Quantum spin Hamiltonians for the SU(2) WZW model

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Workshop:
ADVANCED CONFORMAL FIELD THEORY AND APPLICATIONS

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Overview

An application of CFT is to describe the low energy physics of 1D critical quantum systems, e.g.

- Antiferromagnetic Heisenberg spin chain
- XXZ chain
- Ising model in a transverse field
- Hubbard and t-J chains

CFT, combined with analytical and numerical methods as Bethe ansatz, RG, Lanzcos, DMRG, MPS, etc, give information:

- Correlators of operators
- Susceptibilities, finite T properties, etc
- Finite size corrections to energies
- Entanglement entropies

Another application of CFT is to provide ansatzs for the GS and excitations of the Fractional Quantum Hall effect

- Laughlin-> U(1) CFT
- Moore-Read-> SU(2)@k=2
- Read-Rezayi -> SU(2)@k>2

It has been long known the existence of several analogies between spin systems and the FQHE:
fractionalization of degrees of freedom and non trivial statistics.

In AFH spinons have spin 1/2 instead of spin 1 (=magnons) and behave as semions = $\sqrt{fermions}$

Similar to the quasiparticles of the Laughlin state, which have charge 1/m and anyon statistics.

Spin chains <-- CFT --> FQH

Spin chains: CFT description appears at the “end” of a long analysis.

Alternative: follow the FQH strategy

CFT -> trial wave functions for GS and excitations of spin systems

As in the FHQ one can compute overlaps with the exact GS of microscopic Hamiltonians

This strategy is also similar to the AKLT ansatz or more generally Matrix Produc States (MPS)

MPS/DMRG-> states are “products” of finite dimensional matrices

So CFT gives an infinite dimensional version of MPS (iMPS) where matrices are replaced by operators acting on Fock spaces.

FQH wave functions are the GS of Hamiltonians (contact type)
MPS are the GS of so called Parent Hamiltonians

Question: can we construct the Parent Hamiltonians for the spin wave functions built from CFT?

Answer: Yes we can, when the CFT corresponds to the WZW models

First example:

SU(2)@k=1 -> Haldane-Shastry model for spin 1/2 (uniform version)

We also construct a non-uniform version of this model

Other results:

SU(2)@k=2 -> spin 1 version of the Haldane-Shastry model

SU(2)@k=2 -> spin 1/2 model with degenerate GS's

The construction can be generalized to k>2 and to 2 dimensions !!

Plan of the talk

-- Spin wave functions using vertex operators

-- Applications to spin models

-- Brief review of the Haldane-Shastry model

-- Relation with the SU(2)@k=1WZW model

-- Generalizations to SU(2)@k>1

Based on

- arXiv: 0911.3029 (I. Cirac, G.S.)

- arXiv: 1109.5470 (A.Nielsen, I. Cirac, G.S.)

CFT and Infinite Matrix Product States

Consider a 1D spin 1/2 system with N sites and Hamiltonian

$$H = \sum_{i=1}^{N} h_{i,i+1}$$

The GS wave function is given in a local spin basis by

$$|\psi\rangle = \sum_{s_1,\ldots s_N} \psi(s_1, s_2, \ldots, s_N)\, |s_1, s_2, \ldots, s_N\rangle, \quad s_i = \pm 1$$

Consider an ansatz of the form

$$\psi(s_1, s_2, \ldots, s_N) = \langle v | A^{(1)}(s_1) \cdots A^{(N)}(s_N) | w \rangle$$

MPS: $$s_i \rightarrow A_i(s_i) : \chi \times \chi \; matrix$$

iMPS: $$s_i \rightarrow A_i(s_i) : chiral\ vertex\ operator$$

The iMPS wave functions are conformal block of a CFT

The simplest CFT: massless boson (c=1)

Consider a chiral free boson field $\varphi(z)$

with two-point correlator $\langle \varphi(z_1)\varphi(z_2)\rangle = -\log(z_1 - z_2)$

Take $A_i(s_i) = \chi_{s_i} \; : e^{i s_i \sqrt{\alpha}\, \varphi(z_i)} :$ conformal weight $h = \dfrac{\alpha}{2}$

The wave function is

$$\psi(s_1, s_2, \ldots, s_N) = \prod_i \chi_{s_i} \prod_{i<j} (z_i - z_j)^{\alpha s_i s_j}, \qquad \sum_i s_i = 0$$

Variational parameters $z_i \,, i = 1, \ldots N$

$\sqrt{\alpha}, \quad \chi_{s_i} = \pm 1$

Applications to spin 1/2 Heisenberg like chains:

- Anisotropic (XXZ)
- $J_1 - J_2$
- Random bond

Determining the parameters $\alpha, z_1, \ldots, z_N$ in terms of the couplings

- Overlaps with exact wave functions up to chains with N=20 sites
- Spin-spin correlators
- 2-Renyi entropy

The sign factors given by the Marshall rule of antiferromagnets (Perron-Frobenius theorem)

$$\prod_i \chi_{s_i}, = e^{i\pi/2 \sum_{i:odd}(s_i - 1)}$$

XXZ spin 1/2 model

Hamiltonian periodic BCs

$$H = \sum_{1=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

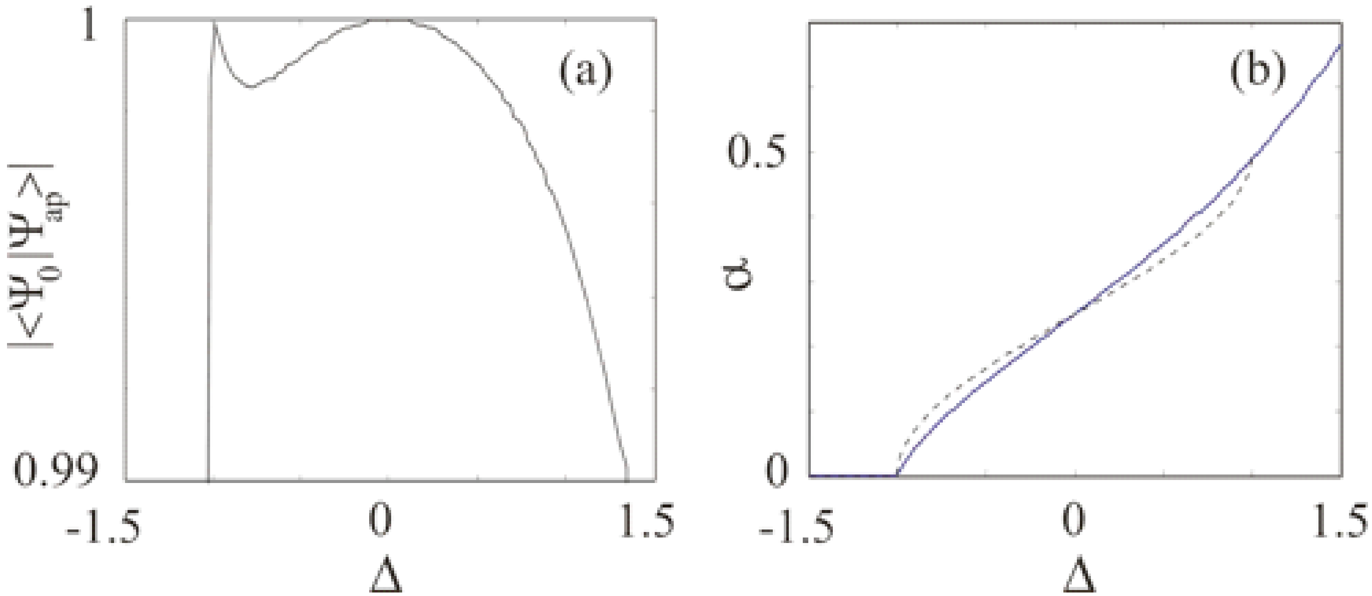
Phases of the model

$$\Delta > 1 \quad \textit{gapped antiferromagnet}$$
$$-1 < \Delta \leq 1 \quad \textit{gapless} \ \ (c = 1\, CFT)$$
$$\Delta \leq -1 \quad \textit{Ferromagnetic}$$

Translational invariant GS $\quad z_n = e^{2\pi i n/N}, \quad n = 1, \ldots, N$

To find α we minimize the energy of the iMPS

Overlap of exact and the iMPS wave functions (N=20)



