Quantum spin Hamiltonians and WZW models

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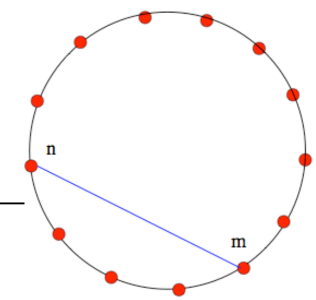
Haldane-Shastry model (1988)

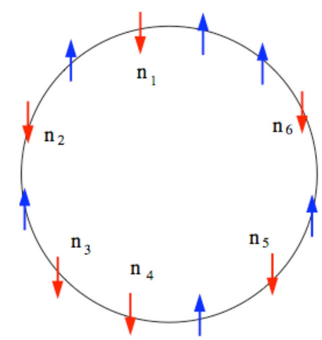
AF Heisenberg spin 1/2 chain with inverse square exchange couplings

$$H = \frac{J\pi^2}{N^2} \sum_{n<m} \frac{\vec{S}_n \cdot \vec{S}_m}{\sin^2(\pi(n-m)/N)}$$



Jastrow type ground state wave function

$$\psi(n_1,\ldots n_{N/2}) \propto e^{i\pi \sum_n n_i} \prod_{i<j} \left| \sin \frac{\pi(n_i - n_j)}{N} \right|^2$$



Obtained from the Gutzwiller projection of the half-filled Fermi state

$$|\psi_G\rangle = P_G |FS\rangle = \prod_{i=1}^{N} (1 - n_{i,\uparrow} n_{i,\downarrow}) \prod_{|k|<k_F} c^*_{k,\uparrow} c^*_{k,\downarrow} |0\rangle$$

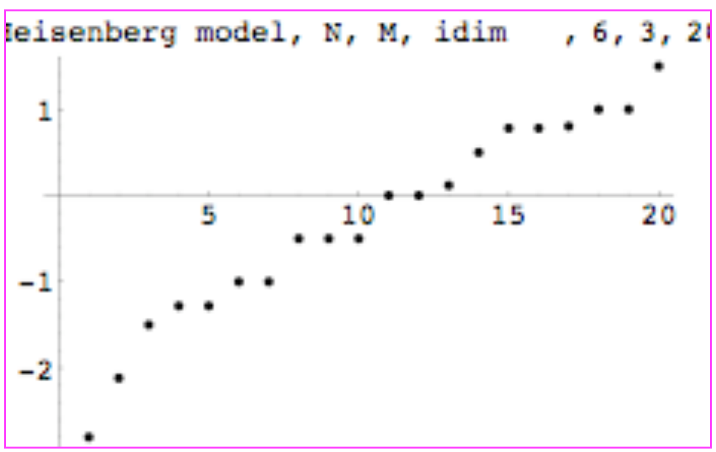
- Two-point correlator (Gebhard-Vollhardt 1987)

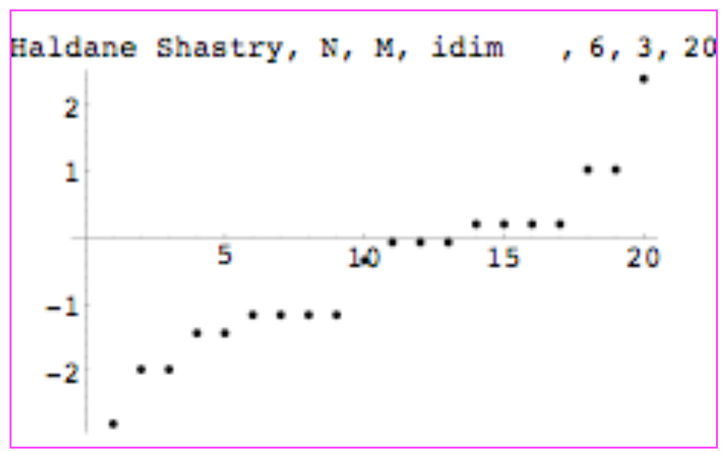
$$\left\langle S_n^a \, S_0^b \right\rangle = (-1)^n \, \delta_{a,b} \, \frac{Si(\pi \, n)}{4\pi \, n}, \qquad Si(x) = \int_0^x dy \, \frac{\sin y}{y}$$

- Elementary excitations: spinons (semions)

$$E(\{m_j\}) = \frac{2\pi^2}{N^2} \sum_{j=1}^{M} m_j (m_j - N), \qquad m_{j+1} \geq m_j + 2$$

- Highly degenerate spectrum-> Yangian symmetry





Alternative form of the HS Hamiltonian

$$z_n = e^{2\pi n i/N} \quad (n = 1,2,\ldots,N) \qquad z_{nm} = z_n - z_m$$

$$H_2 = -\frac{J\pi^2}{N^2}\sum_{i<j}\frac{z_i z_j}{z_{ij}^2}\vec{S}_i\cdot\vec{S}_j$$

Yangian symmetry is generated by the rapidity operator

$$\vec{\Lambda} = \sum_{i,j}\omega_{ij}\vec{S}_i\times\vec{S}_j, \quad \omega_{ij} = \frac{z_i + z_j}{z_i - z_j} \qquad [H_2,\vec{\Lambda}] = 0, \quad [\sum_i \vec{S}_i,\vec{\Lambda}] \neq 0$$

