\ -8(2),\ -5(2),\ 0(1) \\ M = 0,\ E = &\ -19(0),\ -16(0),\ -14(0),\ -13(0),\ -10(0),\ -9(0),\ -8(0),\ -5(0),\ 0(1) \end{aligned}$$

The total number of states, taking into account those with $S^z < 0$ is $2^6 = 64$ and they are organized into SU(2) multiplets as follows

$$\left(\frac{1}{2}\right)^{\otimes 6} = 0(5) \oplus 1(9) \oplus 2(5) + \oplus 3(1)$$

These states can be obtained from the previous construction:

M	I_j	Occupation	$m's$	E	spin	deg
3	(2,3,4)	(1,1,1)	(1,3,5)	-19	0	1
2	$(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2})$	(1,1,0,0)	(1,3)	-14	1	3
2	$(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2})$	(1,0,1,0)	(1,4)	-13	$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$	$1 + 3$
2	$(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2})$	(1,0,0,1)	(1,5)	-10	1	3
2	$(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2})$	(0,1,1,0)	(2,4)	-16	$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$	$1 + 3$
2	$(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2})$	(0,1,0,1)	(2,5)	-13	$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$	$1 + 3$
2	$(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2})$	(0,0,1,1)	(3,5)	-14	1	3
1	(1,2,3,4,5)	(1,0,0,0,0)	(1)	- 5	2	5
1	(1,2,3,4,5)	(0,1,0,0,0)	(2)	- 8	$\frac{1}{2} \otimes \frac{3}{2} = 1 \oplus 2$	$3 + 5$
1	(1,2,3,4,5)	(0,0,1,0,0)	(3)	- 9	$1 \otimes 1 = 0 \oplus 1 \oplus 2$	$1 + 3 + 5$
1	(1,2,3,4,5)	(0,0,0,1,0)	(4)	- 8	$\frac{1}{2} \otimes \frac{3}{2} = 1 \oplus 2$	$3 + 5$
1	(1,2,3,4,5)	(0,0,0,0,1)	(5)	- 5	2	5
0	$(\frac{1}{2}, \ldots, \frac{11}{2})$	(0,0,0,0,0,0)	-	0	3	7

Table .- Eigenstates of the HS Hamiltonian for $N = 6$.

HS with dimerization

If we add a dimerization to the model the energies break as (change the sign)

$$\begin{aligned} M = 3,\ |E| = &\ 19(1),\ 16(1+1),\ 14(2),\ 13(2+2),\ 10(1),\ 9(1+2),\ 8(2+2),\ 5(2),\ 0(1) \\ M = 2,\ |E| = &\ 19(0),\ 16(1),\ 14(2),\ 13(2),\ 10(1),\ 9(1+1),\ 8(2+2),\ 5(2),\ 0(1) \\ M = 1,\ |E| = &\ 19(0),\ 16(0),\ 14(0),\ 13(0),\ 10(0),\ 9(1),\ 8(2),\ 5(2),\ 0(1) \\ M = 0,\ |E| = &\ 19(0),\ 16(0),\ 14(0),\ 13(0),\ 10(0),\ 9(0),\ 8(0),\ 5(0),\ 0(1) \end{aligned}$$

Missing states

The states constructed through the maping onto the Sutherland model are the top of their spin multiplets, i.e. $S = S^z$. This can be seen acting with S^+. Acting with S^- we can recover the rest of the multiplet. The total number of multiplets with $S^z = N/2 - M$ is $g(M, N_{sp})$ which is equal or smaller than the number of irreps

$$g(M,N_{sp})=\begin{pmatrix}N-M\\M\end{pmatrix}\le\begin{pmatrix}N\\M\end{pmatrix}-\begin{pmatrix}N\\M-1\end{pmatrix},\qquad M\le\left[\frac{N}{2}\right]$$

The missing states must be of non polynomial form. Haldane observed from numerical diagonalization that the "missing" states have energies already contained in the set of energies of polynomial-type states. This implies that there is a degeneracy beyond the regular SU(2), which also suggest the existence of a hidden symmetry generated by operators commuting with the Hamiltonian. This symmetry is given by the Yangian. Before constructing the Yangian Haldane found an empirical rule to explain the whole spectrum using the spinons. The rule is the following.

Using the the occupancy description of the $I's$ quantum numbers, one associates an isolated 0 with a spin 1/2 state. If there are n consecutive 0' s separated by 1' s one associate a spin $S=n/2$. The total spin is obtained by the tensor product decomposition of this spins, e.g.

$$(1,0,1,0)\to\frac{1}{2}\otimes\frac{1}{2}=0\oplus 1,\qquad (1,0,0,1)\to 2,\qquad (0,0,1,0,0)\to 1\otimes 1=0\oplus 1\oplus 2$$

In this manner one is able to obtained the whole Hilbert space of states.