$\Delta = -\cos(2\pi\alpha)$

The iMPS is exact in two cases

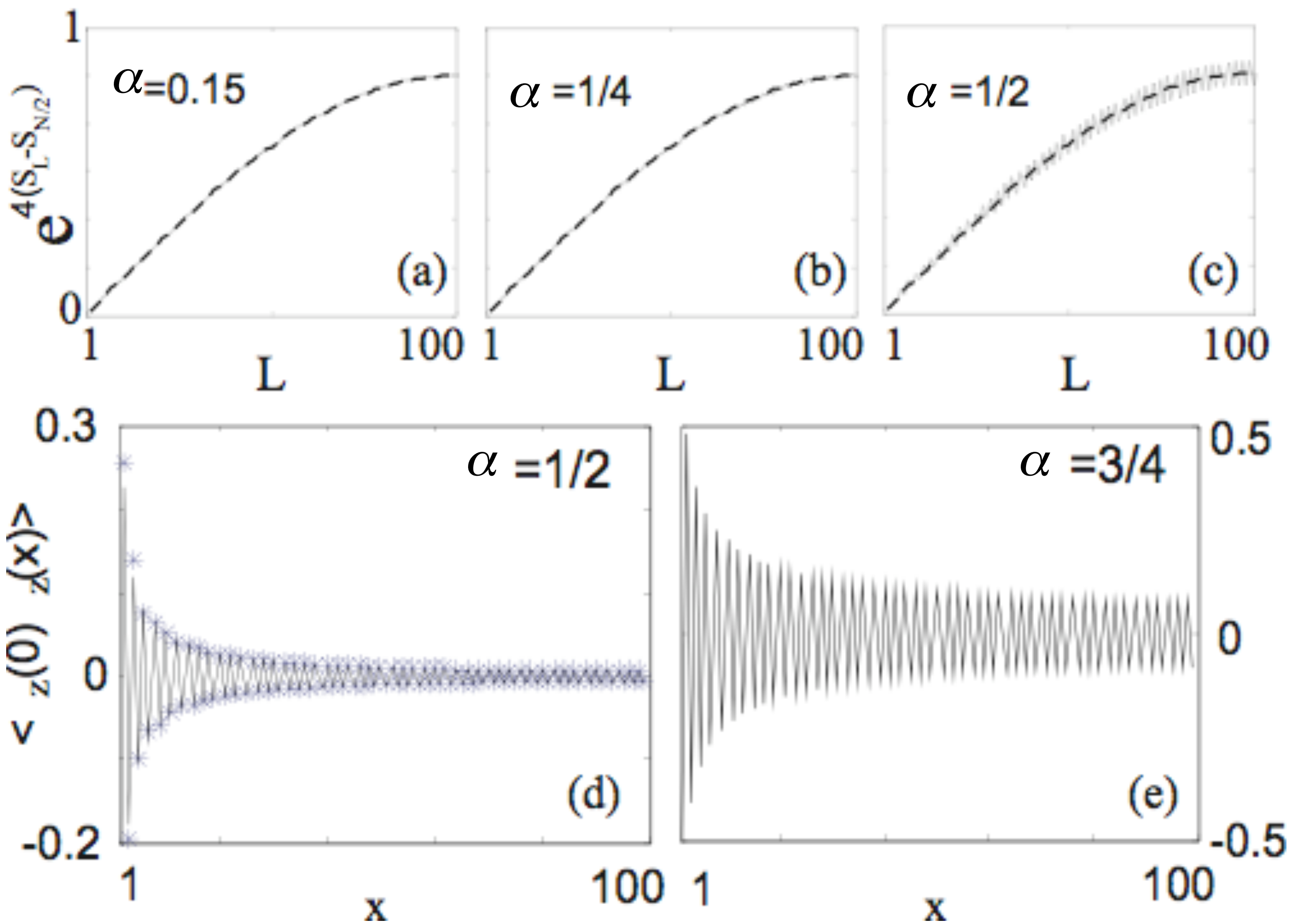
$\Delta = -1 \rightarrow \alpha = 0$ isotropic ferromagnetic chain

$\Delta = 0 \rightarrow \alpha = 1/4$ XX chain

At the isotropic AFH model

$\Delta = 1 \rightarrow \alpha = 1/2$ Haldane-Shastry spin chain

Renyi entropy $S_L = -\log Tr\,\rho_L^2$ and spin correlators (MC method)



In the critical regime it agrees with a c=1 CFT

$J_1 - J_2$ Model (zig-zag chain)

$$H = \sum_{1=1}^{N} J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_i \cdot \vec{S}_{i+2} \qquad (J_1 = 1)$$

$J_2 > 0$ frustrated spin system

Phases

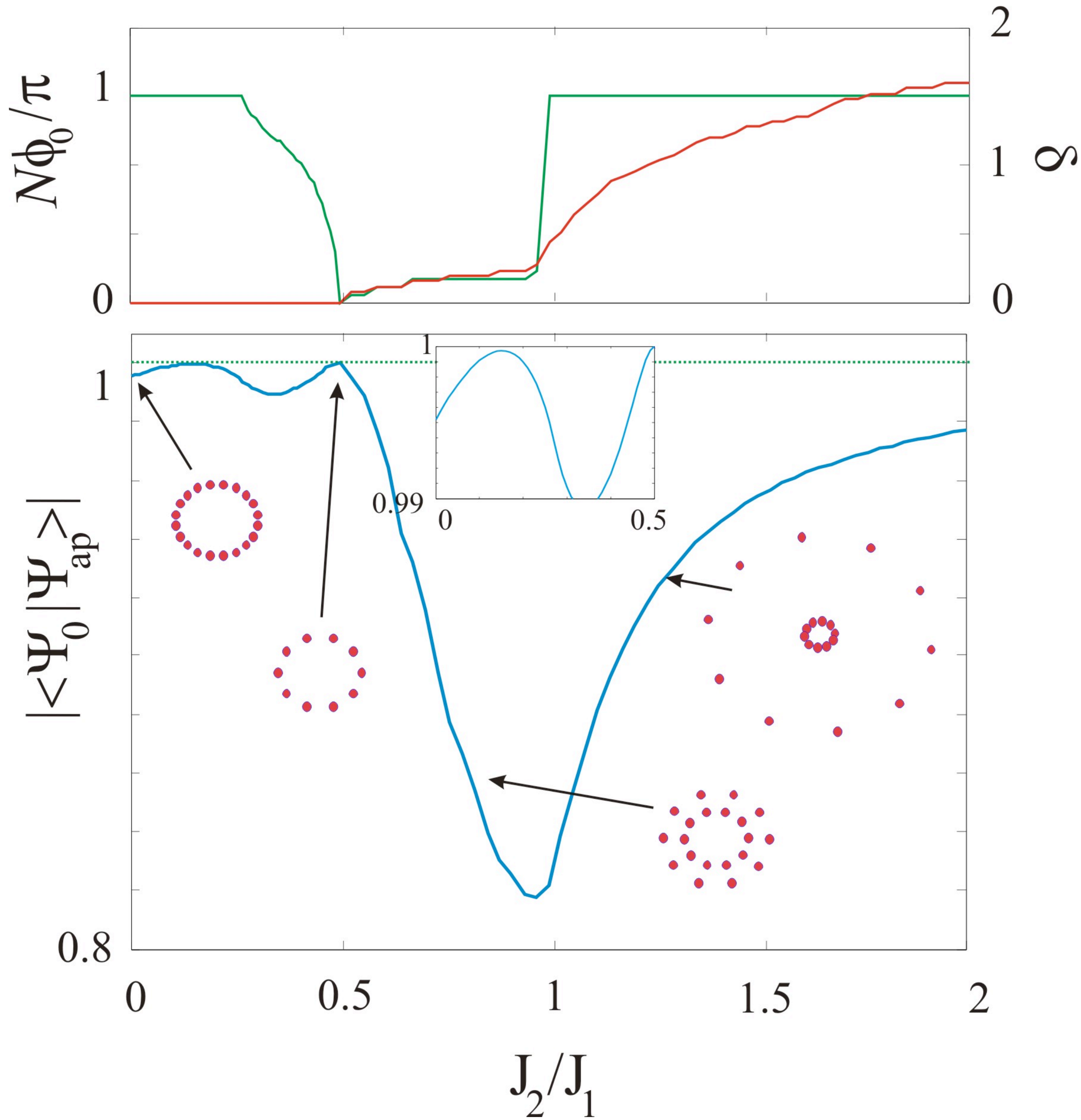
$0 \leq J_2 < J_{2c} \approx 0.241$	Critical c=1
$J_{2c} < J_2 < J_{MG} = 0.5$	Spontaneously dimerized
$J = J_{MG}$	Majumdar-Gosh point
$J_{MG} < J < \infty$	Dimer spiral phase