Integrable but not a a Bethe with high order conserved quantities (Inozemtsev 1990)

$$H_3 = -\sum_{ijk}\frac{z_i z_j z_k}{z_{ij} z_{ik} z_{jk}}\vec{S}_i\cdot(\vec{S}_j\times\vec{S}_k) \qquad [H_2,H_3] = 0,$$

Low energy spectrum

Haldane-Shastry model AF Heisenberg model	→	Universality class WZW model SU(2)@ level k=1

Unlike the Heisenberg model the marginal irrelevant operator is absent

$$HS : \langle S_n^a \, S_0^b \rangle \propto (-1)^n \, \delta_{a,b} \, \frac{1}{n}, \quad Heis : \langle S_n^a \, S_0^b \rangle \propto (-1)^n \, \delta_{a,b} \, \frac{\sqrt{\log n}}{n}$$

HS model is close to the critical J1-J2 model $\quad \frac{J_2}{J_1} = \frac{1}{4} \approx 0.241$

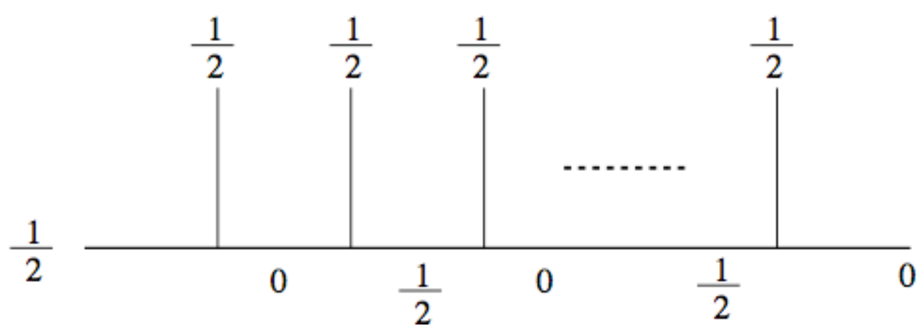
$$H_2 = -\frac{J\pi^2}{N^2} \sum_i (\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{4} \vec{S}_i \cdot \vec{S}_{i+2} + \cdots)$$

The HS wave function as a Conformal block

The $SU(2)_k$ WZW model at level k= 1 has a primary field $\phi_{1/2}$ with spin 1/2 and conformal weight h = 1/4. Using a chiral boson field $\varphi(z)$ it is given by the chiral vertex operator

$$\phi_s(s) = \chi_s :e^{is\,\varphi(z)/2}, \qquad s = \pm 1,\ \chi_s = \pm 1$$

The fusion rules yields only one conformal block



$$\psi\left(z_1, s_1, ,\ldots, z_N, s_N\right) = \left\langle \phi_{s_1}(z_1)\,\phi_{s_2}(z_2)\cdots\phi_{s_N}(z_N)\right\rangle$$
$$= \prod_i \chi_{s_i} \prod_{i<j}\left(z_i - z_j\right)^{s_i s_j/4}$$

Taking

$$z_n = e^{2\pi i n/N}, \quad \chi_{s_n} = 1\ (n : even), \quad e^{i\pi(s_n - 1)/2} \quad (n : odd)$$

one proves

$$\psi(z_1, s_1, \ldots, z_N, s_N) = \prod_i \chi_{s_i} \prod_{i<j} (z_i - z_j)^{s_i s_j/4}$$
$$\propto e^{i\pi \sum_i n_i} \prod_{i<j} \left| \sin \frac{\pi(n_i - n_j)}{N} \right|^2$$

$e^{i\pi \sum_i n_i}$ is the Marshall sign factor which arises in the Heisenberg model from the Perron-Frobenius theorem. In the WZW model it comes from the eq.

$$\left\langle g(z_1, \bar{z}_1)\, g^{-1}(z_2, \bar{z}_2) \ldots g(z_{N-1}, \bar{z}_{N-1})\, g^{-1}(z_N, \bar{z}_N) \right\rangle$$

An inhomogenous generalization of the HS Hamiltonian

The conformal block satisfies the Knizhnik-Zamolodchikov eq

$$\frac{k+2}{2}\frac{\partial}{\partial z_i}\psi(z_1,\ldots,z_N) = \sum_{j\neq i}^{N}\frac{\vec{S}_i\cdot\vec{S}_j}{z_i - z_j}\psi(z_1,\ldots,z_N), \quad (k=1)$$

Making a conformal transformation from the plane to the cylinder

$$\psi_{cyl}(z_1,\ldots,z_N) = \prod_{i=1}^{N} z_i^{1/4}\,\psi(z_1,\ldots,z_N)$$

The KZ equation becomes

$$\frac{k+2}{2}\, z_i\frac{\partial}{\partial z_i}\psi_{cyl} = \sum_{j\neq i}^{N}\frac{z_i+z_j}{z_i-z_j}\vec{S}_i\cdot\vec{S}_j\;\psi_{cyl}(z_1,\ldots,z_N)$$

From the expression of $\psi_{cyl}(z_1,\ldots,z_N)$ one gets the “abelian” KZ eq.

