Infinite Matrix Product States, Conformal Field Theory and the Haldane-Shastry model

Germán Sierra
Instituto de Física Teórica CSIC-UAM, Madrid, Spain

In collaboration with Ignacio Cirac (MPQ,Garching)
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Overview

In recent years low-D quantum lattice systems have been actively investigated using a variety of numerical methods:

- Density Matrix Renormalization Group (DMRG)
- Matrix Product States (MPS)
- Projected Entangled Pair States (PEPS=2D version of MPS)
- Multiscale Entanglement Renormalization Ansatz (MERA)

The performance of the DMRG can now be explained using quantum information tools, e.g. if the entanglement entropy of the subsystems is finite and small, then the DMRG performs well, but for critical systems, where the entropy increases logarithmically, one has to use finite size methods.

The MPS are the variational ansatzs underlying the DMRG method. They can be obtained by means of the Schmidt decomposition of the wave function into two parts A U B. The limitations of the DMRG are explained by the fact that the MPS matrices are finite dimensional.

To overcome this limitation one can try to generalize the MPS to infinite dimensional matrices, so that long range entanglement is not a problem.

We shall show that this be done using CFT. The wave functions so obtained will resemble those used in the Fractional Quantum Hall Effect.

Moreover, in some particular cases these infinite MPS allow us to make contact with the Haldane-Shastry model.

The Haldane-Shastry model (1988)

This model describes a spin 1/2 Heisenberg chain whose ground state is a quantum spin liquid. The exchange interactions are long range decaying as $1/r^2$. This model shares many properties with the nearest-neighbour AF Heisenberg model:

- exotic elementary excitations: spinons
- spin-spin correlation functions decay algebraically
- thermodynamic limit described by a CFT: $SU(2)_{k=1}$ WZW model
- integrable but not à la Bethe
- highly degenerate spectrum described by a Yangian symmetry

Unlike the AFH model, the GS wave function has a simple structure obtained by the Gutzwiller proyection of the half-filled Fermi sea and it has a Jastrow type form which is reminiscent of the Laughlin wave function for the FQHE. The latter wave function can be constructed using the vertex operators of CFT. A similar construction can be done for the HS wave function.

Matrix Product States

Consider a 1D spin 1/2 system with N sites and Hamiltonian

$$H = \sum_{i=1}^{N} (h_{i,i+1} + h_{i,i+2} + \ldots + h_{i,i+r})$$

The GS wave function is given in a local spin basis by

$$|\psi\rangle = \sum_{s_1,\ldots s_N} \psi(s_1, s_2, \ldots, s_N) \, |s_1, s_2, \ldots, s_N\rangle, \quad s_i = \pm 1$$

The number of parameters to describe this wave function grows exponentially as 2^N. The matrix product state is an ansatz for the wave function given by the product of D-dimensional matrices:

$$\psi(s_1, s_2, \ldots, s_N) = \sum_{\alpha_1,\cdots,\alpha_N} A^{(1)}_{\alpha_1,\alpha_2}(s_1)\, A^{(2)}_{\alpha_2,\alpha_3}(s_2) \cdots A^{(N)}_{\alpha_N,\alpha_1}(s_N) = Tr\left(A^{(1)}(s_1) \cdots A^{(N)}(s_N)\right)$$

where $\quad A^{(i)}_{\alpha_i,\alpha_{i+1}}(s_i), \quad s_i = \pm 1 \quad \alpha_i = 1,\ldots,D \quad i = 1,\ldots,N$

Reduction of GS parameters $\quad 2^N \rightarrow 2 \times N \times D^2$

The entanglement entropy of the MPS in a bipartition A U B scales as

$$S_A = Tr_B |\psi\rangle\langle\psi| \propto \log D$$

In a critical system (periodic BCs)

$$\boxed{S_A \approx \frac{c}{3} \log|A| + c_1}$$

(Holzhey et al, Vidal et al. Cardy-Calabrese)

hence one needs very large matrices to describe critical systems

$$N \propto D^{\kappa}, \quad \kappa = \kappa(c)$$

(Tagliacozzo et al, Pollmann et al)

