Quantum Entanglement in Many Body Systems

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Quantum Entanglement

Erwin Schroedinger (1935)

I would call entanglement *the* fundamental trait of Quantum Mechanics, the one that enforces its entire departure from classical lines of thought

PLAN OF THE TALK

- Brief review of entanglement: definition, measures, area law

- Numerical Many Body Methods: DMRG, MPS, PEPS, MERA

- Entanglement in critical spin chain models and Conformal Field Theory

- Infinite MPS and CFT
    * Haldane-Shastry model
    * Kalmeyer-Laughlin wave function → Fractional Quantum Hall

- Conclusions

- What is entanglement ?

- How to quantify it?

DEFINITION

$$\psi \neq \psi_A \otimes \psi_B$$

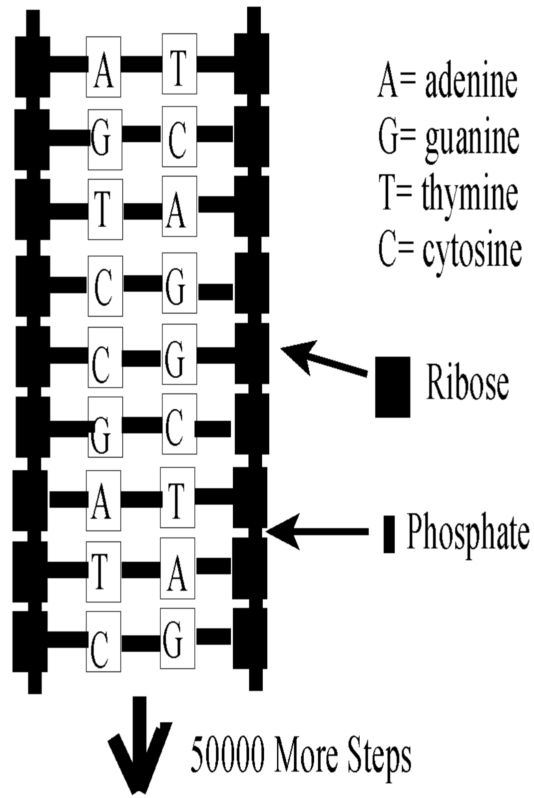
Examples

Not entangled (=product state) $$\psi = \uparrow \otimes \uparrow$$

Entangled states $$\psi = \frac{1}{\sqrt{2}}(\uparrow \otimes \downarrow - \downarrow \otimes \uparrow)$$

Entanglement's DNA : Schmidt decomposition

$$\begin{aligned}\psi = {} & d_1 |1\rangle_A \otimes |1\rangle_B \\ & + d_2 |2\rangle_A \otimes |2\rangle_B \\ & + \cdots \\ & + d_\chi |\chi\rangle_A \otimes |\chi\rangle_B\end{aligned}$$



$$d_1 \geq d_2 \geq \ldots \geq d_\chi > 0, \quad \sum_i d_i^2 = 1$$

Schmidt number $\quad \chi \leq \min\left(\dim H_A, \dim H_B\right)$

Global measure of entanglement : entropy

Reduced Density Matrices:

$$\rho_A = Tr_B \left|\psi\right\rangle\left\langle\psi\right| = \sum_i d_i^2 \left|i\right\rangle_A \left\langle i\right|$$

$$\rho_B = Tr_A \left|\psi\right\rangle\left\langle\psi\right| = \sum_i d_i^2 \left|i\right\rangle_B \left\langle i\right|$$

Scientist knows less than nothing

A MATHEMATICS expert from Bristol has come up with a baffling theory to help explain the complex world of quantum physics.

by Ian Turner
i.turner@bepp.co.uk

Dr Andreas Winter of Bristol University, said it was possible to know "less than nothing" and discovered that "negative knowledge" does exist in the quantum world of extremely small things.

But while the breakthrough may

develop the theory. Their work has now been published in the leading science journal *Nature*.

Quantum physics tries to explain the behaviour of particles, such as electrons, protons, and neutrons, which are smaller than an atom.

Scientists claim that in the quantum

Entanglement entropy = Von Neumann entropy of reduced DM's

$$S_A = -Tr_A\, \varrho_A \ln \varrho_A = -\sum_i d_i^2 \ln d_i^2$$

$$S_A = S_B$$

Other measures of entanglement

Renyi entropies:

$$S_A^{(n)} = \frac{1}{1-n} \ln Tr_A \, \rho_A^n, \quad n = 2,3,\ldots$$

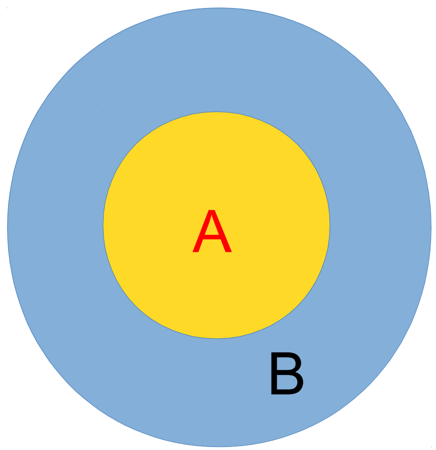
Analytic extension in "n" allows to compute von Neumann

$$S_A = \lim_{n \to 1} S_A^{(n)}$$

Entanglement Hamiltonian and entanglement spectrum

Haldane, Li

B acts as a thermal bath for A (and viceversa)