$$4\, z_i\frac{\partial}{\partial z_i}\psi_{cyl} = \sum_{j\neq i}^{N}\frac{z_i+z_j}{z_i-z_j}s_i\, s_j\;\psi_{cyl}(z_1,\ldots,z_N)$$

Computing $\sum_n z_n^2 \partial^2 / \partial z_n^2$ in two different ways one gets

$$H \psi_{cyl} = E \psi_{cyl}$$

$$H = -\sum_{n \neq m} \left(\frac{z_n z_m}{(z_n - z_m)^2} + \frac{1}{12} w_{n,m} (c_n - c_m) \right) \vec{S}_n \cdot \vec{S}_m$$

$$w_{n,m} = \frac{z_n + z_m}{z_n - z_m}, \quad c_n = \sum_{n \neq m} w_{n,m}, \quad E = \frac{1}{16} \sum_{n \neq m} w_{n,m}^2 - \frac{N(N+1)}{16}$$

In the uniform case $z_n = e^{2\pi i n / N} \rightarrow c_n = 0 \quad \forall n$

and we recover the usual HS Hamiltonian. For other values of z_n we obtain an inhomogenous version of it.

The Inozemtsev operator is also generalized

$$H_3 = -i \sum_{n \neq m \neq l} \frac{z_n z_m z_l}{z_{n,m}^2 z_{m,l}^2 z_{l,n}^2} \vec{S}_n \cdot (\vec{S}_m \times \vec{S}_l) + \sum_{n \neq m} \left(-\frac{1}{12} c_n + \frac{17}{8} c_n^{(2)} w_{n,m} \right) \vec{S}_n \cdot \vec{S}_m$$

$J_1 - J_2$ Model (zig-zag chain)

$$H = \sum_{1=1}^{N} J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_i \cdot \vec{S}_{i+2} \qquad (J_1 = 1)$$

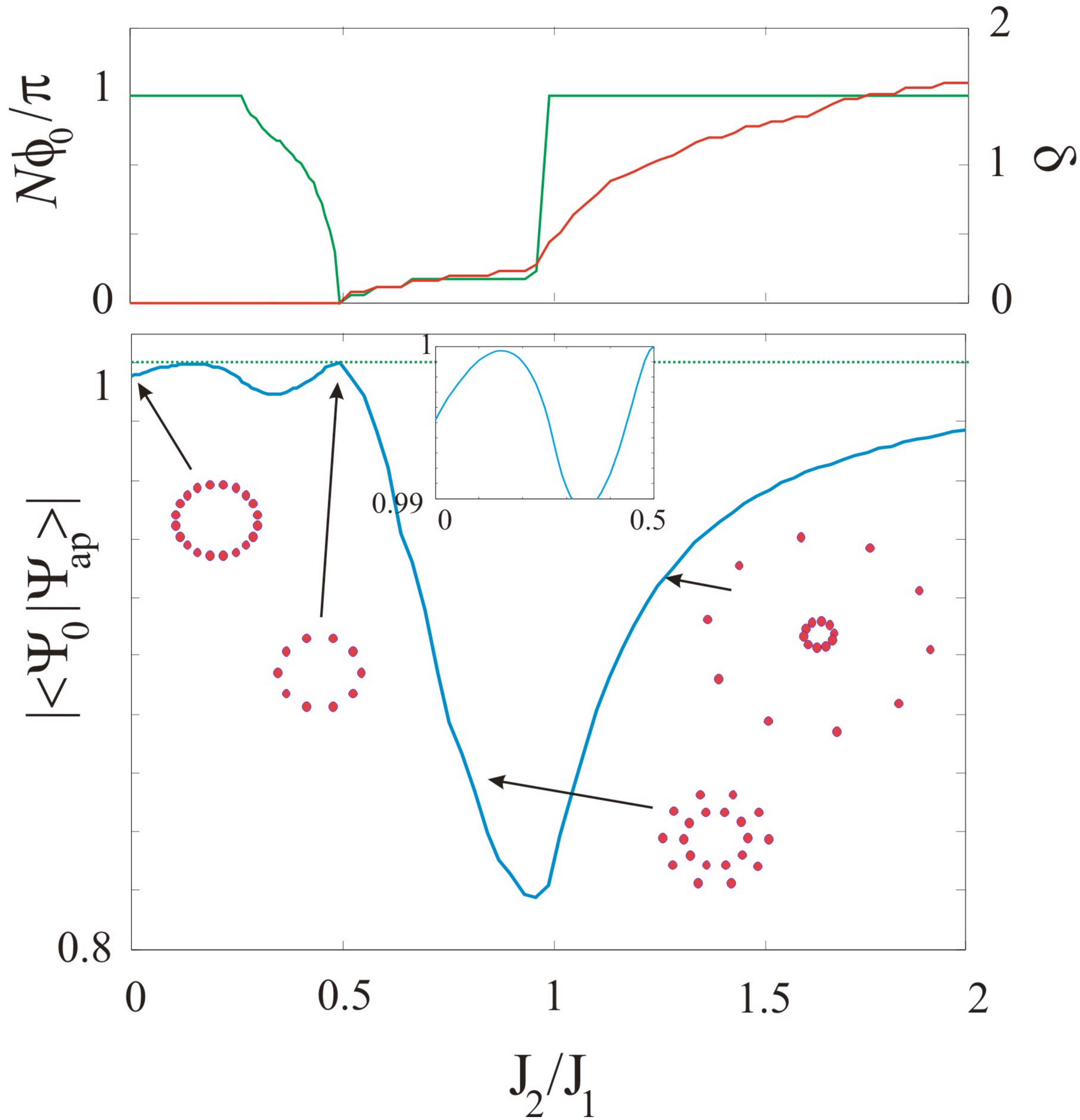
$J_2 > 0$ frustrated spin system

Phases

$0 \le J_2 < J_{2c} \approx 0.241$	Critical c=1
$J_{2c} < J_2 < J_{MG} = 0.5$	Spontaneously dimerized
$J = J_{MG}$	Majumdar-Gosh point
$J_{MG} < J < \infty$	Dimer spiral phase