An alternative to this is to propose a MPS with

$$\boxed{D = \infty}$$

Infinite MPS: Vertex operators in CFT

Consider a chiral free boson field $\varphi(z)$ with a two point correlator

$$\langle \varphi(z_1)\varphi(z_2)\rangle = -\log(z_1 - z_2)$$

Vertex operators are the normal order exponentials

$$V_\beta(z) = :e^{i\beta\varphi(z)}:$$

Two point correlator is given by

$$\left\langle V_{\beta_1}(z_1) V_{\beta_2}(z_2)\right\rangle = (z_1 - z_2)^{\beta_1 \beta_2}$$

The vertex operators act on the Fock spaces generated by all the oscillators of the bosonic field. This allows for an infinite dimensional version of the MPS:

$$A_z(s) = \chi_s :e^{is\sqrt{\alpha}\varphi(z)}: \quad \chi_s = \pm 1$$

The wave function is

$$\psi\left(s_1, s_2, \ldots, s_N\right) = \left\langle A_{z_1}(s_1)\, A_{z_2}(s_2) \cdots A_{z_N}(s_N) \right\rangle$$

Using the vertex correlators one gets

$$\psi\left(s_1, s_2, \ldots, s_N\right) = \chi_{s_1, \ldots s_N} \prod_{i<j} (z_i - z_j)^{\alpha s_i s_j} \times \delta(\sum_i s_i)$$

where charge conservation implies $S^z_{tot} = \frac{1}{2} \sum_{i=1}^{N} s_i = 0, \quad N : even$

$\alpha, z_1, \ldots, z_N$ are variational parameters obtained by minimization of the GS energy. If the GS state is translationally invariant one can choose

$$z_n = e^{2\pi i n / N}, \quad n = 1, \ldots, N$$

$$\boxed{\psi\left(s_1, s_2, \ldots, s_N\right) = C\, \chi_{s_1, \ldots s_N} \prod_{i>j} \left(\sin \frac{\pi(i-j)}{N}\right)^{\alpha s_i s_j} \times \delta(\sum_i s_i)}$$

If $\alpha > 0$ NN antiparallel spins are favoured, which suggests that this wave function describes AF states.

Using the Marshall sign rule we choose

$$\chi_{s_1,\ldots,s_N} = e^{i\pi/2 \sum_{i:odd}(s_i - 1)}$$

We have applied this ansatz to the following models

- Anisotropic Heisenberg model
- $J_1 - J_2$ spin chain
- Random AFH model

Determining the parameters $\alpha, z_1, \ldots, z_N$ in terms of the couplings

- Overlaps with exact wave functions up to N=20 sites
- Spin-spin correlators
- Entanglement entropy

Anisotropic spin 1/2 Heisenberg model

Hamiltonian periodic BCs

$$H = \sum_{1=1}^{N} S_i^x \, S_{i+1}^x + S_i^y \, S_{i+1}^y + \Delta \, S_i^z \, S_{i+1}^z$$