Choice of parameters $\alpha = \frac{1}{2}$ -> rotational invariance

$$z_n = \begin{cases} \exp(\delta - i\phi_0)\exp(2\pi i(0,2,4,\ldots)/N) & \textit{even sites} \\ \exp(-\delta + i\phi_0)\exp(2\pi i(0,2,4,\ldots)/N) & \textit{odd sites} \end{cases}$$

ϕ_0 -> dimerization

δ -> “split” of the chain



$N\phi_0/\pi$
1
0
2
1
0
ϱ
$|<\Psi_0|\Psi_{ap}>|$
1
0.8
1
0.99
0
0.5
0
0.5
1
1.5
2
J_2/J_1

Brief review of the Haldane-Shastry model (1988)

Fermi state of a spin 1/2 particle on a circle at half filling

$$|FS\rangle = \Pi_{|k|<k_F} c^*_{k\uparrow} c^*_{k\downarrow} |0\rangle \qquad k_F = \frac{\pi}{2}$$

Eliminate the states doubly occupied (Gutzwiller projection)

$$|\psi_G\rangle \propto P_G |FS\rangle = \Pi_i (1 - n_{i\uparrow} n_{i\downarrow}) |FS\rangle$$

Spin-spin correlator (Gebhard-Vollhardt 1987)

$$\langle S_n^a S_0^b \rangle = (-1)^n \delta_{ab} \frac{Si(\pi n)}{4\pi n} \approx \delta_{ab} \left[(-1)^n \frac{1}{8n} - \frac{1}{4\pi^2 n^2} \right], (n \to \infty)$$

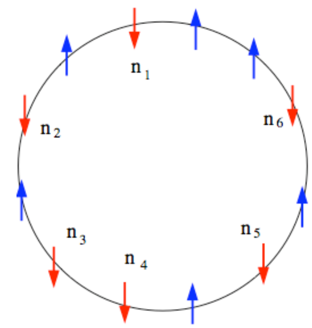
Compare with the correlator in the AF Heisenberg model

$$\langle S_n^a S_0^b \rangle \approx \delta_{ab} \left[(-1)^n \frac{c \sqrt{\log n}}{n} - \frac{1}{4\pi^2 n^2} \right], \ (n \to \infty)$$

The Gutzwiller states has only spin degrees of freedom that can be seen as a hardcore boson

$$|\uparrow\rangle \leftrightarrow |0\rangle \qquad \textit{empty}$$

$$|\downarrow\rangle \leftrightarrow a^* \, |0\rangle \quad \textit{occupied}$$

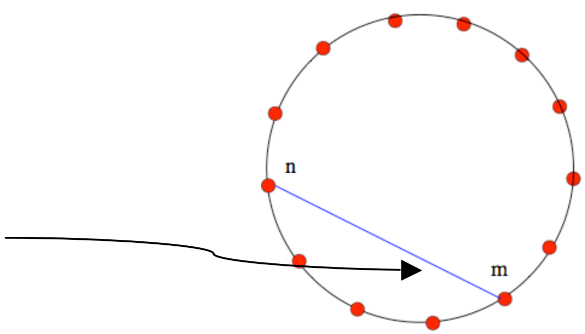


$$|\psi_G\rangle \propto \sum_{n_1,\ldots,n_{N/2}} e^{i\pi \sum_i n_i} \prod_{i<j} \left|\sin \frac{\pi(n_i - n_j)}{N}\right|^2 \; a^*_{n_1} \ldots a^*_{n_{N/2}} |0\rangle$$

n_i : position of the i-boson (i.e. spin down)

$|\psi_G\rangle$ ground state of the Hamiltonian (Haldane-Shastry)

$$H = \frac{J\pi^2}{N^2} \sum_{n<m} \frac{\vec{S}_n \cdot \vec{S}_m}{\sin^2(\pi(n-m)/N)}$$



Properties of the HS model

- spin-spin correlation functions decays algebraically

- elementary excitations: spinons (spin 1/2 with fractional statistics)

- degenerate spectrum described by the Yangian symmetry

- critical theory at the fixed point of the renormalization group

- this fixed point is described a CFT: $SU(2)_{k=1}$ WZW model

The HS model and the AFH model belong to the same universality class described by the WZW model, but the AFH model is a marginal irrelevant perturbation of the WZW which give rise to the log corrections in correlators

The HS state can be written in the spin variables as

$$\psi(s_1, s_2, \ldots, s_N) = \chi_{s_1, \ldots s_N} \prod_{i<j} (z_i - z_j)^{s_i s_j /2} = \left\langle A_{z_1}(s_1) \cdots A_{z_N}(s_N) \right\rangle$$

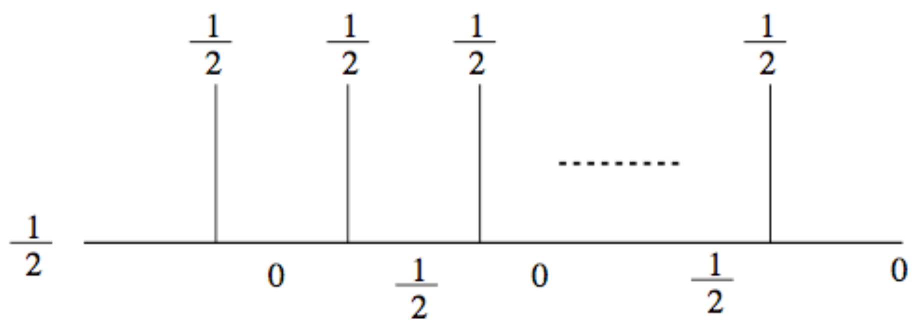
$$A_z(s) = \chi_s :e^{i s \varphi(z)/\sqrt{2}} = \phi_{1/2,s}(z)$$

primary fields of spin 1/2 of and $h = 1/4$ of $SU(2)_{k=1}$

Fusion rule: $\phi_{1/2} \times \phi_{1/2} = \phi_0$

ψ is the unique conformal block

(N even)



The Haldane-Shastry state is a conformal block

The HS- Hamiltonian can be written as

$$H = -\sum_{n \neq m} \frac{z_n \, z_m}{(z_n - z_m)^2} \vec{S}_n \cdot \vec{S}_m , \qquad z_n = e^{2\pi i n / N}$$

<u>Questions:</u>

- can one derive this Hamitonian using CFT methods?

- can one find a Hamiltonian when z’s are non uniform?

The conformal block satisfies the Knizhnik-Zamolodchikov eq

$$\frac{k+2}{2}\frac{\partial}{\partial z_i}\psi(z_1,\ldots,z_N)=\sum_{j\neq i}^{N}\frac{\vec{S}_i\cdot\vec{S}_j}{z_i-z_j}\psi(z_1,\ldots,z_N)$$ For k=1

Making a conformal transformation to the cylinder $z=e^w$

$$\psi_{cyl}(w_1,\ldots,w_N)=\prod_{i=1}^{N}z_i^{1/4}\,\psi_{plane}(z_1,\ldots,z_N)$$

The KZ equation becomes

$$\frac{k+2}{2}\,z_i\frac{\partial}{\partial z_i}\psi_{cyl}=\sum_{j\neq i}^{N}\frac{z_i+z_j}{z_i-z_j}\vec{S}_i\cdot\vec{S}_j\;\psi_{cyl}(z_1,\ldots,z_N)$$

From explicit computation one also has the “abelian” KZ eq.

$$4\,z_i\frac{\partial}{\partial z_i}\psi_{cyl}=\sum_{j\neq i}^{N}\frac{z_i+z_j}{z_i-z_j}s_i\,s_j\;\psi_{cyl}(z_1,\ldots,z_N)$$

Taking another derivative and combining these two eqs one gets

$$H\psi_{cyl} = E\psi_{cyl}$$

$$H = -\sum_{n\neq m}\left(\frac{z_n z_m}{(z_n - z_m)^2} + \frac{1}{12} w_{n,m}(c_n - c_m)\right)\vec{S}_n \cdot \vec{S}_m$$