$$\rho_A = e^{-H_E}$$

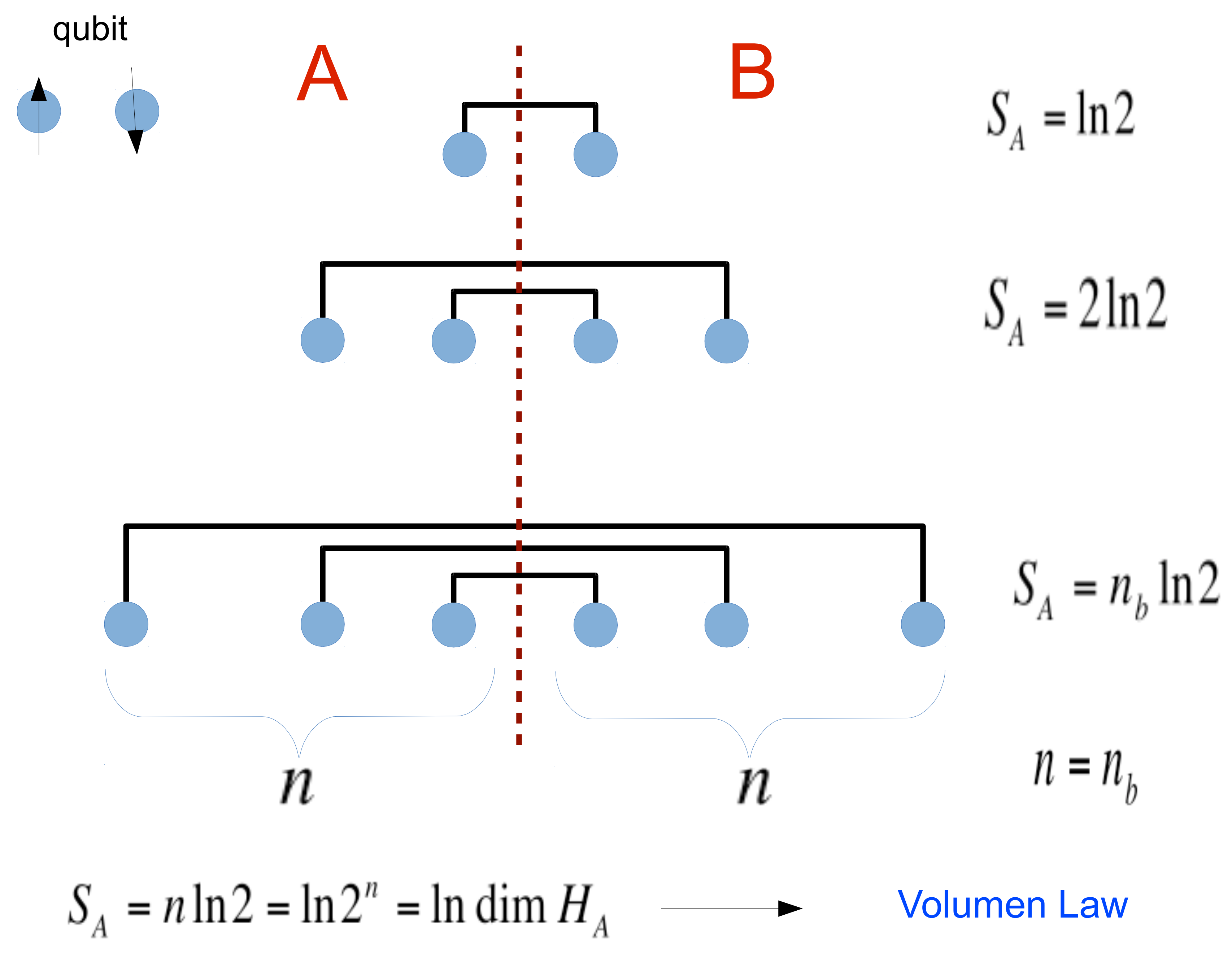
Entanglement Hamiltonian

$$d_i^2 = e^{-\varepsilon_i}$$

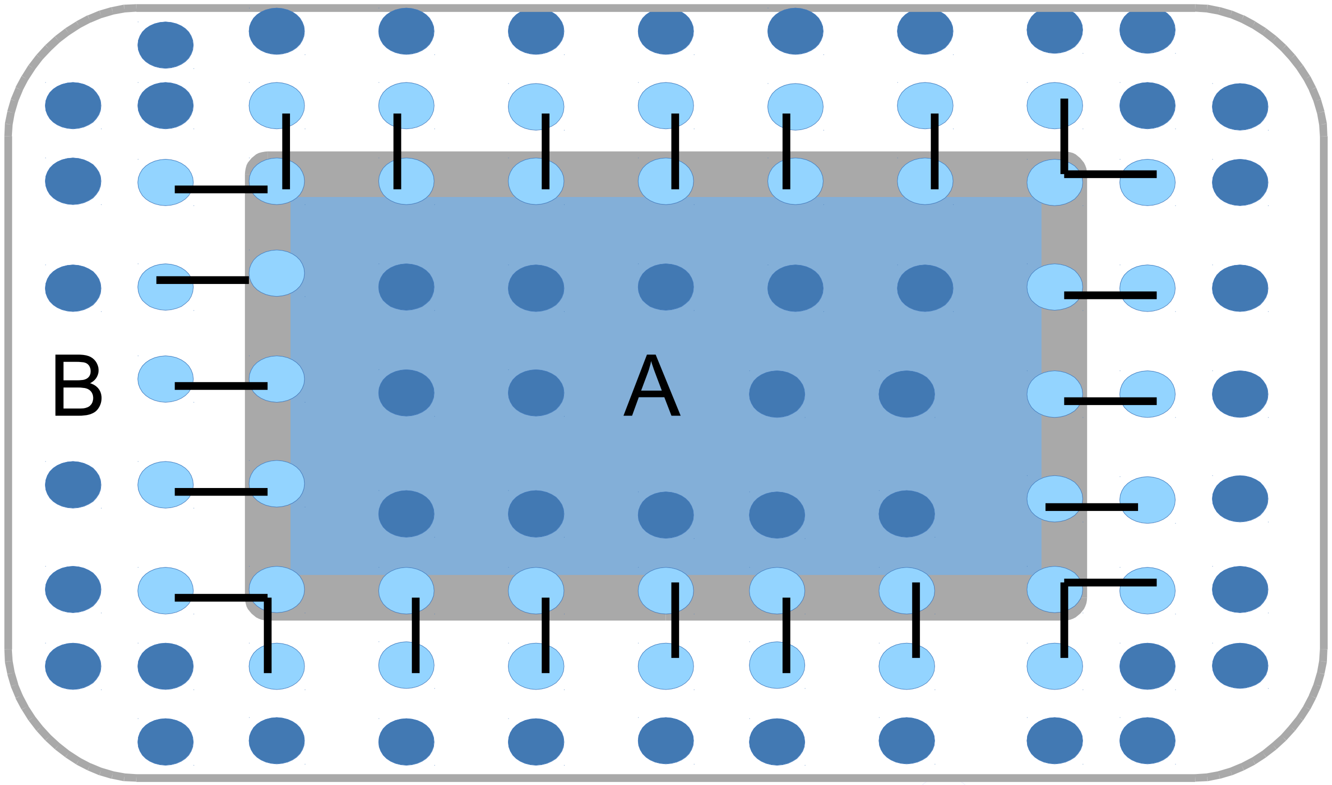
Entanglement energies

The lowest “energy” states are the most probable ones

H_E → associated to the boundary of A → relation to holography


qubit
A
B
$S_A = \ln 2$
$S_A = 2\ln 2$
$S_A = n_b \ln 2$
n
n
$n = n_b$
$S_A = n\ln 2 = \ln 2^n = \ln \dim H_A$
Volumen Law