The Yangian and conserved quantities

In 1990 Inozemtsev found two invariants commuting with the HS Hamiltonian. At third order they are given by

$$H_3=\sum_{ijk}\frac{z_i z_j z_k}{z_{ij}z_{jk}z_{ki}}\vec{\mathbf{S}}_i\cdot\vec{\mathbf{S}}_j\times\vec{\mathbf{S}}_k$$

where the prime indicates the exclusion of coincident indices and $z_{ij}=z_i-z_j$. The other one is

$$\begin{aligned}I_3 &= \sum_{i,j,k}{}'\,(w_{ij}+w_{jk}+w_{ki})\,\vec{\mathbf{S}}_i\cdot\vec{\mathbf{S}}_j\times\vec{\mathbf{S}}_k\\ &\propto \vec{\mathbf{S}}\cdot\left(\sum_{i,j} w_{ij}\vec{\mathbf{S}}_i\times\vec{\mathbf{S}}_j\right)=\vec{\mathbf{S}}\cdot\vec{\Lambda}\end{aligned}$$

where $w_{ij}=(z_i+z_j)/(z_i-z_j)$. $\vec{\Lambda}$ is called the rapidity vector. Indeed Λ^z measures the total rapidity of the polynomial type states, i.e. $\sum_j m_j$. The operator $\vec{\Lambda}$ commutes with the Hamiltonian but not with $\vec{\mathbf{S}}^2$. Both operators $\vec{\mathbf{S}}$ and $\vec{Lambda}$ generates the so called Yangian $\mathcal{Y}_2$ Hopf algebra associated to the group $SU(2)$. The highest weight vectors of $\mathcal{Y}_2$ are the polynomial eigenstates constructed above. The missing states in the spectrum are obtained applying the remaining generators of $\mathcal{Y}_2$.

CFT and the HS model

In reference[21] the HS model is derived from the WZW model $SU(2)_k$ at level $k=1$. The HS wave function is shown to be proportional to the chiral conformal block of the primary fields of spin label $j=1/2$ and conformal weights $h=1/4$. The fusion rules

$$\phi_0\times\phi_0=\phi_0,\qquad \phi_0\times\phi_{1/2}=\phi_{1/2},\qquad \phi_{1/2}\times\phi_{1/2}=\phi_0,$$

where ϕ_0 is the primary field of the identity, show that there is only one conformal block at genus 0 for the N point correlator (N even)

$$\psi_{s_1,\ldots,s_N}(z_1,\ldots,z_N)=\langle\phi_{1/2,s_1}(z_1)\ldots\phi_{1/2,s_N}(z_N)\rangle,\qquad \sum s_i=0$$

where $s_i=\pm 1$ is twice the third component of the spin. The HS Hamiltonian is derived from the KZ equation satisfied by these conformal blocks.

Derivation of the GS

Let us denote by ψ the conformal block of N primary fields of the $SU(2)_{k=1}$ WZW model. The KZ eq. is

$$\frac{k+2}{2}\frac{\partial}{\partial z_i}\,\psi = \sum_{j(\neq i)} \frac{\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j}{z_i - z_j},\ \psi \qquad i = 1, \ldots, N$$

Using vertex operators, the conformal block ψ is given by

$$\psi = \chi \prod_{i>j} (z_i - z_j)^{s_i s_j/2}$$

where χ is the Marshall sign factor. Using the conformal transformation $z = e^w$, one can map the complex plane into the cylinder where the conformal blocks becomes

$$\psi_{\rm cyl}(w_1, \ldots, w_N) = \left(\frac{dw_1}{dz_1}\right)^{-h} \cdots \left(\frac{dw_N}{dz_N}\right)^{-h} \psi_{\rm plane}(z_1, \ldots, z_N)$$

and $h = 1/4$ is the conformal weight of the spin 1/2 primary field. Expressing $\psi_{\rm cyl}$ in terms of the z_i variables one has

$$\psi_{\rm cyl}(z_1, \ldots, z_N) = \prod_{i=1}^{N} z_i^{1/4}\ \psi_{\rm plane}(z_1, \ldots, z_N) = \chi \prod_{i>j} (z_i - z_j)^{s_i s_j/2} \prod_i z_i^{1/4}$$