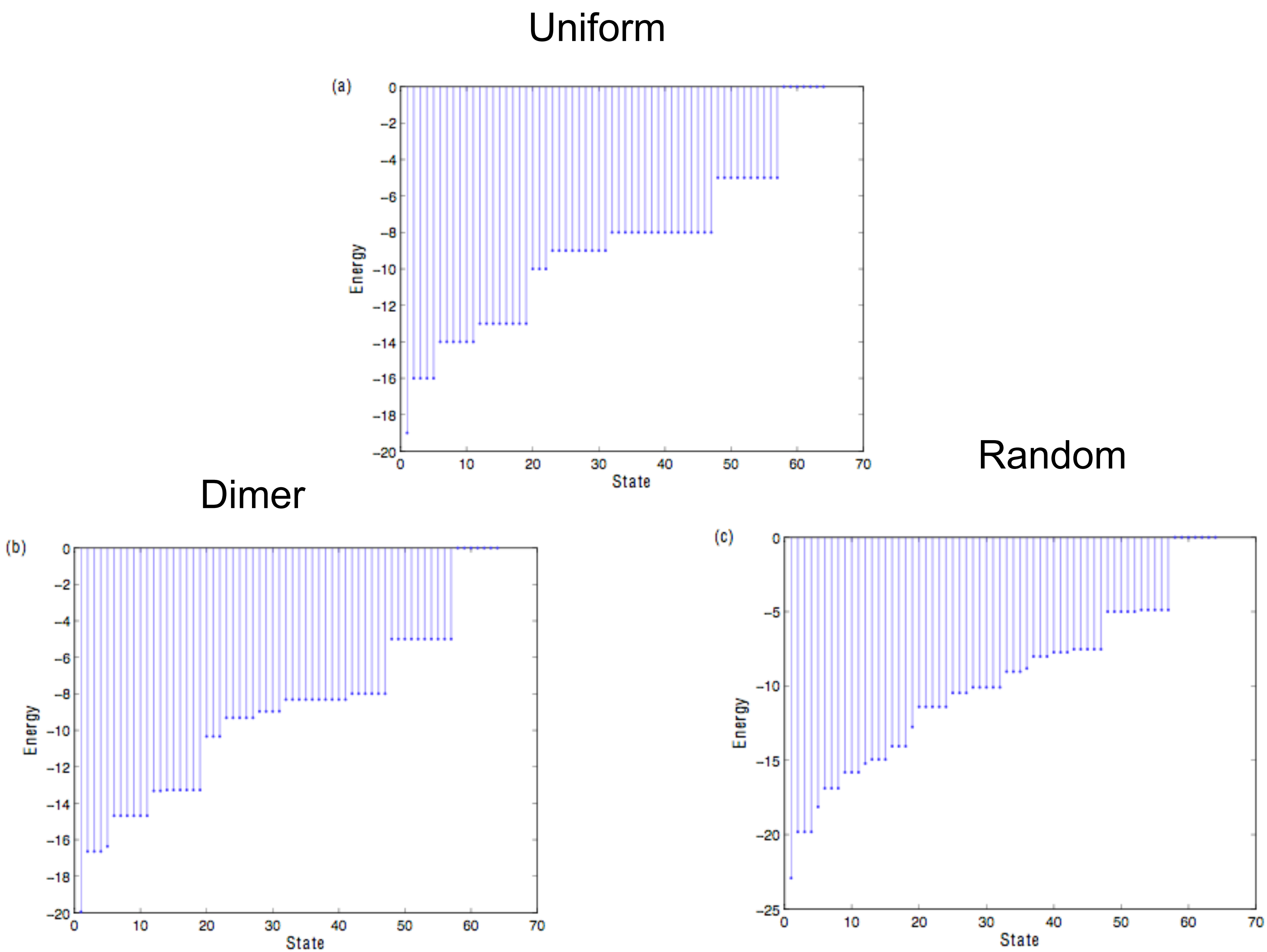
$$w_{n,m} = \frac{z_n + z_m}{z_n - z_m}, \quad c_n = \sum_{n\neq m} w_{n,m}, \quad E = \frac{1}{16}\sum_{n\neq m} w_{n,m}^2 - \frac{N(N+1)}{16}$$

In the uniform case $z_n = e^{2\pi i n/N} \rightarrow c_n = 0 \quad \forall n$

and we recover the HS Hamiltonian.

For other values of z_n we obtain an non uniform version

Except for z'n uniform the spectrum has no accidental degeneracies-> Yangian symmetry is broken


Uniform
(a)
Energy
State
Dimer
(b)
Energy
State
Random
(c)
Energy
State

Question: How to generalize this construction to
SU(2)@k with k > 1?

1st step Wave function for spin systems = conformal blocks

2nd step: Construct the Hamiltonian for which these Conformal
blocks are ground states

They key lies in the NULL VECTORS

Hamiltonians from null vectors

Kac-Moody algebra SU(2)@k

$$[J_n^a, J_m^b] = i\varepsilon^{abc} J_{n+m}^c + \frac{k}{2} n \delta^{ab} \delta_{n+m}$$

Integrable irreps correspond to the primary fields

$$\phi_{j,m}, \quad j = 0, \frac{1}{2}, \cdots, \frac{k}{2}, \quad m = j, \cdots - j$$

In each irrep there is a null vector given by (Gepner-Witten)

$$\left|\chi_{j_*, j_*}\right\rangle = \left(J_{-1}^{+}\right)^{n_*} \left|\phi_{j\ ,j}\right\rangle, \quad n_* = k + 1 - 2j, \quad j_* = k + 1 - j$$

This is the highest weight vector of a multiplet with spin j_*

Clebsch-Gordan decomposition $V_{j_*} \rightarrow V_1^{\otimes n_*} \otimes V_j$

To describe the multiplet one defines the projectors

$$K_{n_*,j} : V_1^{\otimes n_*} \otimes V_j \rightarrow V_{j_*} \rightarrow V_1^{\otimes n_*} \otimes V_j$$

So that the null fields can be written as

$$\chi_{a_1 \cdots a_{n_*},jm}(z) = \sum \left(K_{n_*,j}\right)^{a_1 \cdots a_{n_*},m}_{b_1 \cdots b_{n_*},m'} J^{b_1}_{-1} \cdots J^{b_{n_*}}_{-1} \ \phi_{jm'}(z)$$

Impose the decoupling of null fields in a correlator of primary fields

$$\left\langle \phi_j(z_1) \cdots \chi_{a_1 \cdots a_{n_*} j}(z_i) \cdots \phi_j(z_n) \right\rangle = 0$$

Using the Ward identity

$$\left\langle \phi_j(z_1) \cdots (J^a_{-1}\psi)(z_i) \cdots \phi_j(z_n) \right\rangle = \sum_{i_1 (\neq i)}^{n} \frac{t_i^a}{z_i - z_{i_1}} \left\langle \phi_j(z_1) \cdots \psi(z_i) \cdots \phi_j(z_n) \right\rangle$$

To find that the conformal blocks

$$\psi(z_1 \cdots z_n) = \left\langle \phi_j(z_1) \cdots \phi_j(z_n) \right\rangle$$

satisfy

$$C_{n_* \; j}^{i, a_1 \cdots a_{n*}} \left(z_1, \ldots z_n \right) \psi(z_1 \cdots z_n) = 0, \quad i = 1, \ldots n, \quad a = 1,2,3$$

where

$$C_{n_* \; j}^{i, a_1 \cdots a_{n*}} \left(z_1, \ldots z_n \right) = \sum \left(K_{n_* \; j}^{(i)} \right)_{b_1 \cdots b_{n*}}^{a_1 \cdots a_{n*}} w_{i\, i_1} \cdots w_{i\, i_1} \, t_{i_1}^{b_1} \cdots t_{i_{n*}}^{b_{n*}}$$

$$w_{ij} = \frac{z_i + z_j}{z_i - z_j}$$

Define the operators $H_{n_* j}^{(i)} = \sum_a \left(C_{n_* j}^{i, a_1 \cdots a_{n*}} \right)^* C_{n_* j}^{i, a_1 \cdots a_{n*}}$