Phases of the model

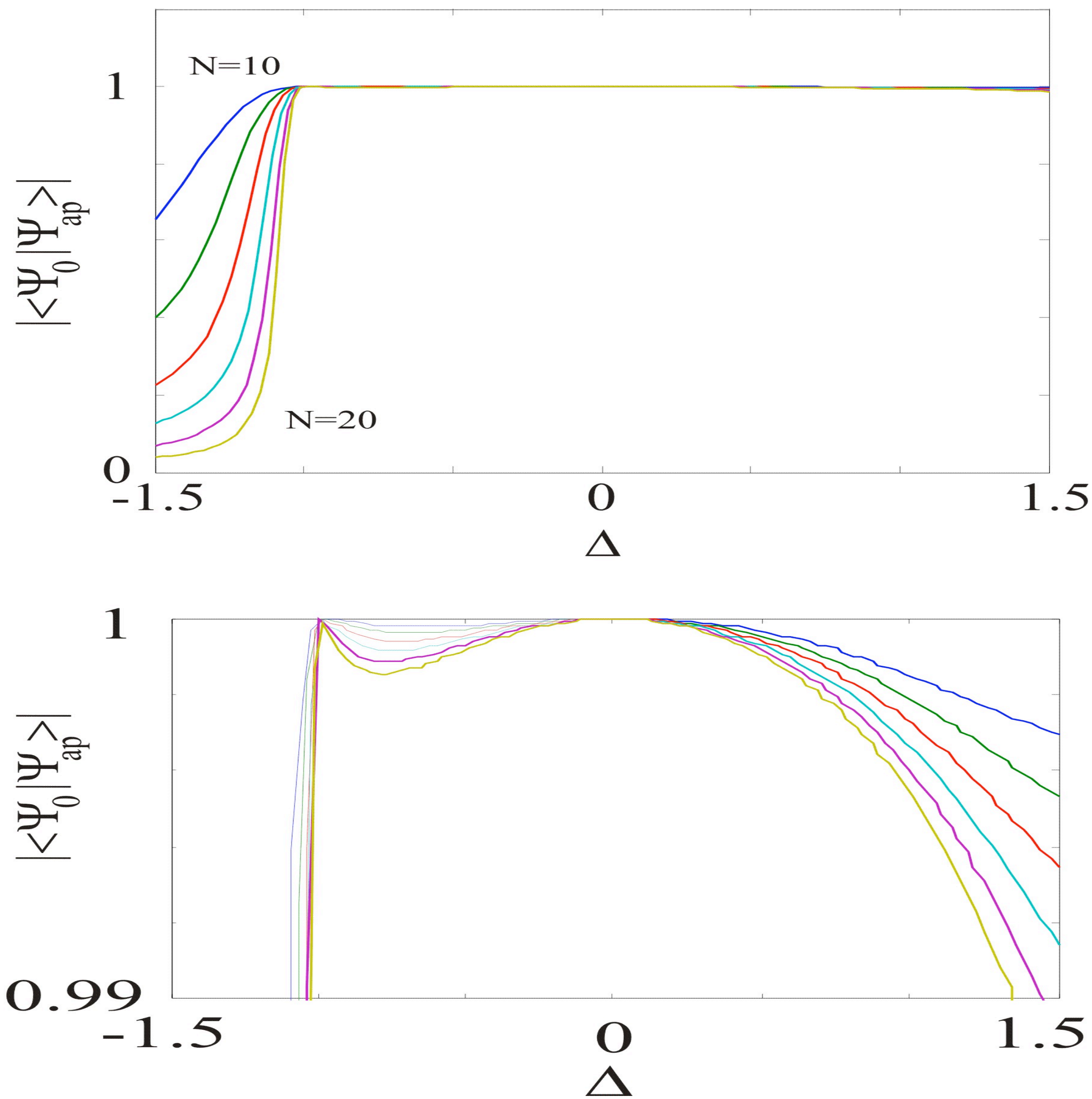
$$\Delta > 1 \quad gapped\ Neel$$

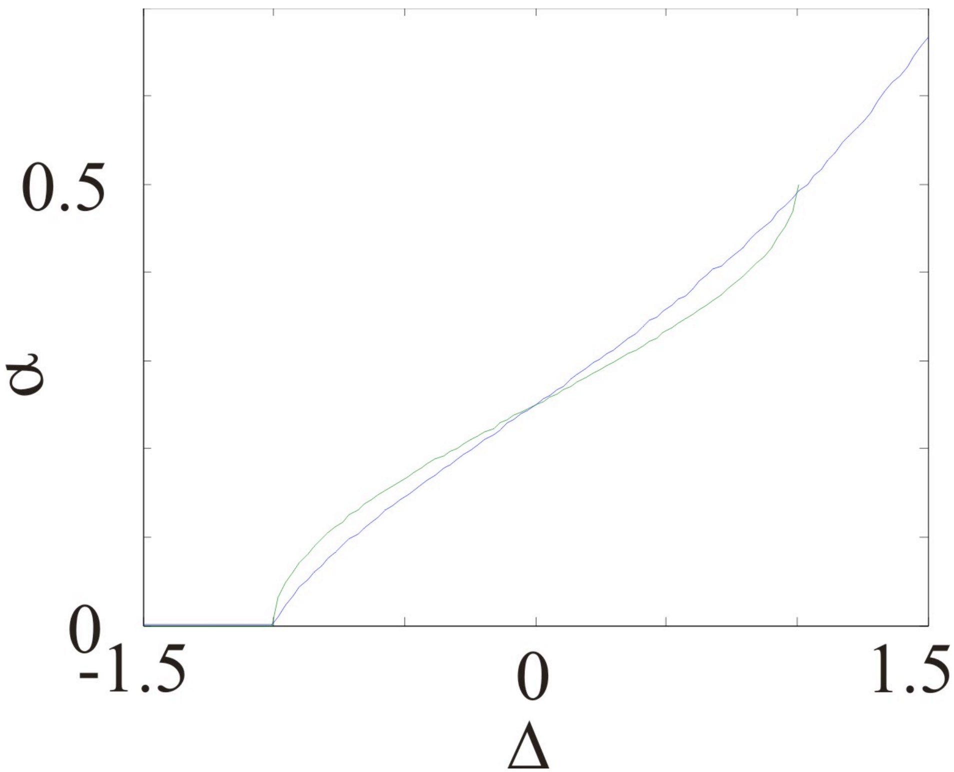
$$-1 < \Delta \leq 1 \quad gapless \ \ (c = 1\, CFT)$$

$$\Delta \leq -1 \quad Ferromagnetic$$

To find α we maximize the overlap between the exact numerical GS wave function and the ansatz for N=10 up to 20