The KZ eq. for this wave function can be written as

$$3 z_i \frac{\partial}{\partial z_i}\,\psi = \sum_{j(\neq i)} w_{ij}\ \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j,\ \psi \qquad i = 1, \ldots, N$$

with

$$w_{ij} = \frac{z_i + z_j}{z_i - z_j}$$

We skip from now one the index *cyl*. We shall show that ψ is an eigenstate of the HS Hamiltonian. Applying directly the derivative one finds

$$z_i \frac{\partial}{\partial z_i}\,\psi = \left(\frac{1}{2}\sum_{j(\neq i)} \frac{z_i}{z_i - z_j} s_i s_j + \frac{1}{4}\right)\psi = \frac{1}{2}\sum_{j(\neq i)}\left(\frac{z_i}{z_i - z_j} - \frac{1}{2}\frac{z_i - z_j}{z_i - z_j}\right) s_i s_j \psi$$

where we have used

$$\sum_{j(\neq i)} s_i s_j = \sum_j s_i s_j - 1 = -1, \qquad \sum_j s_j = 0$$

Simplifying the above expression one gets an abelian version of the KZ eq.

$$z_i \frac{\partial}{\partial z_i}\,\psi = \frac{1}{4}\sum_{j(\neq i)} w_{ij}\ s_i s_j\ \psi$$

Taking another derivative one finds

$$\left(z_i \frac{\partial}{\partial z_i}\right)^2 \psi = -\frac{1}{2} \sum_{j(\neq i)} \frac{z_i z_j}{z_{ij}^2}\ s_i s_j\ \psi + \frac{1}{16} \sum_{j(\neq i)} \sum_{k(\neq i)} w_{ij} w_{ik}\ s_j s_k\ \psi$$

where we have used $s_i^2 = 1$ and

$$z_i \frac{\partial}{\partial z_i} w_{ij} = -\frac{2 z_i z_j}{z_{ij}^2}$$

Summing over i in the last expression one gets

$$\Delta\psi \equiv \sum_i \left(z_i \frac{\partial}{\partial z_i}\right)^2 \psi = -\frac{1}{2} \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2}\ s_i s_j\ \psi + \frac{1}{16} \sum_i \sum_{j(\neq i)} \sum_{k(\neq i)} w_{ij} w_{ik}\ s_j s_k\ \psi$$

The triple sum can be written as

$$\sum_i \sum_{j(\neq i)} \sum_{k(\neq i)} = \sum_{i \neq j} \delta_{jk} + \sum_{j \neq k} \sum_{i(\neq j,k)}$$

so that

$$\Delta\psi = -\frac{1}{2} \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2}\ s_i s_j\ \psi + \frac{1}{16} \sum_{i \neq j} w_{ij}^2\ \psi + \frac{1}{16} \sum_{j \neq k} \sum_{i(\neq j,k)} w_{ij} w_{ik}\ s_j s_k\ \psi$$

Exchanging $i \leftrightarrow k$ this becomes

$$\Delta\psi = -\frac{1}{2} \sum_{i \neq j} \frac{z_i z_j}{z_{ij}^2}\ s_i s_j\ \psi + \frac{1}{16} \sum_{i \neq j} w_{ij}^2\ \psi + \frac{1}{16} \sum_{i \neq j} s_i s_j \sum_{k(\neq i,j)} w_{ki} w_{kj}\ \psi$$

Next we use the identity

$$w_{ki} w_{kj} + w_{ij} w_{ik} + w_{jk} w_{ji} = 1, \qquad i \neq j \neq k \neq i$$

To derive

$$\begin{aligned} \sum_{k(\neq i,j)} w_{ki} w_{kj} &= \sum_{k(\neq i,j)} (1 + w_{ij} w_{ki} - w_{ij} w_{kj}) = N - 2 + w_{ij} \sum_{k(\neq i,j)} (w_{ki} - w_{kj}) \\ &= N - 2 + w_{ij} \left[\sum_{k(\neq i)} w_{ki} - w_{ji} - \sum_{k(\neq j)} w_{kj} + w_{ij} \right] \end{aligned}$$

and

$$\sum_{k(\neq i,j)} w_{ki} w_{kj} = N - 2 + 2 w_{ij}^2 + w_{ij}(c_i - c_j), \qquad c_i \equiv \sum_{k(\neq i)} w_{ki}$$

Plugging this eq. above we find

$$
\begin{aligned}
\frac{\Delta\psi}{\psi} &= -\frac{1}{2}\sum_{i\neq j}\frac{z_i z_j}{z_{ij}^2}\, s_i s_j + \frac{1}{16}\sum_{i\neq j} w_{ij}^2 + \frac{1}{16}\sum_{i\neq j} s_i s_j (N-2+2w_{ij}^2 + w_{ij}(c_i - c_j)) \\
&= -\frac{1}{2}\sum_{i\neq j}\left(\frac{z_i z_j}{z_{ij}^2} - \frac{1}{4}w_{ij}^2\right)\, s_i s_j + \frac{1}{16}\sum_{i\neq j} w_{ij}^2 + \frac{N-2}{16}\sum_{i\neq j} s_i s_j + \frac{1}{16}\sum_{i\neq j} s_i s_j\, w_{ij}(c_i - c_j) \\
&= \frac{1}{8}\sum_{i\neq j} s_i s_j + \frac{N-2}{16}\sum_{i\neq j} s_i s_j + \frac{1}{16}\sum_{i\neq j} w_{ij}^2 + \frac{1}{16}\sum_{i\neq j} s_i s_j w_{ij}(c_i - c_j) \\
&= -\frac{N^2}{16} + \frac{1}{16}\sum_{i\neq j} w_{ij}^2 + \frac{1}{16}\sum_{i\neq j} s_i s_j w_{ij}(c_i - c_j)
\end{aligned}
$$