Satisfy:

$$1)\ \left(H_{n_* j}^{(i)} \right)^* = H_{n_* j}^{(i)}$$

$$2)\ H_{n_* j}^{(i)} \geq 0$$

$$3)\ \left[H_{n_* j}^{(i)}, \sum_i t_i^a \right] = 0$$

$$4)\ H_{n_* j}^{(i)} \psi = 0$$

Define the total Hamiltonian as

$$H_{n_* j} = \sum_{i=1}^{n} H_{n_* j}^{(i)}$$

Whose GS is $\psi(z_1 \cdots z_n)$ with zero eigenvalue

SU(2)@k=1

Here $j = 1/2, \quad n_* = 1$

$$K^{a,m}_{b,m'} = \frac{2}{3}\left(\delta_{a,b}\,\delta_{m,m'} - i\sum_c \varepsilon_{abc}\, t^c_{m,m'}\right), \quad t^a = \frac{\sigma^a}{2}$$

$$C^{i,a}(z_1,\ldots z_n) = \sum_{j(\neq i)}^{n} w_{ij}(t^a_j + i\,\varepsilon_{abc}\, t^b_i\, t^c_j)$$

We recover the HS model

$$H_{1,1/2} \propto -\sum_{n\neq m}\left(\frac{z_n\, z_m}{(z_n - z_m)^2} + \frac{1}{12} w_{n,m}(c_n - c_m)\right) t^a_n\, t^a_m$$

Equations for spin correlators

From the decoupling eq.

$$C^{i,a}\left(z_1,\ldots z_N\right)\psi(z_1\cdots z_N)=0,\quad i=1,\ldots N,\quad a=1,2,3$$

One gets a linear system of equations for spin-spin correlators

$$w_{ij}\left\langle t_i^a\, t_j^a\right\rangle+\sum_{k(\neq i,j)} w_{ik}\left\langle t_j^a\, t_k^a\right\rangle+\frac{3}{4}w_{ij}=0,\quad i\neq j$$

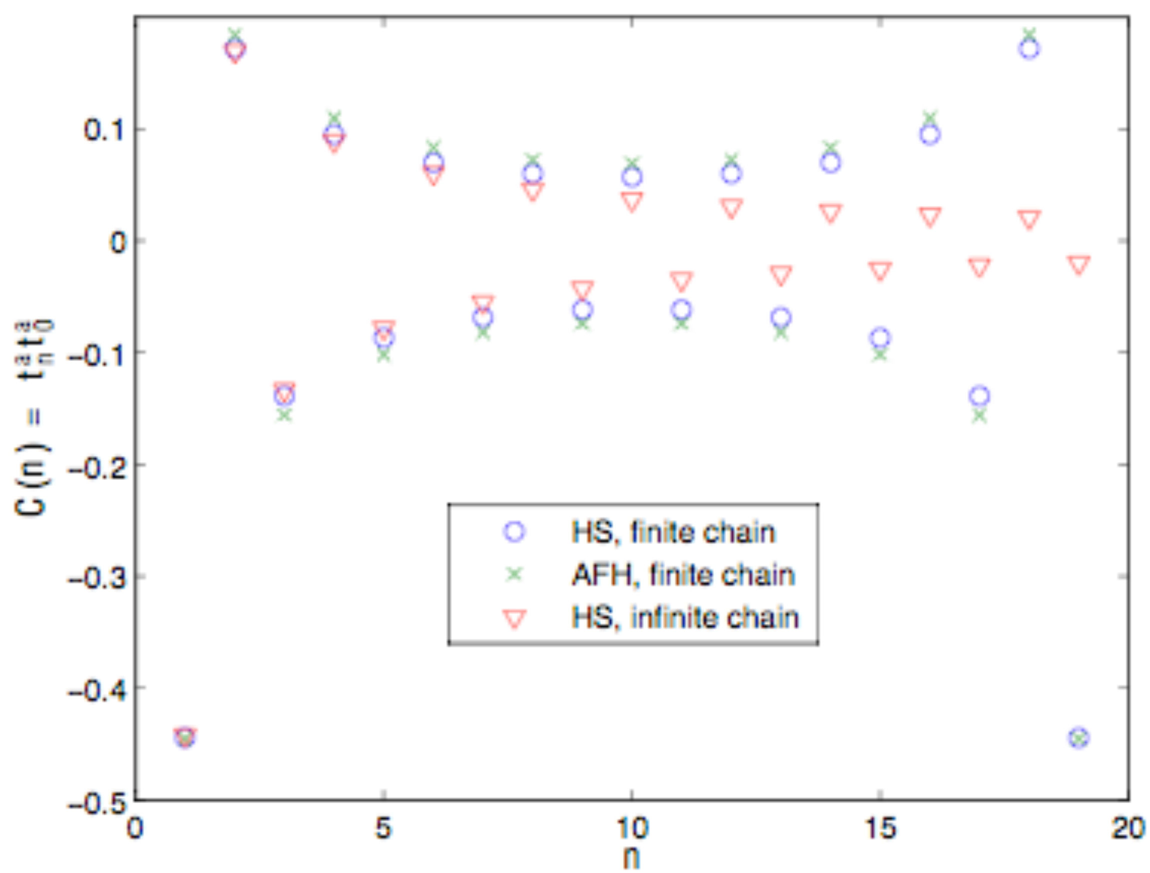
We recover the Gebhard-Vollhardt result for N-> infinity

$$\left\langle t_n^a\, t_0^b\right\rangle=(-1)^n\delta_{ab}\frac{Si(\pi n)}{4\pi n},\qquad Si(z)=\int_0^z dt\,\frac{\sin t}{t}$$