Overlap of exact wave function and the ansatz





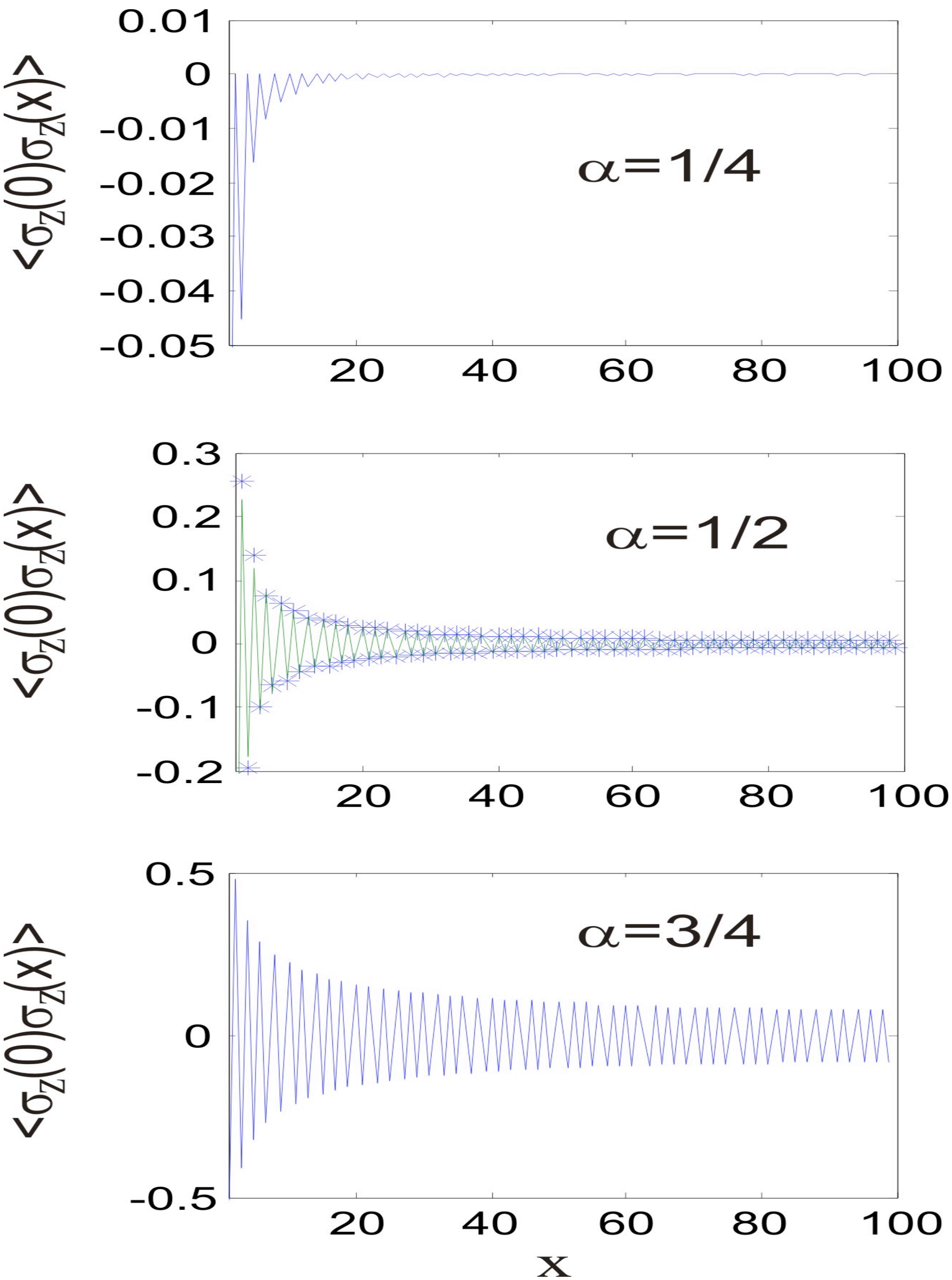
The ansatz is exact at two values

$$\Delta = -1 \rightarrow \alpha = 0$$ Isotropic Ferromagnetic chain

$$\Delta = 0 \rightarrow \alpha = 1/4$$ XX chain

At the isotropic AFH model

$$\Delta = 1 \rightarrow \alpha = 1/2$$ Haldane-Shastry chain



Spin-spin
Correlations
N=200

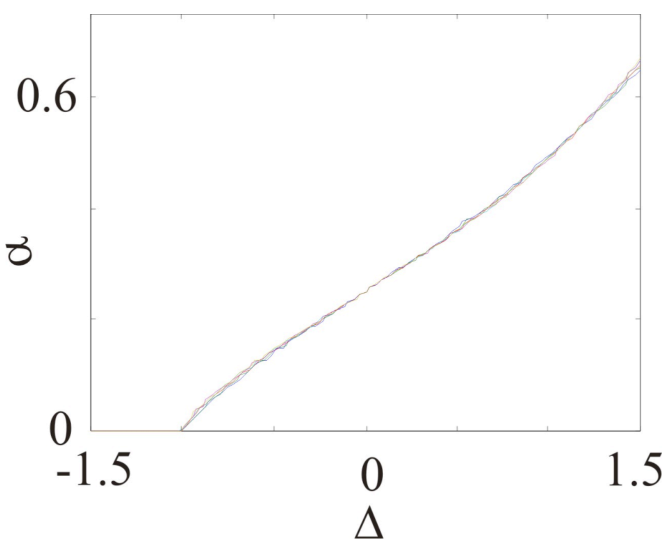
Algebraic decay in the critical region $-1 < \Delta \leq 1 \rightarrow 0 < \alpha \leq 1/2$