Entropic Area law

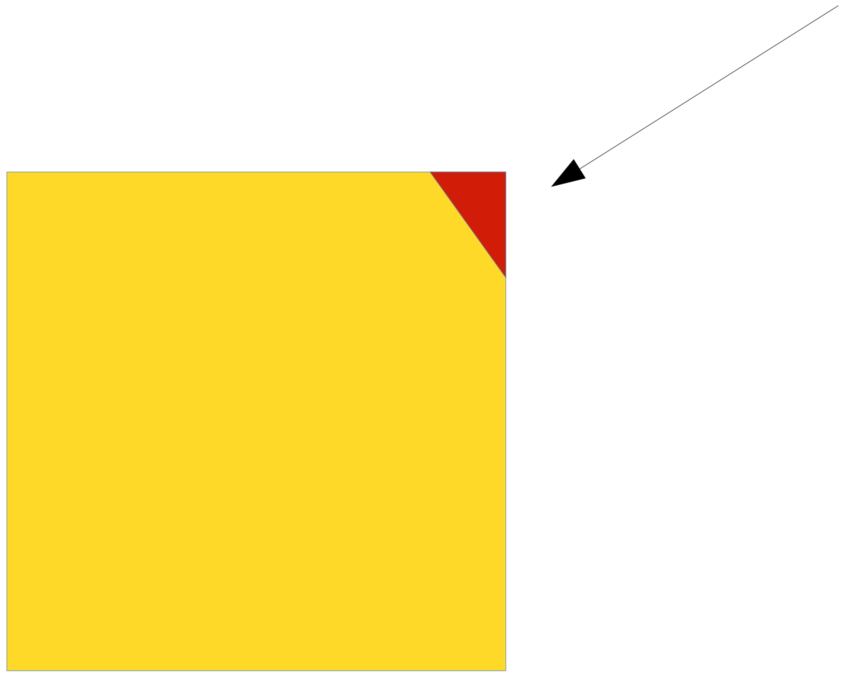


$S_A = S_B \propto$ Area of boundary between A and B $= \#$ $|$ and $-$

Area law holds for low energy states of local Hamiltonians

Low energy states of local Hamiltonians are here

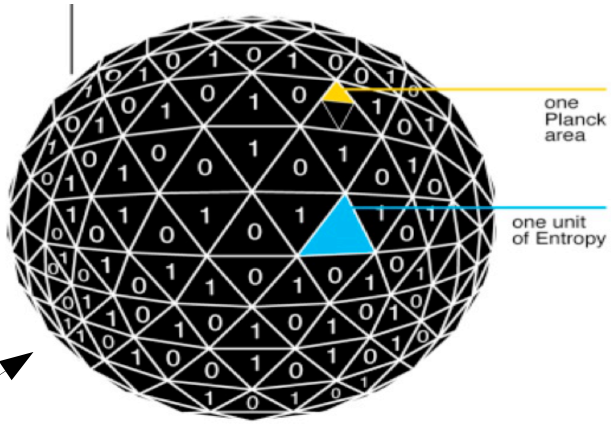
Hilbert space =



Black hole entropy

Bekenstein- Hawking 70's

$g_{\mu\nu}(x)$



horizon

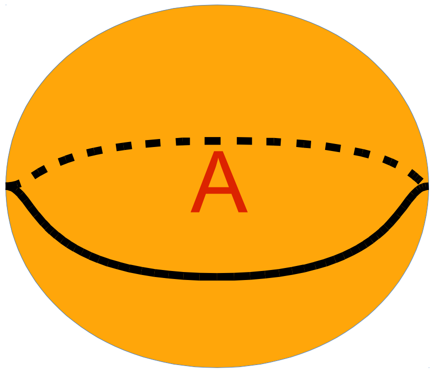
$$S_{BH} = \frac{Area}{4\,\ell_P^2}$$

Planck length

Area law in Quantum Field Theory

Bombelli et al (86), Srednicki (93)

$\phi(x^\mu)$



$$S_A = C\,\frac{Area}{\varepsilon} + \dots$$

UV cutoff

1D Area law

$$S_A \propto cte$$

Independent of size of A or B

Characteristic of systems with a gap in the spectrum or finite correlation length

Explains the success of the DMRG method (Steve White 1992)

Why?

DMRG = variational ansatz = Matrix Product States

Ostlund, Rommer

$$H = \sum_{i-1}^{N} h_{i,i+1}$$

$$|\psi\rangle = \sum_{s_1,\ldots s_N} \psi(s_1, s_2, \ldots, s_N)\, |s_1, s_2, \ldots, s_N\rangle, \quad s_i = \pm 1$$

A Matrix Product State (MPS)

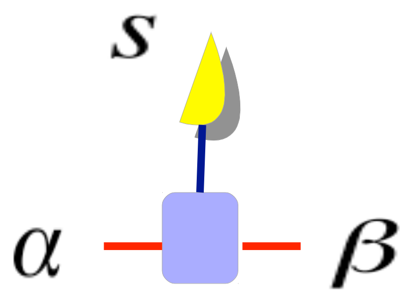
open $$\psi(s_1, s_2, \ldots, s_N) = \langle v | A^{(1)}(s_1) \cdots A^{(N)}(s_N) | w \rangle$$