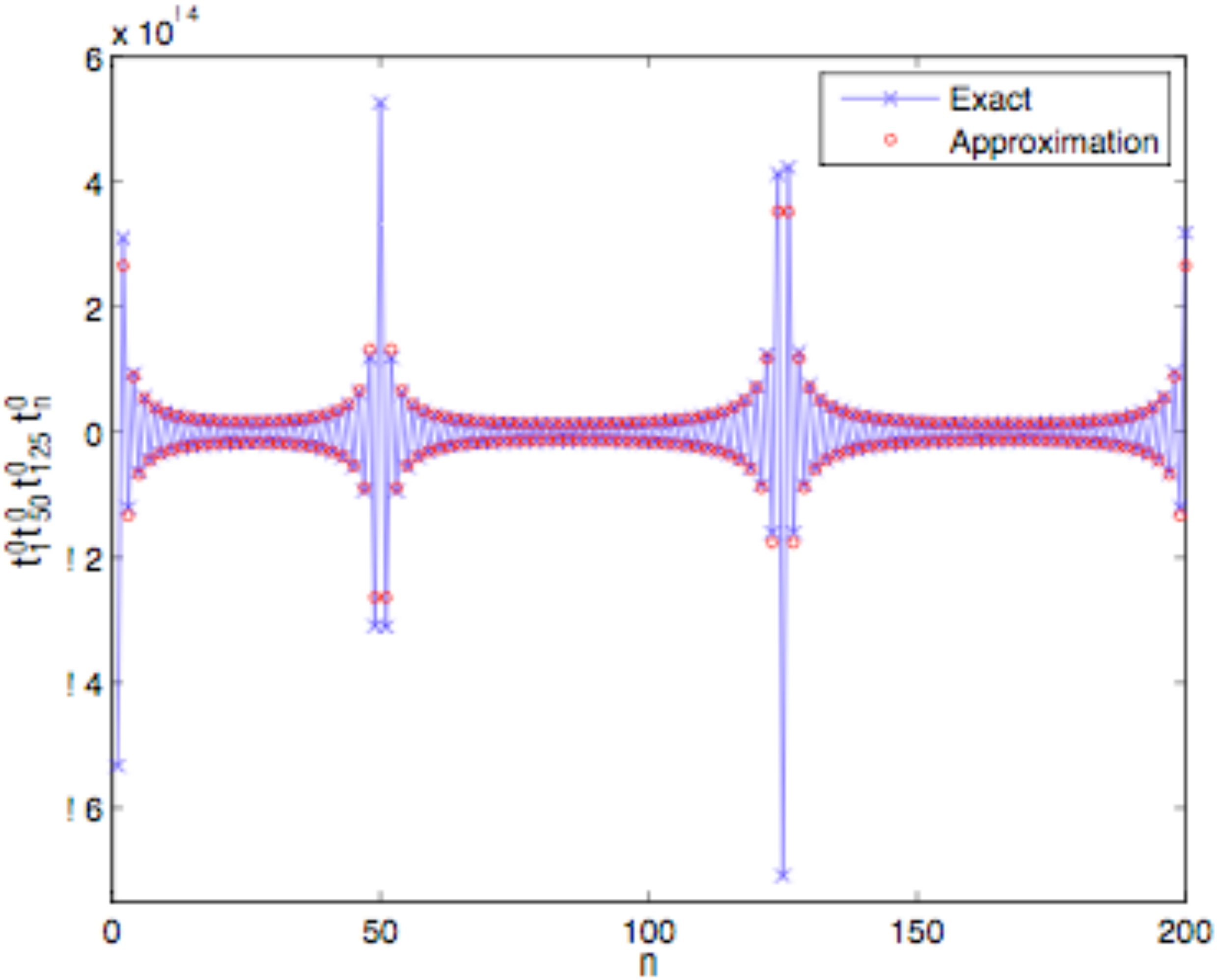
and a finite N expression

$$\left\langle t_n^a\, t_0^b\right\rangle=(-1)^n\delta_{ab}\frac{(-1)^n}{4N\sin(\pi n/N)}\sum_{m=1}^{N/2}\frac{\sin(2\pi n(m-1/2)/N)}{m-1/2}$$

Comparison of spin-spin correlators: AFH, HS(N=infty), HS(N)



Four point spin correlator $\left\langle t_1^0\, t_{50}^0\, t_{125}^0\, t_n^0 \right\rangle \quad n = 1,\cdots,200$



SU(2)@k=2

Primary fields: $\phi_0,\ \phi_{1/2},\ \phi_1$

Fusion rules $\phi_{1/2}\times\phi_{1/2}=\phi_0+\phi_1, \quad \phi_1\times\phi_1=\phi_0, \quad \phi_1\times\phi_{1/2}=\phi_{1/2}$

The spin 1 field is a simple current so there is only one conformal block involving an even number of fields

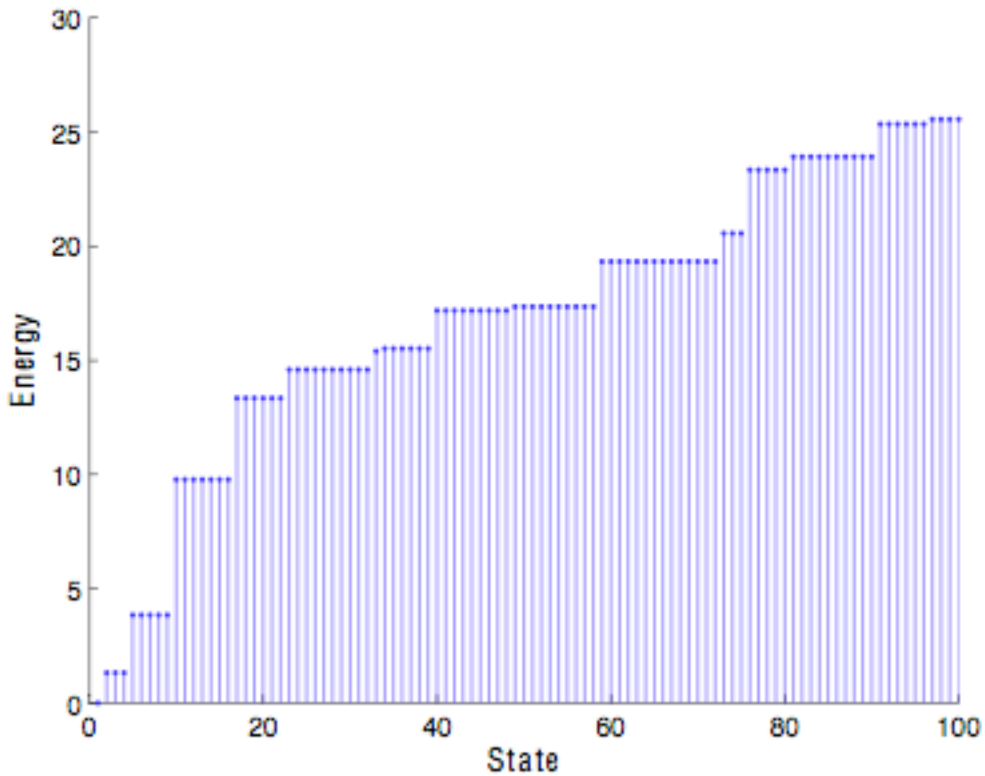
$$\psi_{s_1\ldots s_N}=\left\langle \phi_{s_1}(z_1)\cdots\phi_{s_N}(z_N)\right\rangle, \qquad s_i=0,\pm 1$$

This is the GS of the Hamiltonian

$$H=-\frac{4}{3}\sum_{i\neq j}w_{ij}^2-\frac{1}{3}\sum_{i\neq j}\left(w_{ij}^2+2\sum_{k(\neq i,j)}w_{ki}\,w_{kj}\right)t_i^a\,t_j^a+\frac{1}{6}\sum_{i\neq j}w_{ij}^2\left(t_i^a\,t_j^a\right)^2+\frac{1}{6}\sum_{i\neq j\neq k}w_{ij}\,w_{ik}t_i^a\,t_j^a\,t_i^b\,t_k^b$$

(See also a recent paper by Greiter for a s=1 Hamiltonian)

Spectrum in the uniform case



There are not accidental degeneracies-> No Yangian symmetry

SU(2)@k=2 = Boson + Ising (c= 3/2 = 1+ 1/2)

Primary spin 1 fields (h=1/2)

$$\phi_{\pm 1}(z_j) = e^{\pm i\varphi(z_j)}, \quad \phi_0(z_j) = (-1)^j \chi(z_j)$$