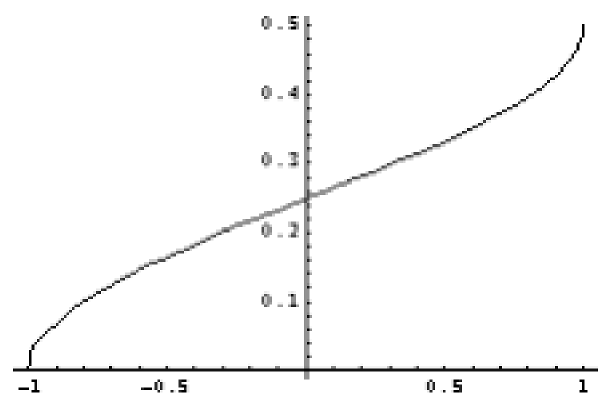
Exacts results of the XXZ chain (Luther-Emery, Lukyanov)

$$\left\langle \sigma_n^x \sigma_0^x \right\rangle = \frac{F}{n^\eta}, \quad \Delta = -\cos(\pi \eta)$$

Correlator of the Calogero-Sutherland model computed approximately using a replica trick (Astrakharchik, et al.)

$$\left\langle \sigma_n^x \sigma_0^x \right\rangle \propto \frac{1}{n^{2\alpha}} \quad \rightarrow \quad \Delta = -\cos(2\pi \alpha)$$





At $\alpha = 1/2$ the exact expression of the correlators are given by (Gebhard and Vollhardt)

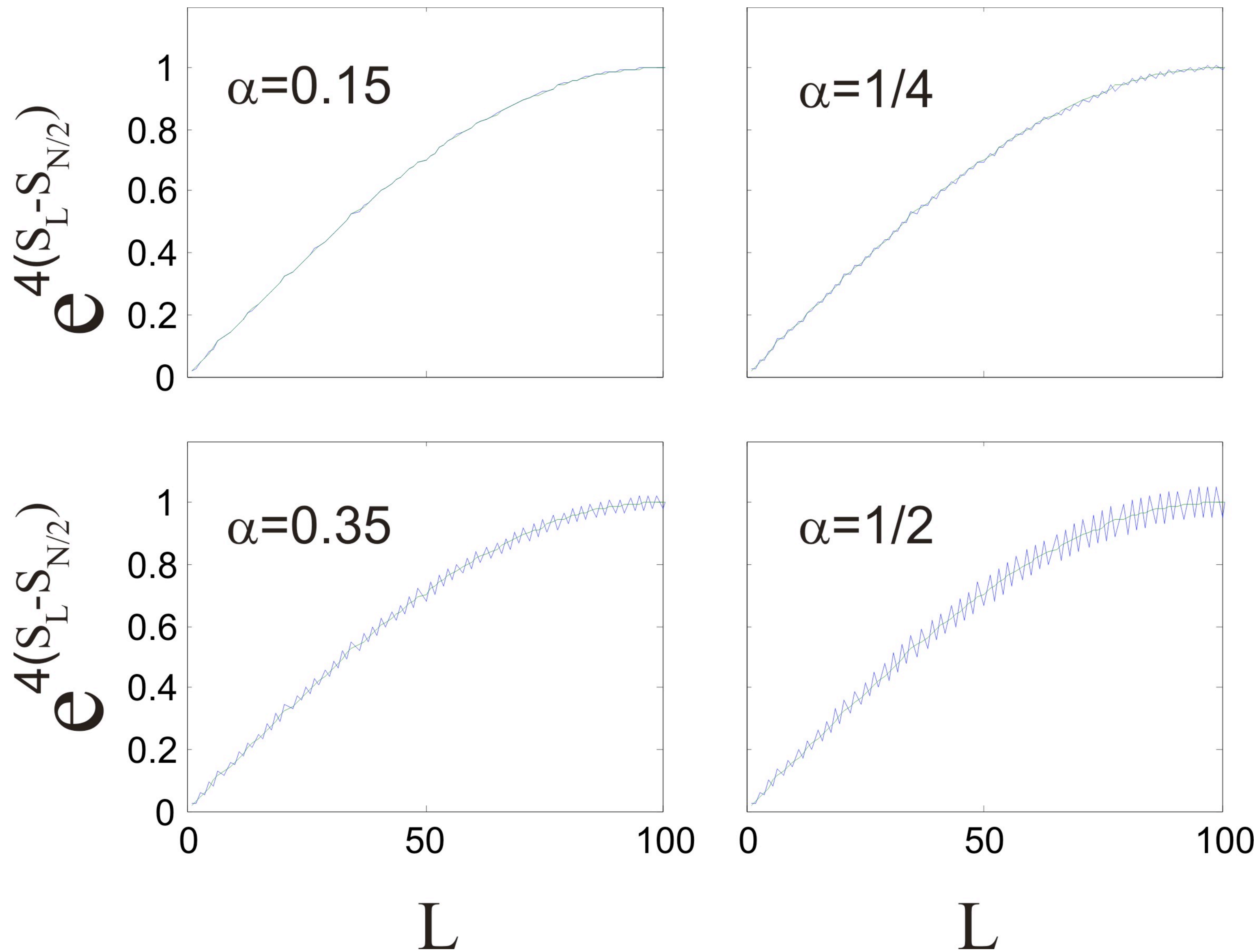
$$\left\langle \sigma_n^a \, \sigma_0^a \right\rangle = (-1)^n \, \frac{Si(\pi \, n)}{\pi \, n}, \quad a = x, y, x$$