Use z' s as variational parameters

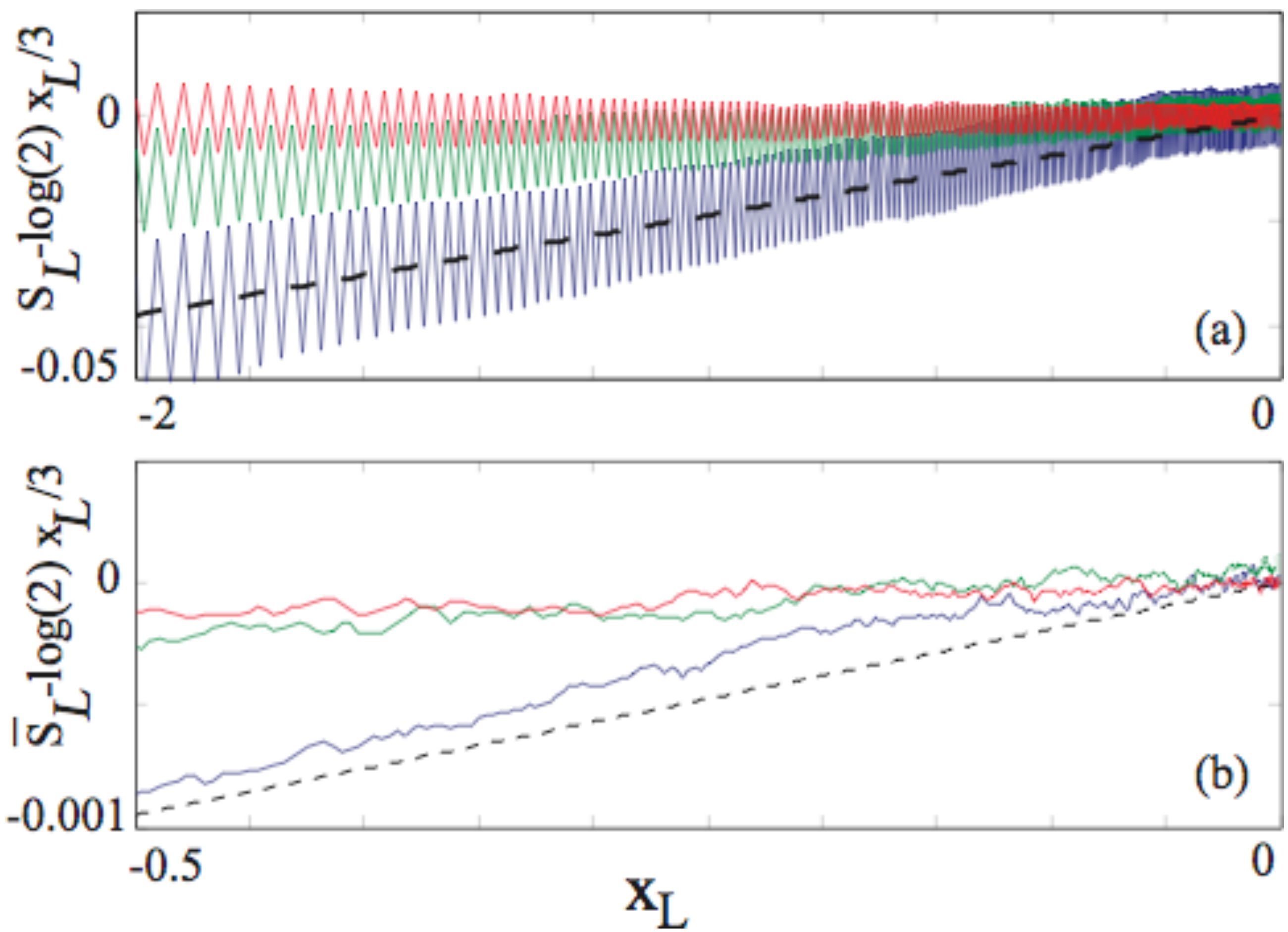
$$z_n = \begin{cases} \exp(\delta - i\phi_0)\exp(2\pi i(0,2,4,\ldots)/N) & \textit{even sites} \\ \exp(-\delta + i\phi_0)\exp(2\pi i(0,2,4,\ldots)/N) & \textit{odd sites} \end{cases}$$



$N\phi_0/\pi$
ϱ
$|\langle\Psi_0|\Psi_{ap}\rangle|$
J_2/J_1
0
0.5
1
1.5
2
0.8
0.99

Random AFH model

Renyi entropy $S_L \propto \frac{\log 2}{3} \log L$ (Refael-Moore)



Can we generalize the previous construction to SU(2)@k for k > 1 and other primary fields?