closed $$\psi(s_1, s_2, \ldots, s_N) = Tr(A^{(1)}(s_1) \cdots A^{(N)}(s_N))$$

$$A_i(s_i) : \chi \times \chi \; matrix$$

Graphical representation of MPS

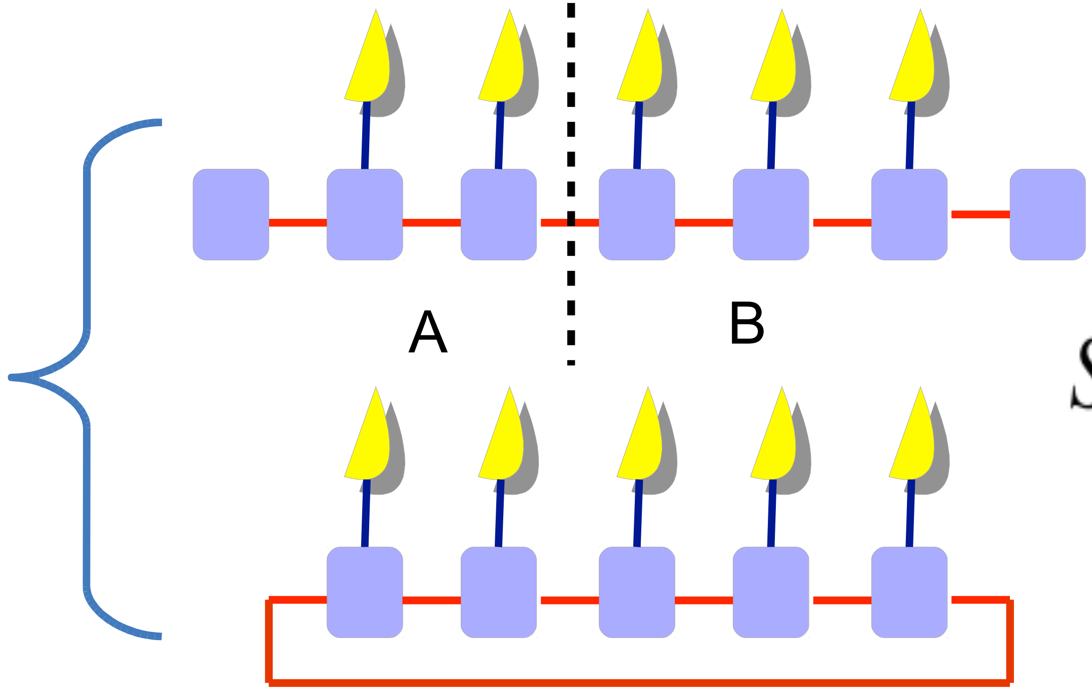
$A_{\alpha,\beta}(s) =$



$s = +,-$

$\alpha,\beta = 1,2,\cdots,\chi$

$\psi(s_1, s_2, \ldots, s_N) =$

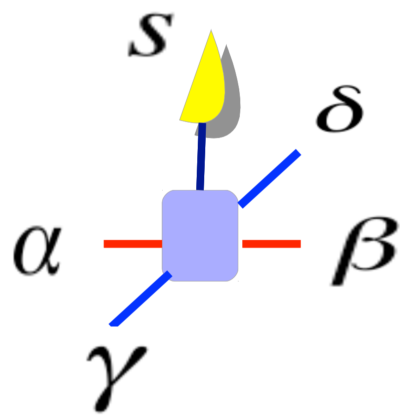


$S_A \leq \ln \chi$

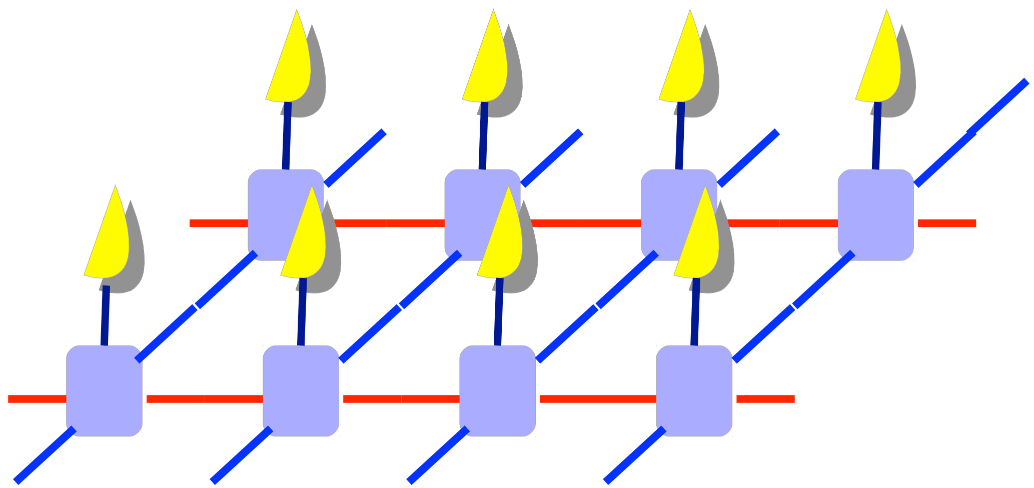
2D Area Law : Proyected Entangled Pair States (PEPS)

Cirac and Verstraete 2004

$A^{\gamma,\delta}_{\alpha,\beta}(s) =$

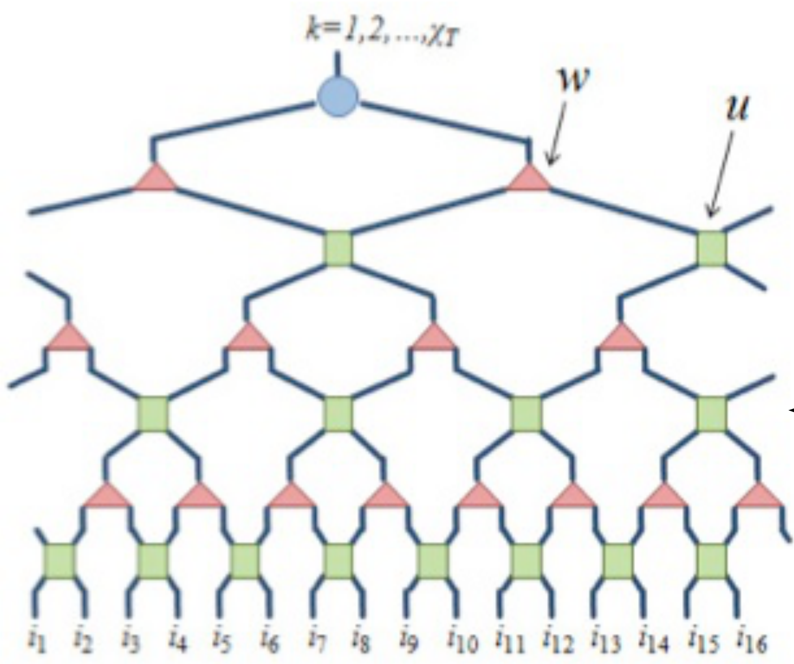


$\alpha,\beta = 1,2,\cdots,\chi$



Multiscale Entangled Renormalization Ansatz (MERA)

Vidal 2005



RG -time

disantangler

isometry

MPS, PEPS, MERA $\rightarrow$ Tensor Networks States

Entanglement in spin chains

Vidal, Latorre, Rico, Kitaev (03)

$$H_{ISING} = \sum_i \sigma_i^X \sigma_{i+1}^X - h\sigma_i^Z$$

$$H_{XY} = \sum_i \frac{1+\gamma}{2}\sigma_i^X \sigma_{i+1}^X + \frac{1-\gamma}{2}\sigma_i^Y \sigma_{i+1}^Y - h\sigma_i^Z$$

$$H_{XXZ} = \sum_i \sigma_i^X \sigma_{i+1}^X + \sigma_i^Y \sigma_{i+1}^Y + \Delta\sigma_i^Z \sigma_{i+1}^Z - h\sigma_i^Z$$

When the chain is critical

$$S_A = \frac{c}{3}\ln \ell + cte, \quad \ell << L$$

c =1/2 (ISING), 1 (XY and XXZ)

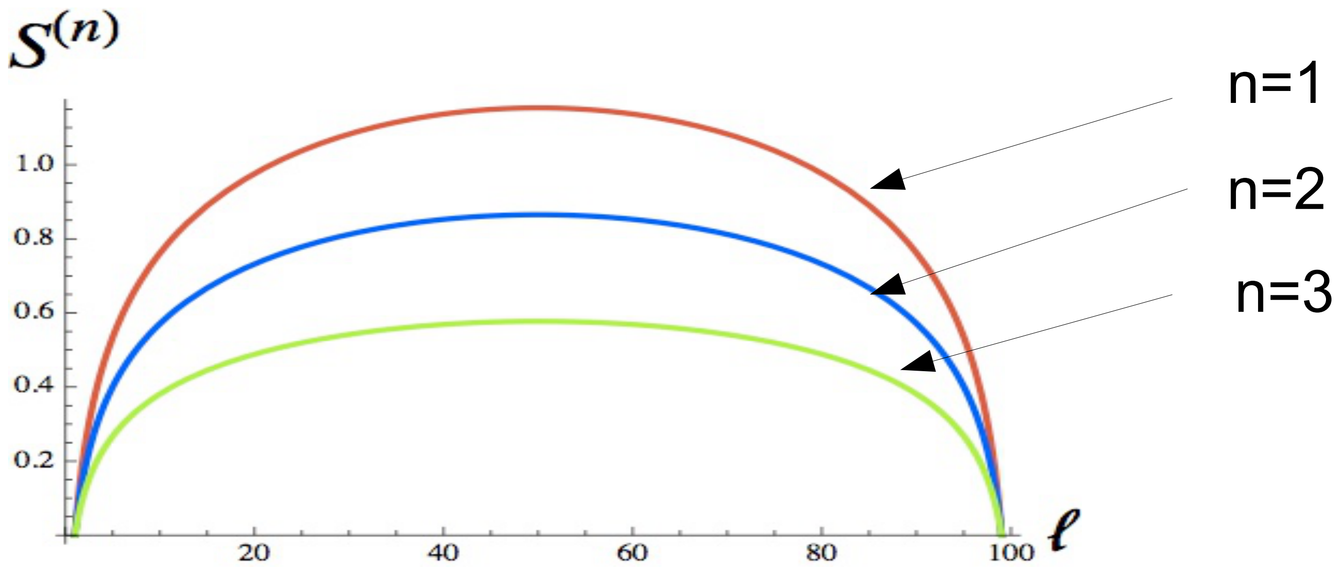
c: central charge of the Conformal Field Theory

Entanglement in Conformal Field Theory

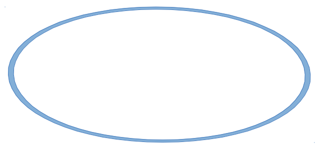
Holzhey, Larsen, Wilczek (94)
Calabrese, Cardy (04)