Where Si(x) is the sine integral

The correlators show AF long range order

$$\Delta > 1 \rightarrow \alpha > 1/2$$

Reny entropy $S_L = -\log Tr\,\rho_L^2$ using MonteCarlo simulations: agree with CFT prediction with c = 1



Exact GS wave function at $\alpha = 1/4 \rightarrow \Delta = 0$

$$H = J \sum_{1=1}^{N} \left(S_i^x \, S_{i+1}^x + S_i^y \, S_{i+1}^y \right)$$

Apply the unitary transformation $U = \prod_{i:odd} (2 S_i^z) \quad J \rightarrow -J$

The Hamiltonian with J = -1 describes free hard-core bosons.
The GS is the absolute value of the Slater determinant

$$f(x_1,\ldots,x_{N/2}) \propto \prod_{n<m} \left| e^{2\pi i x_n / N} - e^{2\pi i x_m / N} \right| \propto \prod_{n<m} \left| \sin \frac{\pi (x_n - x_m)}{N} \right|$$

where $x_1,\ldots,x_{N/2}$ are the positions of the bosons in the lattice

The many body state can be written as

$$|\psi\rangle \propto \sum_{q_1,\ldots,q_N} \prod_{n<m} \left| \sin \frac{\pi (x_n - x_m)}{N} \right|^{q_n q_m} \left(a_1^* \right)^{q_1} \ldots \left(a_N^* \right)^{q_N} |0\rangle$$

where $q_n = 0, 1$ is the n-th site is empty or occupied

Map: hard core boson --> spin 1/2 system

$$|0\rangle \to |s_n = 1\rangle, \quad a_n^* |0\rangle \to |s_n = -1\rangle \qquad s_n = 1 - 2q_n$$

In the spin variables the GS is

$$|\psi\rangle \propto \sum_{s_1,\dots,s_N} e^{i\frac{\pi}{2}\sum_{n:\,odd} s_n} \prod_{n<m} \left| \sin\frac{\pi(x_n - x_m)}{N} \right|^{s_n s_m/4} |s_1,\dots,s_N\rangle$$

$J_1 - J_2$ Model (zig-zag chain)

$$H = \sum_{1=1}^{N} J_1 \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \vec{S}_i \cdot \vec{S}_{i+2} \qquad (J_1 = 1)$$