Review: Null vectors of the SU(2)@k WZW model

Kac-Moody algebra

$$[J_n^a, J_m^b] = i\,\varepsilon^{abc}\,J_{n+m}^c + \frac{k}{2}\,n\,\delta^{ab}\,\delta_{n+m}$$

Null vector at level k

$$\chi_{k+1} = \left(J_{-1}^{+}\right)^{k+1}\phi_0, \qquad J_n^a\,\chi_{k+1} = 0, \quad \forall n > 0$$

Decoupling of this null vector yields the primary fields (Gepner-Witten)

$$\left\langle \chi_{k+1}(z)\,\phi_1(z_1)\cdots\phi_n(z_n)\right\rangle = 0 \rightarrow j = 0, \frac{1}{2}, \cdots, \frac{k}{2}$$

and the fusion rules of the model

$$\phi_{j_1} \otimes \phi_{j_2} = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2,\,k-j_1-j_2)} \phi_j$$

The null vector $\chi_{k+1} = \left(J_{-1}^{+}\right)^{k+1} \phi_0$ is the highest weight vector of a multiplet with total spin m = k+1.

$$\chi_m^{a_1 \ldots a_m} = C^{(m)}_{a_1 \ldots a_m\, b_1 \ldots b_m} J_{-1}^{b_1} \cdots J_{-1}^{b_m} \phi_0$$

$\chi_m^{a_1 \ldots a_m}$ is totally symmetric and traceless in the a's indices

$$\chi_{k+1} = \chi_m^{++\ldots+}$$

The tensors $C^{(m)}_{a_1 \ldots a_m b_1 \ldots b_m}$ are proyectors in this space

$$C^{(2)}_{a_1 a_2 b_1 b_2} = \frac{1}{2}\left(\delta_{a_1 b_1}\delta_{a_2 b_2} + \delta_{a_1 b_2}\delta_{a_2 b_1}\right) - \frac{1}{3}\delta_{a_1 a_2}\delta_{b_1 b_2}$$

$$C^{(3)}_{a_1 a_2 a_3 b_1 b_2 b_3} = \frac{1}{6}\left(\delta_{a_1 b_1}\delta_{a_2 b_2}\delta_{a_3 b_3} + permutations\right) - \frac{1}{15}\left(\delta_{a_1 a_2}\delta_{b_1 b_2}\delta_{a_3 b_3} + \cdots\right)$$

Decoupling of null vectors in correlators of primary fields

$$\left\langle \chi_m^{a_1\dots a_m}(z)\,\phi_1(z_1)\cdots\phi_n(z_n)\right\rangle = 0$$

Using the Ward identity

$$\left\langle (J_{-1}^a\chi)(z)\,\phi_1(z_1)\cdots\phi_n(z_n)\right\rangle = \sum_{j=1}^{n}\frac{t_j^a}{z-z_j}\left\langle \chi(z)\,\phi_1(z_1)\cdots\phi_n(z_n)\right\rangle$$

and the SU(2) invariance of the conformal block

$$\psi(z_1\cdots z_n) = \left\langle\, \phi_1(z_1)\cdots\phi_n(z_n)\right\rangle$$

one finds $R^{(m)}_{a_1\dots a_m}\left(z,z_1,\dots z_n\right)\psi(z_1\cdots z_n) = 0, \quad \forall\, z$

$$R^{(m)}_{a_1\dots a_m}\left(z,z_1,\dots z_n\right) = \sum_{j_1\dots j_m}^{n} C^{(m)}_{a_1\dots a_m b_1\dots b_m}\,\frac{(z+z_{j_1})\dots(z+z_{j_m})}{(z-z_{j_1})\dots(z-z_{j_m})}\,t_{j_1}^{b_1}\dots t_{j_m}^{b_m}$$

Taking residues $\left(w_{ij} = \frac{z_i + z_j}{z_i - z_j} \right)$

$$R^{(m,i)}_{a_1 \ldots a_m} \equiv \oint_{z_i} \frac{dz}{z} R^{(m)}_{a_1 \ldots a_m}\left(z, z_1, \ldots z_n\right) = \sum_{j_2 \ldots j_m \neq i}^{n} C^{(m)}_{a_1 \ldots a_m b_1 \ldots b_m} w_{i\, j_2} \ldots w_{i\, j_m} t_i^{b_1} \ldots t_{j_m}^{b_m}$$

Define the operators

$$H^{(m,i)} \equiv \sum_{a_1 \ldots a_m} \left(R^{(m,i)}_{a_1 \ldots a_m} \right)^* R^{(m,i)}_{a_1 \ldots a_m}$$

Properties:

1) $\left(H^{(m,i)} \right)^* = H^{(m,i)}$

2) $H^{(m,i)} \geq 0$

3) $\left[H^{(m,i)}, \sum_i t_i^a \right] = 0$

4) $H^{(m,i)} \psi = 0$

where $\psi(z_1 \cdots z_n) = \left\langle \phi_1(z_1) \cdots \phi_n(z_n) \right\rangle$

$H^{(m,i)}$

Set of Hamiltonians whose ground state are the conformal blocks of the WZW model.