Replica trick -> Renyi entropies

$$S_A^{(n)} = \frac{1}{1-n} \ln Tr\, \rho_A^n = \frac{c}{6}\left(1+\frac{1}{n}\right) \ln\left(\frac{L}{\pi a}\sin\frac{\pi \ell}{L}\right) + \gamma_n$$



Entanglement entropy of periodic systems



$$S_A^{(1)} = \frac{c}{3}\ln\left(\frac{L}{\pi a}\sin\frac{\pi\ell}{L}\right) + c'_1$$

Entanglement entropy of open systems

$$S_A^{(1)} = \frac{c}{6}\ln\left(\frac{L}{\pi a}\sin\frac{\pi\ell}{L}\right) + \frac{c'_1}{2} + 2g$$

Half of the close system

Boundary entropy (Affleck, Ludwig)

Entanglement entropy at finite temperature

$$S_A = \frac{c}{3}\ln\left(\frac{\beta}{\pi a}\sinh\frac{\pi\ell}{\beta}\right) + c'_1, \qquad \beta = \frac{1}{T}$$

High temperature $$S_A \approx \frac{\pi c}{3}\frac{\ell}{\beta}$$ extensive

Perturbing the CFT → Massive theory with finite correlation length ξ

$$S_A \approx \frac{c}{6}\ln\frac{\xi}{a}$$

Finite size effects in CFT

Blote, Cardy, Nighingale (86)
Affleck (86)

Central charge gives the finite size corrections of the ground state energy

$$E_0 \approx e_\infty L - \frac{\pi c}{6L}$$

Excited states are caracterized by the conformal weights (h, h*) and their level (n,n*)

$$E_{exc} - E_0 \approx \frac{2\pi}{L}\left(h + \bar{h} + n + \bar{n}\right)$$

Primary states : n=n*=0 are created by a primary field

$$\left|\Phi\right\rangle = \Phi(0,0)\left|0\right\rangle$$

Entanglement in excited states of CFT

Alcaraz, Ibañez, GS (11)

Renyi entropies of primary states are given by 2n point correlators

$$F_{\Phi}^{(n)}(x) = \frac{Tr_A\left(\rho_A^{(\Phi)}\right)^n}{Tr_A\left(\rho_A^{(1)}\right)^n} \propto \frac{\left\langle \Phi\Phi^* \ldots^{n} \ldots \Phi\Phi^* \right\rangle}{\left\langle \Phi\Phi^* \right\rangle^n}$$

Von Neumann entropy $\quad S_{exc}(\ell) - S_{GS}(\ell) = -dF_{\Phi}^{(n)}/dn\,|_{n=1}$

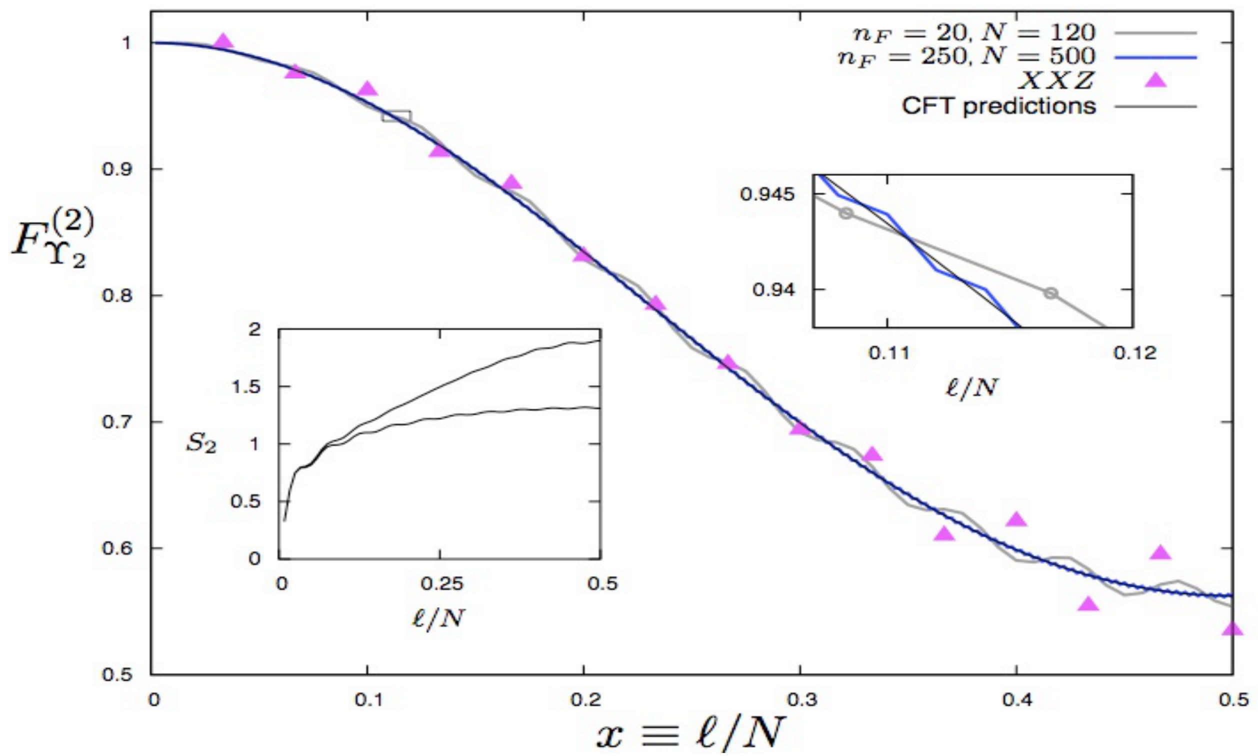
By analytic extension one obtains generically

$$S_{exc}(\ell) - S_{GS}(\ell) \approx \frac{2\pi^2}{3}\left(h+\bar{h}\right)\left(\frac{\ell}{L}\right)^2, \quad \ell << L$$

Example: XXZ spin chain ⟶ c=1 free boson

Lowest excitation with $p = 2\pi/L$ ⟶ $\partial\varphi(z)$

$$F^{(2)}_{\partial\phi}(x) = 1 - 2s^2 + 3s^4 - 2s^6 + s^8, \quad s = \sin\frac{\pi x}{2}, \quad x = \frac{\ell}{L}$$



General Renyi entropies for $\partial\varphi(z)$ Essler, Lauchli, Calabrese (12)

$$F_{\partial\phi}^{(n)}(x) = \left[\left(\frac{2\sin\pi x}{n}\right)^n \frac{\Gamma\left(\frac{1+n+n\csc\pi x}{2}\right)}{\Gamma\left(\frac{1+n+n\csc\pi x}{2}\right)}\right]^2$$