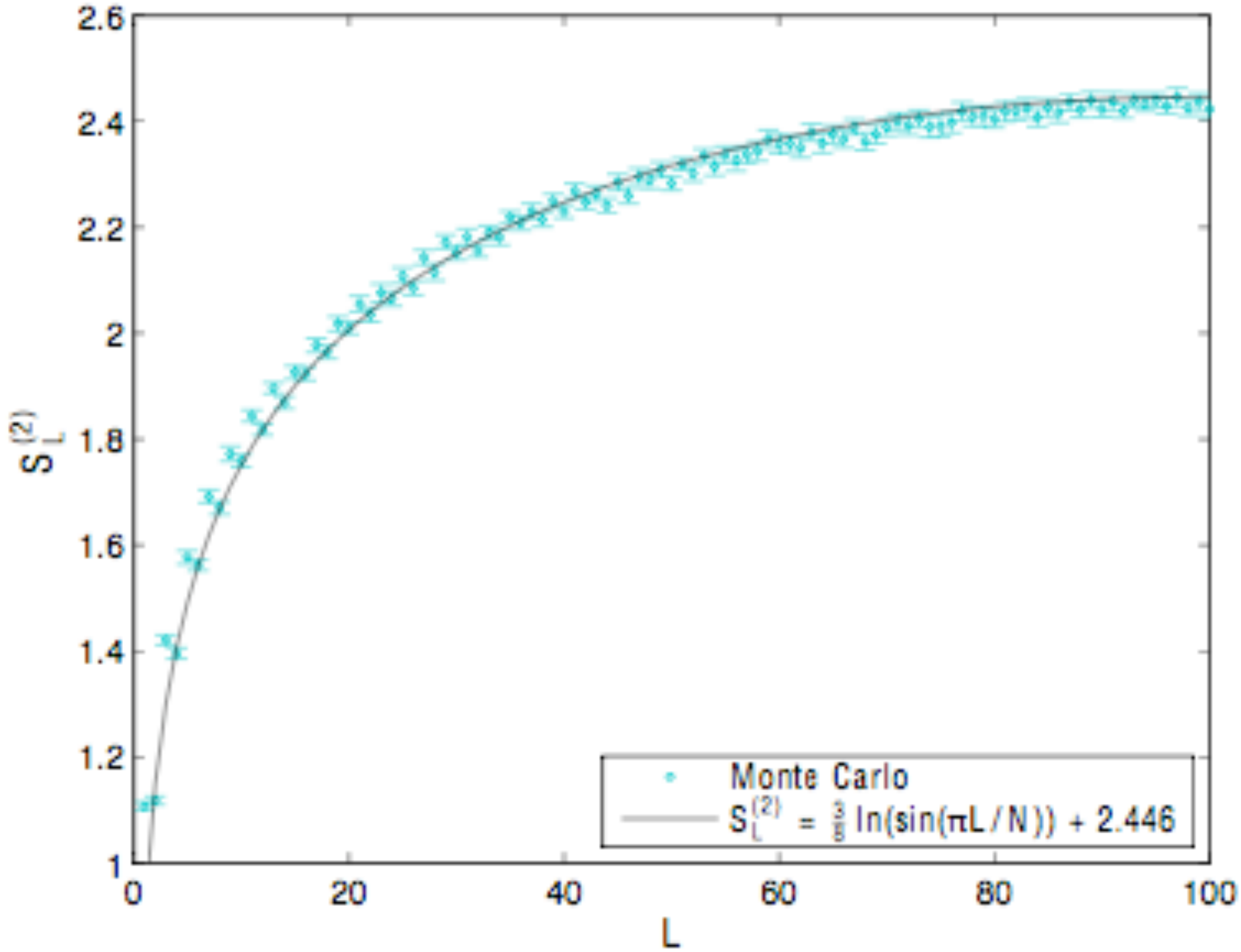
$\chi(z)$ Majorana fermion

$$\psi_{s_1 \cdots s_N} = (-1)^{\sum_{i:\,odd} s_i} \prod_{i<j} \left(z_i - z_j\right)^{s_i s_j} Pf_0 \frac{1}{z_i - z_j}, \quad \sum_i s_i = 0, \quad N : even$$

In the uniform case we expect the low energy spectrum of this model to be described by SU(2)@k=2 model

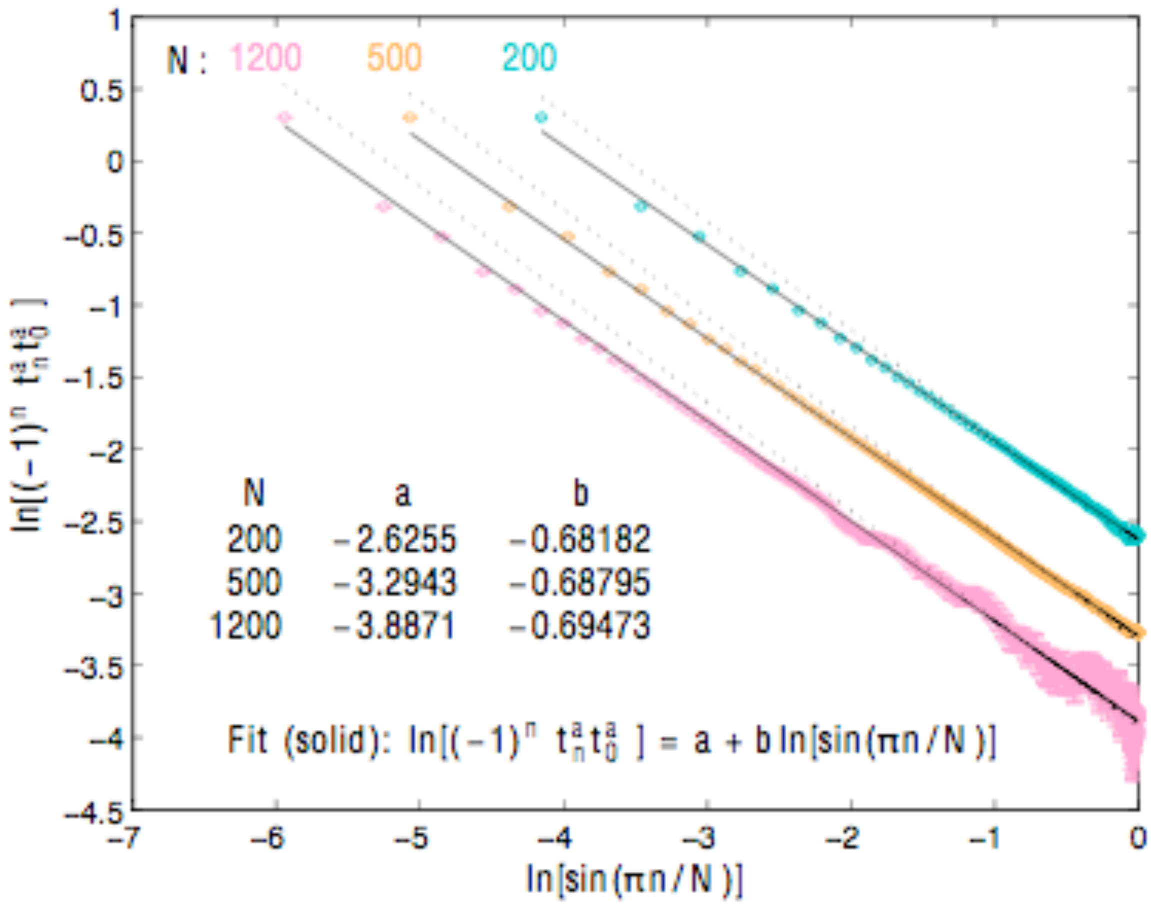
Look at-> Renyi entropy and spin-spin correlator

Renyi entropy $S_L = -\log Tr\, \rho_L^2$



Spin-spin correlator

CFT prediction $$\langle t_n^a t_0^a \rangle \approx (-1)^n \left(\sin \frac{\pi n}{N} \right)^b, \quad b = -\frac{3}{4}$$



Suggest existence of log corrections (Narajan and Shastry)

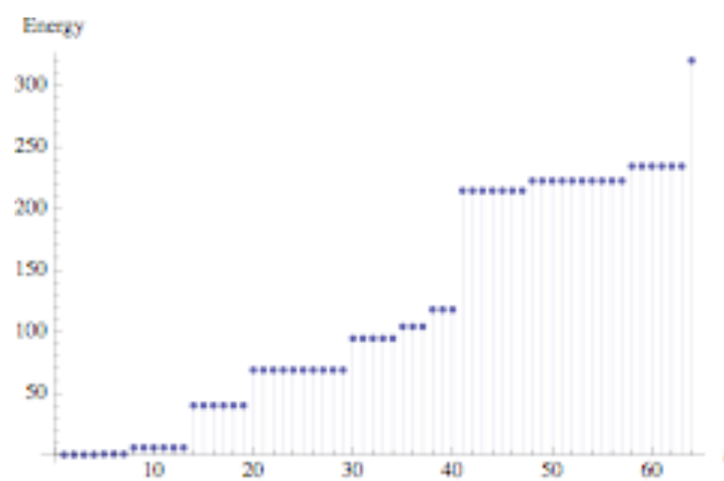
Take again SU(2)@k=2

Fusion rule of spin 1/2 field $\phi_{1/2} \times \phi_{1/2} = \phi_0 + \phi_1$

Number of chiral correlators of N spin 1/2 fields = $2^{N/2-1}$

Now the GS is NOT unique but degenerate !!

Example N = 6 -> 4 GS



The spin Hamiltonian contains 4 body terms

Mixing spin 1/2 and spin 1 for SU(2)@k=2

$$\left\langle \phi_{1/2} \ldots^{N_{1/2}} \ldots \phi_{1/2}\, \phi_1 \ldots^{N_1} \ldots \phi_1 \right\rangle_p \quad \text{->} \quad 2^{\frac{1}{2}N_{1/2}-1}$$

The degeneracy only depends on the number of spin 1/2 fields

SU(2)@2 = Boson + Ising
c =3/2 = 1 + 1/2

Spin 1 field $\quad \phi_{1,\pm 1}(z) = e^{\pm i\varphi(z)}, \quad \phi_{1,0}(z) = \chi(z)$

Spin 1/2 field $\quad \phi_{1/2,\pm 1/2}(z) = \sigma(z)\, e^{\pm i\varphi(z)/2}$

$\chi(z)$ is the Majorana field and $\sigma(z)$ is the spin field of the Ising model

Ising fusion rules $\quad \chi \times \chi = id, \quad \chi \times \sigma = \sigma, \quad \sigma \times \sigma = id + \chi$

Moore-Read wave function for FQHE @5/2 (1992)

CFT = boson (c=1) + Ising (c=1/2)