The degeneracy of the GS manifold is given by the fusion rules of SU(2)@k

Example 1: k=1, j=1/2

$$H^{(2,i)} = -\sum_{j_2 \ldots j_m \neq i}^{n} C^{(2)}_{a_1 a_2 b_1 b_2} w_{i\,j} w_{i\,k}\; t_i^{a_1} t_j^{a_2} t_i^{b_1} t_k^{b_2}$$

$$= -\left[\frac{3}{4} \sum_{j \neq i} w_{ij}^2 + \sum_{j \neq k(\neq i)} w_{ij}\, w_{ik}\, t_j^a\, t_k^a + \sum_{j \neq i} w_{ij}^2\, t_i^a\, t_j^a \right]$$

t_i^a spin 1/2 matrices

The inhomogenous Haldane-Shastry Hamiltonian is recovered as

$$H = \sum_i H^{(2,i)}$$

A further generalization is

$$H = \sum_i g_i\, H^{(2,i)}, \quad g_i \geq 0, \quad \forall i$$

Example 2: k=2, j=1

Fusion rule $\phi_1 \times \phi_1 = \phi_0$ yields only one conformal block

The Hamiltonian contains 3 body terms (t_i^a spin 1 matrices)

$$H^{(3)} = -4 \sum_{i_1 \neq i_2} w_{i_1 i_2}^2 - \sum_{i_1 \neq i_2} (w_{i_1 i_2}^2 + 2 \sum_{k \neq i_1 i_2} w_{k i_1} w_{k i_2}) t_{i_1}^a t_{i_2}^a$$

$$+ \frac{1}{2} \sum_{i_1 \neq i_2} w_{i_1 i_2}^2 \left(t_{i_1}^a t_{i_2}^a \right)^2 + \frac{1}{2} \sum_{i_1 \neq i_2 \neq i_3} w_{i_1 i_2} w_{i_1 i_3} t_{i_1}^a t_{i_2}^a t_{i_1}^b t_{i_3}^b$$

Since c = 3/2 = 3 x Ising model

Spin 1 field -> triplet of Majorana fermions

$$\psi_{a_1 a_2 \ldots a_n} = \left\langle \chi_{a_1}(z_1) \cdots \chi_{a_n}(z_n) \right\rangle$$

Expect this system to have a spin gap

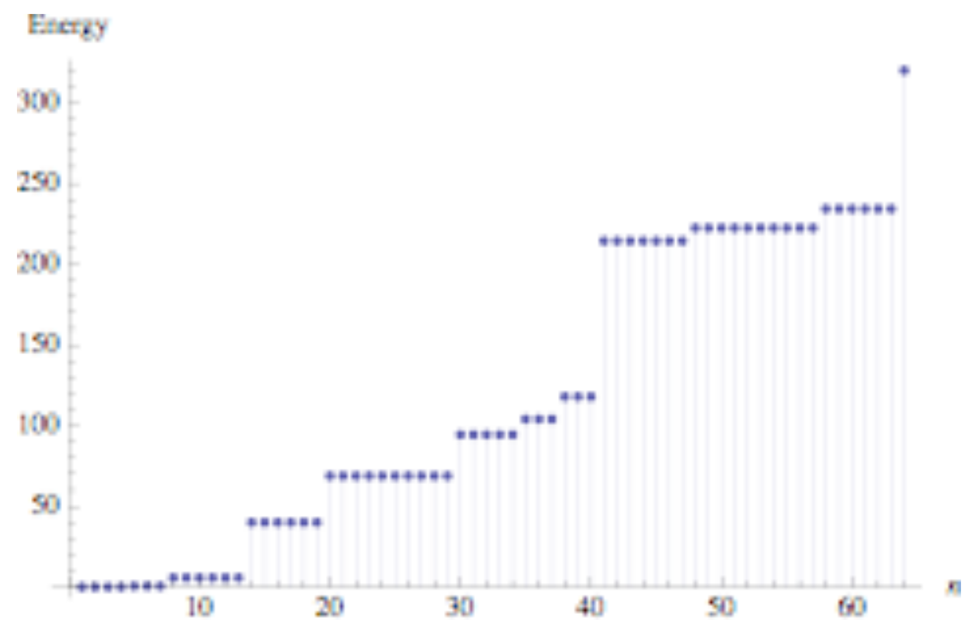
Example 3: k=2, j=1/2

Fusion rules $\phi_{1/2} \times \phi_{1/2} = \phi_0 + \phi_1$ dimension GS manifold is $2^{N/2-1}$

The Hamiltonian contains 4 body terms

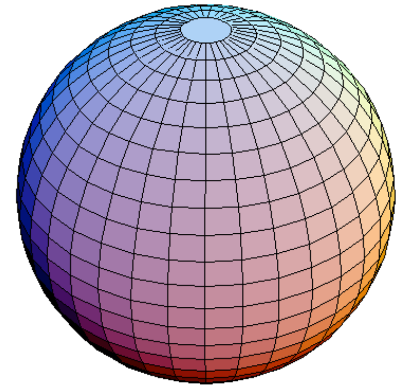
$$H = J_0 + \sum_{i_1 i_2} J_{i_1 i_2} \, t^a_{i_1} \, t^a_{i_2} + \sum_{i_1 i_2 i_3 i_4} J_{i_1 i_2 i_3 i_4} \, t^a_{i_1} \, t^a_{i_2} t^b_{i_3} \, t^b_{i_4}$$