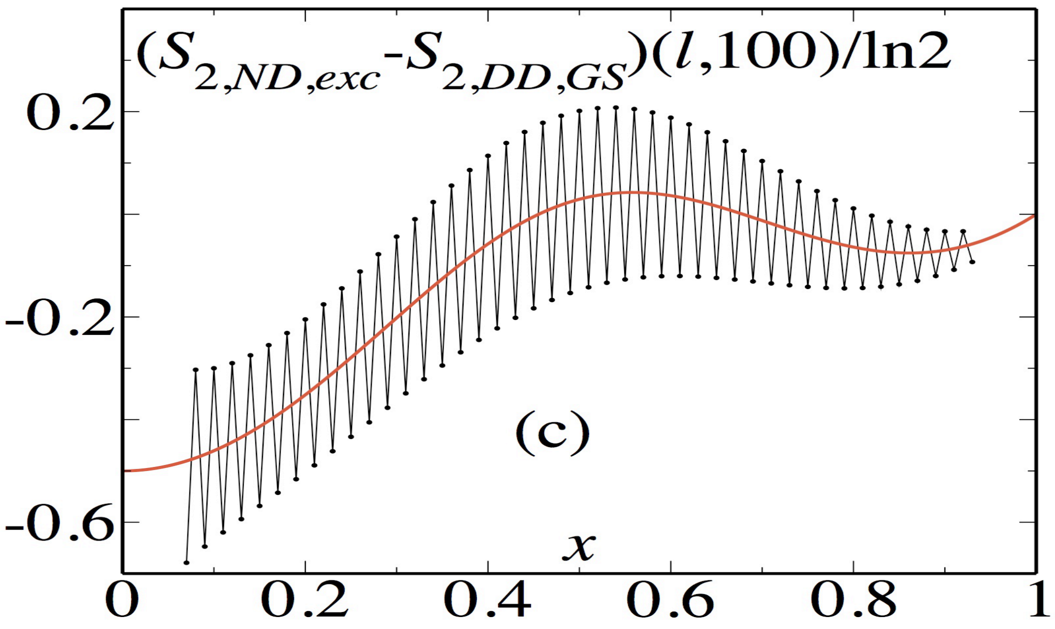
$$S_{exc} - S_{GS} = \sin\pi x + \ln|2\sin\pi x| + \psi\left(\frac{1}{2\sin\pi x}\right), \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

Entanglement in open systems CFT

Taddia, Xavier, Alcaraz, GS (13)

$$H_B = -\sum_{j=1}^{L-1} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^x \sigma_{j+1}^x\right) + \\ -\frac{1}{2}(\alpha_- \sigma_1^- + \alpha_+ \sigma_1^+ + \alpha_z \sigma_1^z + \beta_- \sigma_L^- + \beta_+ \sigma_L^+ + \beta_z \sigma_L^z),$$

Boson with
Neumann or Dirichlet
Boundary conditions



Infinite Matrix Products and CFT

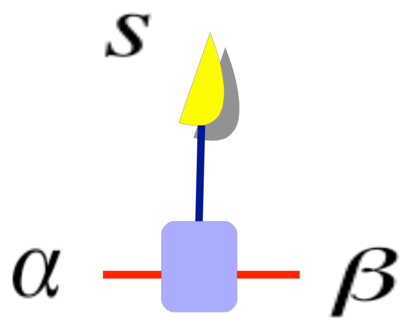
Cirac, GS, 2009
Nielsen, Cirac, GS

1D area law → $S_A \leq \ln \chi$

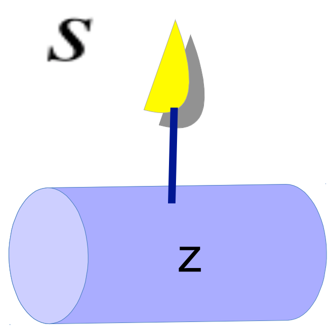
Critical systems → $S_A \approx \frac{c}{3} \ln \ell$

$\chi = \infty$

$A_{\alpha,\beta}(s) =$



$\phi_s(z) =$

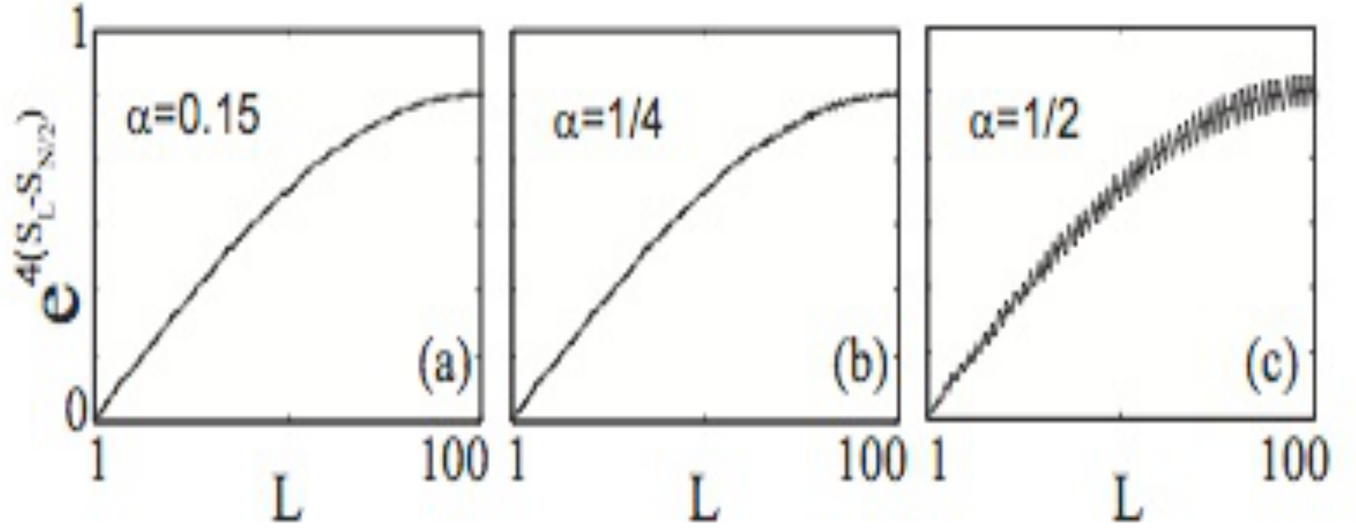


Operator in a CFT

Vertex operators $\phi_s(z) \propto e^{is\sqrt{\alpha}\,\varphi(z)}, \quad s = \pm 1$

The Infinite MPS wave functions have a Jastrow form

$$\psi(s_1,\ldots,s_N) = \left\langle \phi_{s_1}(z_1)\cdots\phi_{s_N}(z_N) \right\rangle \propto \prod_{i<j}(z_i - z_j)^{\alpha\, s_i\, s_j}$$