Electron operator $\psi_e(z) = \chi(z)\, e^{i\sqrt{2}\varphi(z)}$

Ground state wave function

$$\langle \psi_e(z_1)\ldots\psi_e(z_N)\rangle = \prod_{i<j}(z_i - z_j)^2 \langle \chi(z_1)\ldots\chi(z_N)\rangle$$

$$\langle \chi(z_1)\ldots\chi(z_N)\rangle = Pfaffian \frac{1}{z_i - z_j} = \sqrt{\det \frac{1}{z_i - z_j}}$$

Quasihole operator $\psi_{qh}(z) = \sigma(z)\, e^{\frac{i}{2\sqrt{2}}\varphi(z)}$

$$\langle \psi_{qh} \ldots^{N_{qh}} \ldots \psi_{qh}\, \psi_e \ldots^{N_e} \ldots \psi_e \rangle_p$$ -> Degeneracy $2^{\frac{1}{2}N_{qh}-1}$

Fusion rules of Ising

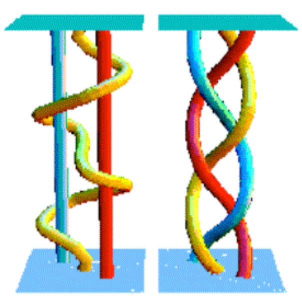
The quasiholes of the Moore-Read state: non abelian anyons

$$\left\langle \psi_{qh}(z_i)\ldots\psi_{qh}(z_j)\ldots\psi_e \ldots\psi_e \right\rangle_p = \sum_q B^{\pm}_{pq} \left\langle \psi_{qh}(z_j)\ldots\psi_{qh}(z_i)\ldots\psi_e \ldots\psi_e \right\rangle_q$$

The degenerate wave functions mix under the braiding operations

Braidring matrices: $B^{\pm}_{pq} : M \times M \;\; matrices, \quad M = 2^{N_{qh}/2-1}$

Basis for Topological Quantum Computation
(braids -> gates)



An analogy via CFT

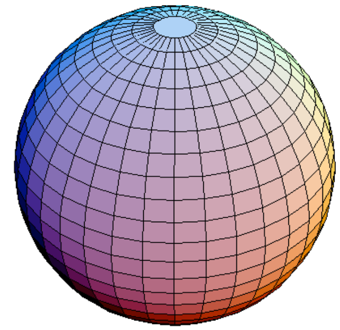
FQHE ←	CFT	→ Spin Models
Electron	χ field	spin 1
Quasihole	σ field	spin 1/2
Braiding of quasiholes	Monodromy of correlators	Adiabatic change of H

In the FQHE braiding is possible because electrons live effectively in 2 dimensions

To have “braiding” for the spin systems we need to generalize these models to 2D

SU(2)@k=1, spin 1/2, D=2

The wave function is defined in the sphere



$$\psi(s_1,,\ldots,s_N) = \prod_i \chi_{s_i} \prod_{i<j} \left(u_i v_j - u_j v_i\right)^{s_i s_j /4} = \prod_i \chi_{s_i} \prod_{i<j} \left(\rho_{i j}\right)^{-s_i s_j /4}$$

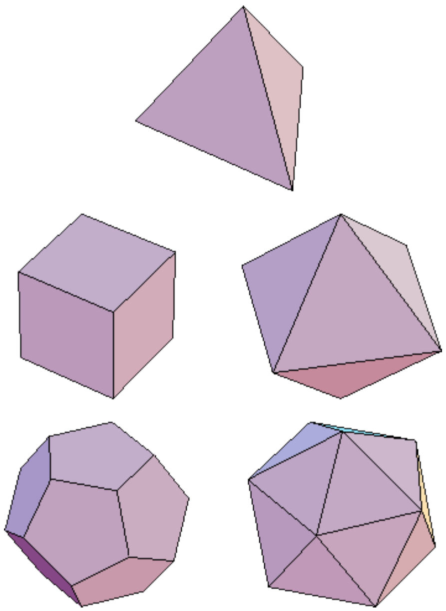
u and v are the spinor coordinates. This is the GS of the Hamiltonian

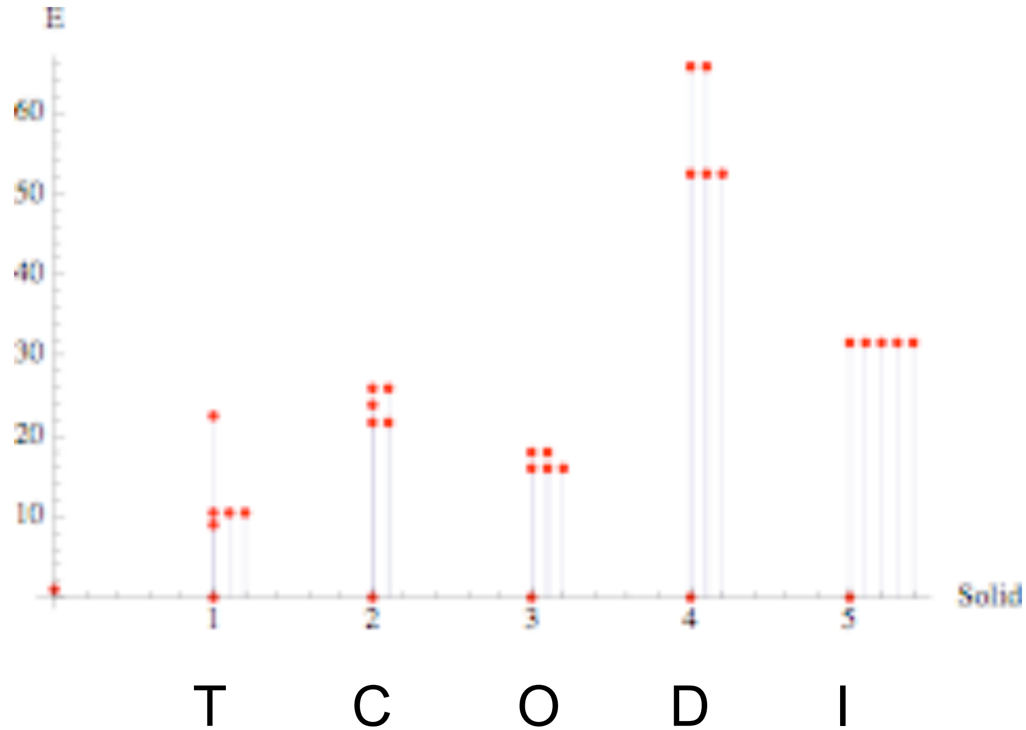
$$H = \frac{3}{4} \sum_{i_1 \neq i_2} \left|\rho_{i_1 i_2}\right|^2 + \sum_{i_1 \neq i_2} [\left|\rho_{i_1 i_2}\right|^2 + \sum_k \bar{\rho}_{k i_1} \rho_{k i_2} (\bar{u}_{i_1} u_{i_2} + \bar{v}_{i_1} v_{i_2})] t^a_{i_1} t^a_{i_2}$$

$$- i \sum_{i_i \neq i_2 \neq i_3} \sum_k \bar{\rho}_{i_1 i_2} \rho_{i_1 i_3} (\bar{u}_{i_2} u_{i_3} + \bar{v}_{i_2} v_{i_3})] \varepsilon^{a b c} t^a_{i_1} t^b_{i_2} t^c_{i_3}$$

2D generalization of the Haldane-Shastry model

Low energy spectrum on the Platonic Solids





The SU(2)@k=2 in 2D is the analogue of the Moore-Read state

In the FQHE the z's are the positions of the electrons or quasiholes

In the spin models the z's parametrize the couplings of the Hamiltonian. They are not real positions of the spins.

Braiding amounts to change these couplings is a certain way.

So in principle one can do topological quantum computation in these spin systems.

But one has first to show that Holonomy = Monodromy

This problem has been recently solved for the Moore-Read state (Bonderson, Gurarie, Nayak, 2010)

Conclusions

- Using CFT we extended the MPS to infinite dimensional matrices

- Description of critical and non critical systems

- Generalization of the Haldane-Shastry model in several directions
    1) non uniform
    2) higher spin
    3) degenerate ground states
    4) 1D -> 2D
    5) analogues of non abelian FQHE

Prospects

- Physics of the generalized HS Hamiltonians

- TCQ with HS models

Thanks