N=6
dim GS = 4



Example 4: k=1, j=1/2, D=2

$$\psi(z_1,s_1,,\ldots,z_N,s_N) = \prod_i \chi_{s_i} \prod_{i<j} (z_i - z_j)^{s_i s_j/4}$$



Map z into the spinor coordinates of the sphere

$$z = \frac{v}{u}, \quad u = \cos\frac{\theta}{2} e^{i\phi/2}, \quad v = \sin\frac{\theta}{2} e^{-i\phi/2}$$

$$\psi(z_1,s_1,,\ldots,z_N,s_N) = \prod_i \chi_{s_i} \prod_{i<j} (u_i v_j - u_j v_i)^{s_i s_j/4} = \prod_i \chi_{s_i} \prod_{i<j} (\rho_{ij})^{-s_i s_j/4}$$

GS of the Hamiltonian

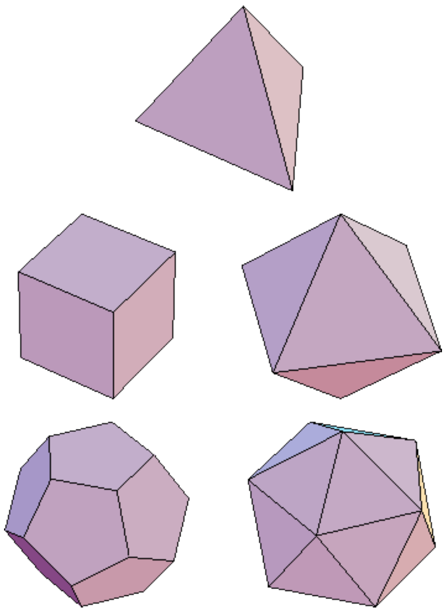
$$H = \frac{3}{4}\sum_{i_1 \neq i_2} |\rho_{i_1 i_2}|^2 + \sum_{i_1 \neq i_2} [|\rho_{i_1 i_2}|^2 + \sum_k \bar{\rho}_{k i_1} \rho_{k i_2} (\bar{u}_{i_1} u_{i_2} + \bar{v}_{i_1} v_{i_2})] t^a_{i_1} t^a_{i_2}$$

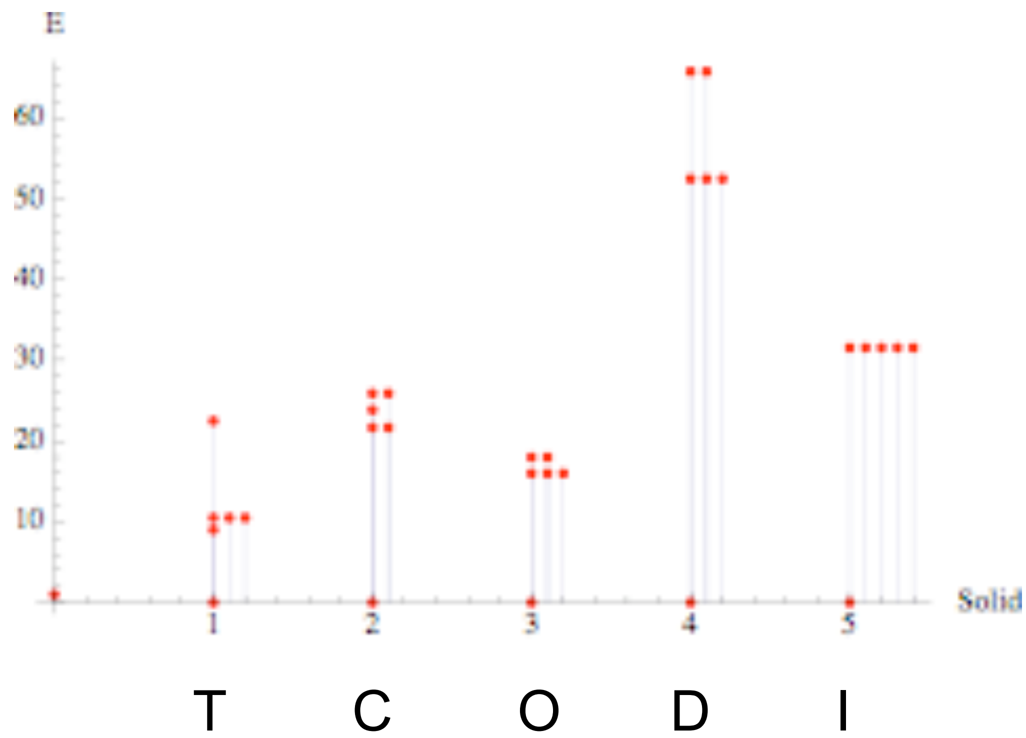
$$- i \sum_{i_i \neq i_2 \neq i_3} \sum_k \bar{\rho}_{i_1 i_2} \rho_{i_1 i_3} (\bar{u}_{i_2} u_{i_3} + \bar{v}_{i_2} v_{i_3})] \varepsilon^{abc} t^a_{i_1} t^b_{i_2} t^c_{i_3}$$