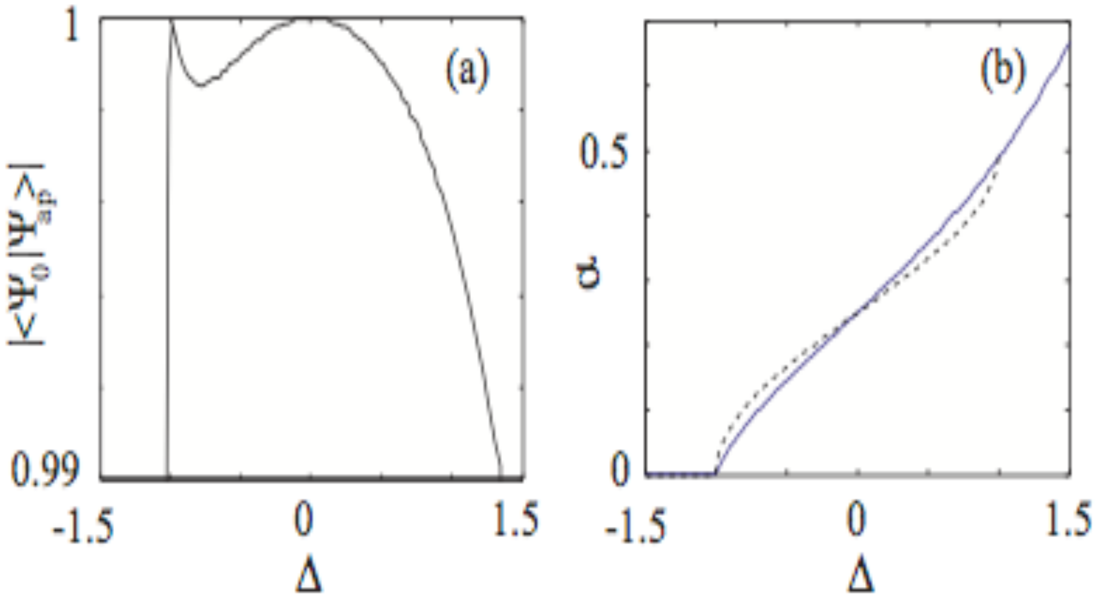


Agrees with a c =1 CFT

Trial wave functions for the XXZ model

$$H_{XXZ} = \sum_i \sigma_i^X \sigma_{i+1}^X + \sigma_i^Y \sigma_{i+1}^Y + \Delta \sigma_i^Z \sigma_{i+1}^Z$$

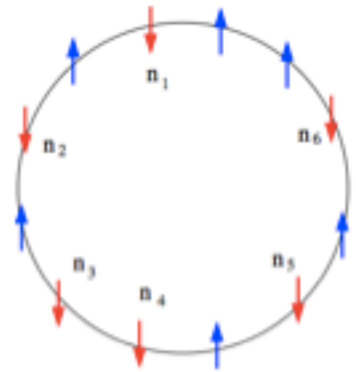
Overlap



$$\alpha = \frac{1}{4} \leftrightarrow \Delta = 0$$ Exact wave function

Haldane-Shastry model and CFT

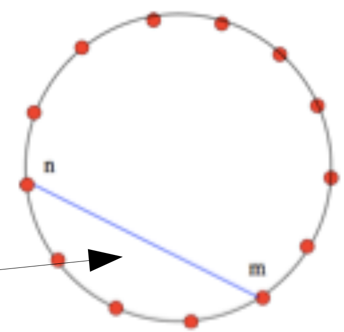
$\alpha = \frac{1}{2}$ Is equivalent to the Haldane-Shastry wave function



$$\psi_{HS}(n_1,\ldots,n_{N/2}) = \prod_i z_{n_i} \prod_{i<j} (z_{n_i} - z_{n_j})^2, \quad z_n = e^{2\pi i n/N}$$

Ground state of the Hamiltonian

$$H_{HS} = -\frac{4J\pi^2}{N^2} \sum_{n<m} \frac{z_n z_m}{(z_n - z_m)^2} \vec{S}_n \cdot \vec{S}_m, \quad z_n = e^{2\pi i/N}$$



The HS model belongs to the same universality class as AF Heisenbrg model

Kalmeyer-Laughlin wave function and CFT

Bosonic version of Laughlin wave function at filling = ½

$$\psi_{KL}(z_1,\cdots z_{N/2}) = \prod_n \chi(z_n) \prod_{n<m} (z_n - z_m)^2 e^{-\sum_q |z_q|^2/4}$$

$$\lim_{N\to\infty} \frac{\psi_{CFT}}{\psi_{KL}} = 1$$

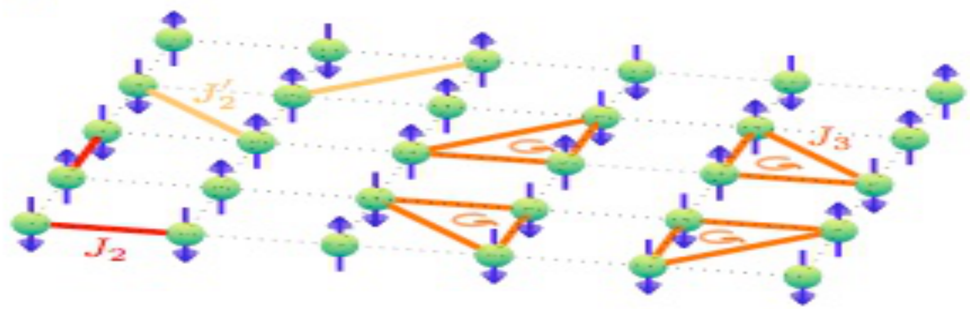
Parent Hamiltonian of the CFT wave function

$$H = \frac{1}{2}\sum_{i\neq j} |w_{ij}|^2 + \frac{2}{3}\sum_{i\neq j} |w_{ij}|^2 S_i^a S_j^a + \frac{2}{3}\sum_{i\neq j\neq k} w_{ij}^* w_{ik} S_j^a S_k^a$$
$$- \frac{2i}{3}\sum_{i\neq j\neq k} w_{ij}^* w_{ik} \varepsilon^{abc} S_i^a S_j^b S_k^c,$$

Truncated Hamiltonian

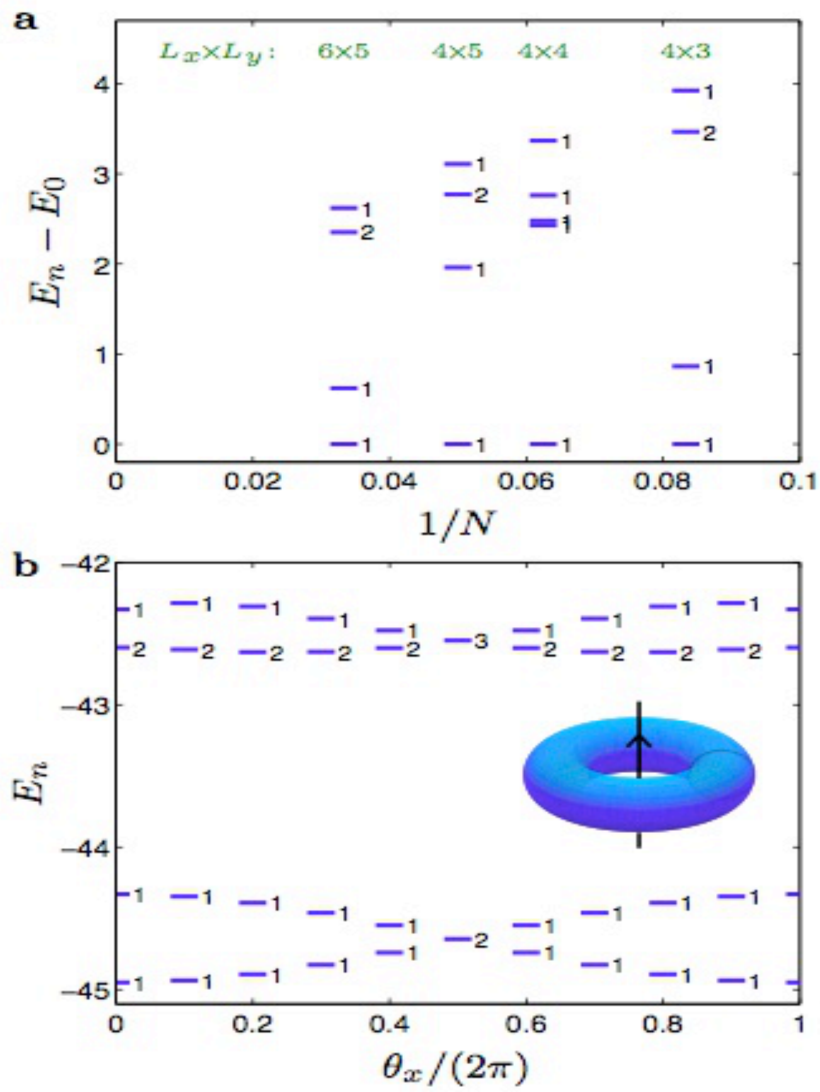
$$H = J_2 \sum_{<n,m>} 2\vec{S}_n \cdot \vec{S}_m + J_2' \sum_{\ll n,m \gg} 2\vec{S}_n \cdot \vec{S}_m - J_3 \sum_{<n,m,p>_\circlearrowleft} 4\vec{S}_n \cdot \left(\vec{S}_m \times \vec{S}_p\right)$$