where we have used

$$
\frac{z_i z_j}{z_{ij}^2} - \frac{1}{4}w_{ij}^2 = -\frac{1}{4}, \qquad \sum_{i\neq j} s_i s_j = -N
$$

From the abelian KZ equation we find

$$
\frac{1}{2}\sum_i c_i z_i \frac{\partial}{\partial z_i}\ \psi = \frac{1}{16}\sum_{i\neq j} s_i s_j w_{ij}(c_i - c_j)
$$

So finally

$$
\left(\Delta - \frac{1}{2}\sum_i c_i z_i \frac{\partial}{\partial z_i}\right)\ \psi = \left(-\frac{N^2}{16} + \frac{1}{16}\sum_{i\neq j} w_{ij}^2\right)\psi
$$

The operator on the LHS can also be computed using the KZ equation.

$$
3 z_i \frac{\partial}{\partial z_i}\ \psi = \sum_{j(\neq i)} w_{ij}\ \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j,\ \psi \qquad i = 1, \ldots, N
$$

Taking another derivative and following the same steps as before one finds

$$
9\ \Delta\psi = -6\sum_{i\neq j}\frac{z_i z_j}{z_{ij}^2}\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j \psi + \sum_{i\neq j} w_{ij}^2 (\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j)^2 \psi + \sum_{i\neq j}\sum_{k(\neq i,j)} w_{ki} w_{kj} (\vec{\mathbf{S}}_k \cdot \vec{\mathbf{S}}_i)(\vec{\mathbf{S}}_k \cdot \vec{\mathbf{S}}_j)\psi
$$

The following identities are needed

$$
\begin{aligned}
(\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j)^2 &= \frac{3}{16} - \frac{1}{2}\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j, \qquad i \neq j \\
(\vec{\mathbf{S}}_k \cdot \vec{\mathbf{S}}_i)(\vec{\mathbf{S}}_k \cdot \vec{\mathbf{S}}_j) &= \frac{1}{4}\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j + \frac{i}{2}\,\vec{\mathbf{S}}_k \cdot (\vec{\mathbf{S}}_i \times \vec{\mathbf{S}}_j), \qquad i \neq j \neq k \neq i \\
\sum_{i\neq j}\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j &= -\frac{3}{4}N
\end{aligned}
$$

which finally lead to

$$
\Delta\psi = \left[-\frac{2}{3}\sum_{i\neq j}\frac{z_i z_j}{z_{ij}^2}\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j + \frac{1}{48}\sum_{i\neq j} w_{ij}^2 - \frac{N(N-2)}{48} + \frac{1}{36}\sum_{i\neq j}\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j w_{ij}(c_i - c_j)\right]\psi
$$

Similarly

$$\frac{1}{2}\sum_i c_i z_i \frac{\partial}{\partial z_i}\psi = \frac{1}{12}\sum_{i\neq j} \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j w_{ij}(c_i - c_j)\psi$$

Hence

$$\left[\Delta - \frac{1}{2}\sum_i c_i z_i \frac{\partial}{\partial z_i}\right]\psi = \left[-\frac{2}{3}\sum_{i\neq j}\left(\frac{z_i z_j}{z_{ij}^2} + \frac{1}{12}w_{ij}(c_i - c_j)\right)\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j + \frac{1}{48}\sum_{i\neq j} w_{ij}^2 - \frac{N(N-2)}{48}\right]\psi$$

Equating this with the previous eq. found from the abelian KZ eq. we arrive at

$$-\sum_{i\neq j}\left(\frac{z_i z_j}{z_{ij}^2} + \frac{1}{12}w_{ij}(c_i - c_j)\right)\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j \;\psi = \left(\frac{1}{16}\sum_{i\neq j} w_{ij}^2 - \frac{N(N+1)}{16}\right)\psi$$

In the case where $z_n = e^{2\pi i n/N}$, the constants c_n vanish and we recover the standard Haldane and Shastry with translational invariances. Moreover we have found a generalization of the HS model for another choices of the HS Hamiltonian.

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