H is invariant under the SU(2) rotations

$$\begin{pmatrix} u \\ v \end{pmatrix} \to U \begin{pmatrix} u \\ v \end{pmatrix}$$

Low energy spectrum on the Platonic Solids





Example 5: k=2, j=1/2 and 1, D=2

SU(2)@2 = Boson + Ising
(3/2 = 1 + 1/2)

spin j=1 field $\phi_{1,\pm1}(z) = e^{\pm i\varphi(z)}, \quad \phi_{1,0}(z) = \chi(z), \quad h_1 = h_\chi = \frac{1}{2}$

spin j=1/2 field $\phi_{1/2,\pm1/2}(z) = \sigma(z)\, e^{\pm i\varphi(z)/2}, \quad h_\sigma = \frac{1}{16}, \quad h_{1/2} = \frac{3}{16}$

The conformal blocks of this WZW model can be obtained from those of the Ising model

N spins 1

$$\psi(s_1,,\ldots,s_N) = \chi_s \prod_{i<j} (z_i - z_j)^{s_i s_j} \, Pf_0\left(\frac{1}{z_i - z_j}\right)$$

The Pfaffian comes from the correlator of Majorana fields

2m Majorana + 2 σ fields (Moore-Read)

$$\left\langle \sigma(v_1)\sigma(v_2)\prod_{i=1}^{2m}\chi(z_i)\right\rangle = 2^{-m} v_{12}{}^{-1/8}\prod_{i=1}^{2m}\left((z_i - v_1)(z_i - v_2)\right)^{-1/2} Pf\,\frac{(z_i - v_1)(z_j - v_2) + (z_i - v_2)(z_j - v_1)}{z_i - z_j}$$

2m Majorana + 4 σ fields (Nayak-Wilczek)

$$\left\langle \sigma(v_1)\cdots\sigma(v_4)\prod_{i=1}^{2m}\chi(z_i)\right\rangle = C_m \prod_{a<b} v_{ab}{}^{-1/8}\left(\sqrt{v_{13}v_{24}} \pm \sqrt{v_{14}v_{23}}\right)^{-1/2}$$

$$\times \left[\sqrt{v_{13}v_{24}}\,Pf\,\frac{h_{(13)(24)}(z_i,z_j)}{z_i - z_j} \pm \sqrt{v_{14}v_{23}}\,Pf\,\frac{h_{(14)(23)}(z_i,z_j)}{z_i - z_j}\right]$$

$$h_{(ab)(cd)}(z_i,z_j) = \left[\frac{(z_i - v_a)(z_i - v_b)(z_j - v_c)(z_j - v_d)}{(z_i - v_c)(z_i - v_d)(z_j - v_a)(z_j - v_b)}\right]^{1/2} + (i \leftrightarrow j)$$

Based on these expressions we have found a general formula for arbitrary number of Majorana and σ fields

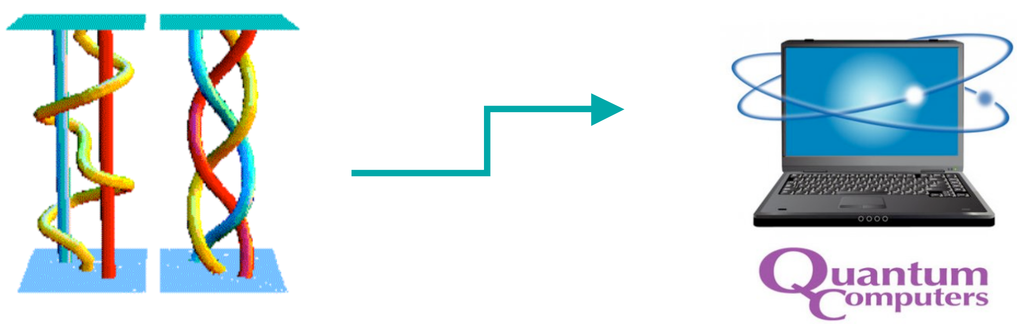
The later conformal blocks have appeared in connection with the Fractional Quantum Hall effect at filling fraction 5/2. This is the so called Pfaffian state due to Moore and Read.

FQHE/CFT correspondence

electron = $\chi(z)\, e^{i\sqrt{2}\varphi(z)}$ quasihole -> $\sigma(z)\, e^{\frac{i}{2\sqrt{2}}\varphi(z)}$

Quasiholes are non abelian anyons because their wave Functions (conformal blocks) mix under braiding of their positions.

Basis for Topological Quantum Computation
(braids -> gates)



FQHE → Spin Hamiltonians

electron → spin 1
quasihole → spin 1/2

slow braid of quasiholes → adiabatic change of H

Under appropiate conditions this will mix the GS wave functions in terms of the braiding matrices.

Holonomy = Monodromy (Wilczek,Wen, Read,..)

This may provide a "spin" scenario of TQC analogous to the FQHE

Conclusions

-Using CFT we extended the MPS to infinite dimensional matrices

-Description of critical and non critical systems

-Generalization of the Haldane-Shastry model in several directions

1) inhomogenous
2) higher spin
3) degenerate ground states
4) 2D: sphere

-Constructed the conformal blocks of the Ising model and SU(2)@2

Prospects

-Physics of the higher spin and degenerate Hamiltonians

-Show if holonomy = monodromy

-Infinite dimensional version of PEPS