a



N	$L_x \times L_y$	Plane $\lvert\langle\psi_{P0}\vert\psi_{P0}^{CFT}\rangle\rvert$	Torus $\lvert\langle\psi_{T0}\vert\psi_{T0}^{CFT}\rangle\rvert$	$\lvert\langle\psi_{T1}\vert\psi_{T1}^{CFT}\rangle\rvert$
12	4×3	0.9860	0.9818	0.9533
16	4×4	0.9812	0.9747	0.9572
20	4×5	0.9728	0.9655	0.9200
30	6×5	-	0.9258	0.9361

Topological properties



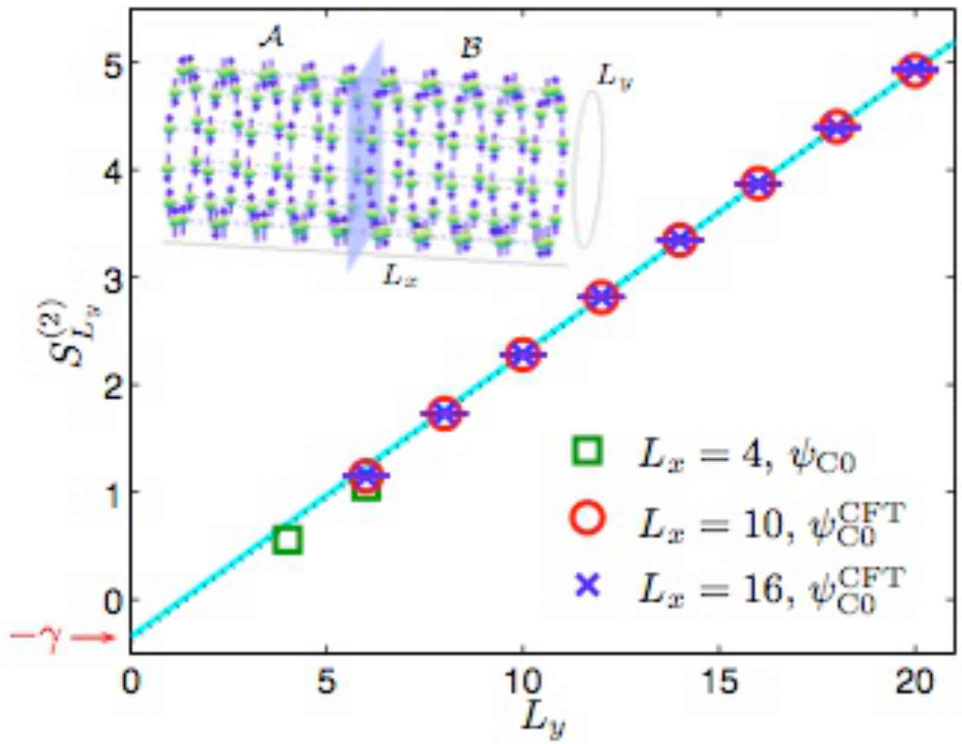
Topological entanglement entropy

Kitaev-Preskill
Levin-Wen

$$S_A \approx c\,L_y - \gamma$$

$$\gamma = \frac{1}{2}\ln 2$$

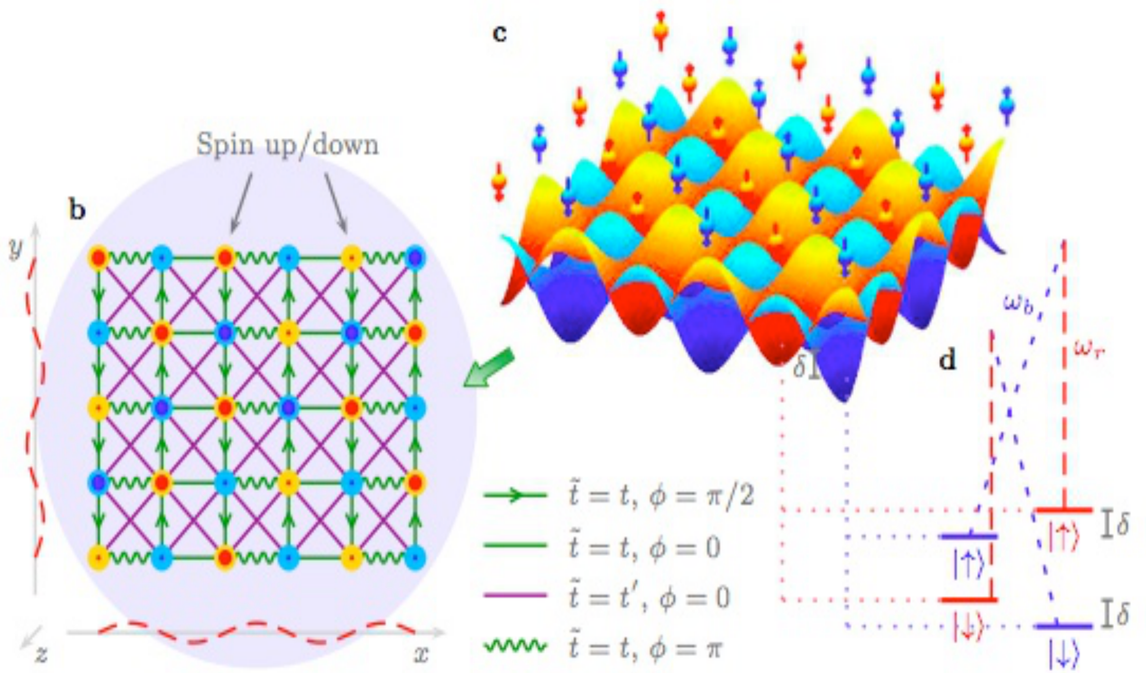
Quantum dimension of anyons



Fermi-Hubbard Hamiltonian → Spin Hamiltonian

$$H_{\mathrm{FH}} = \sum_{\sigma} H_{\mathrm{kin},\sigma} + U \sum_{n=1}^{N} \hat{a}^{\dagger}_{n\uparrow}\hat{a}_{n\uparrow}\hat{a}^{\dagger}_{n\downarrow}\hat{a}_{n\downarrow}$$

Optical lattice realization of the FQH at fillin 1/2



Conclusions

- The concept of entanglement has deepen our understanding of Quantum Physics in the last years

- It lead to the understanding of the DMRG method and it turn to the invention of the PEPS, MERA, etc.

- New characterization of the phase transitions and new phases of matter in particular the topological phases like the FQHE.

- Intriguing links between black hole physics, holography and many body physics