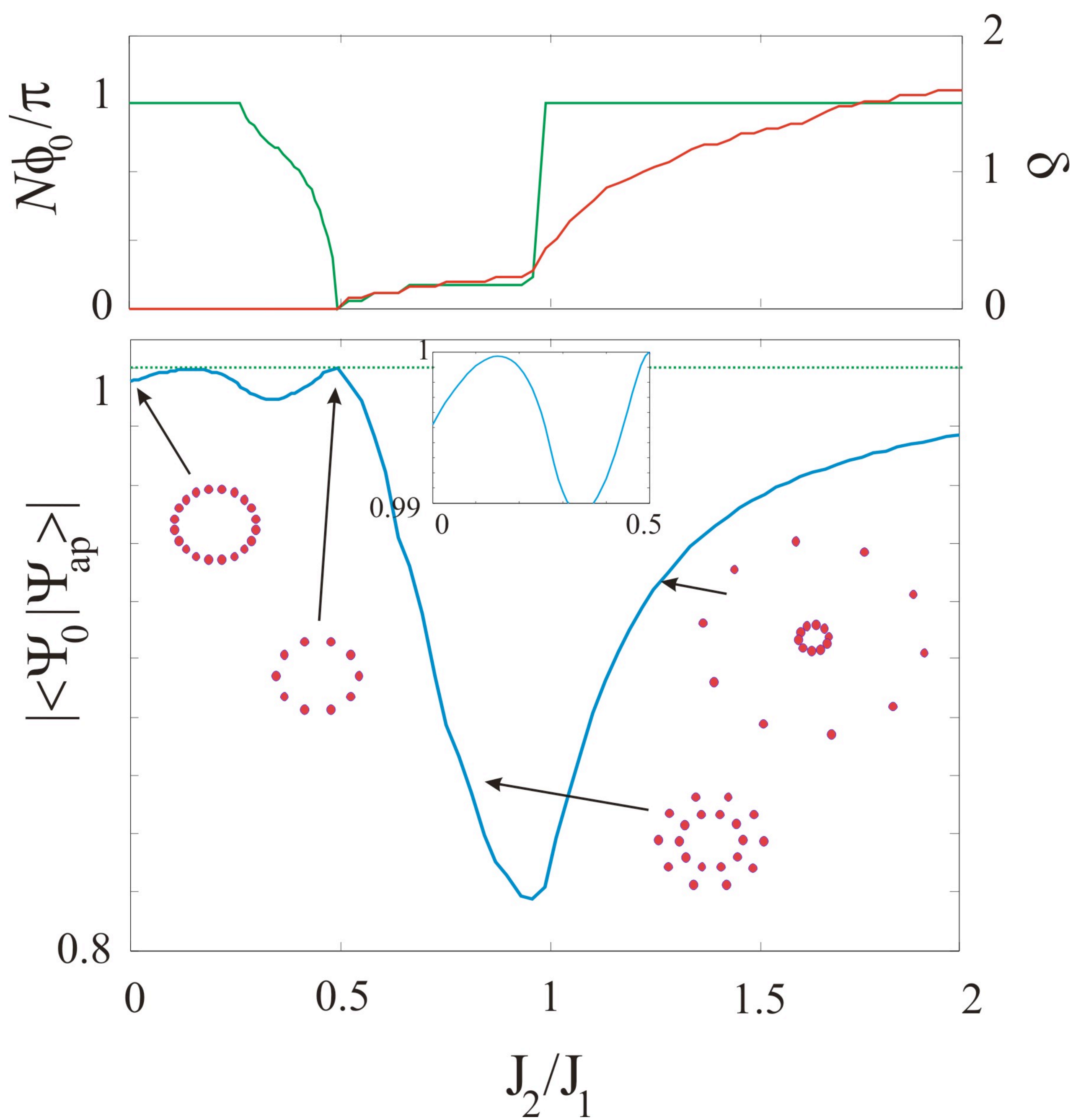
$J_2 > 0$ frustrated spin system

Phases

$0 \le J_2 < J_{2c} \approx 0.241$	Critical c=1
$J_{2c} < J_2 < J_{MG} = 0.5$	Spontaneously dimerized
$J = J_{MG}$	Majumdar-Gosh point
$J_{MG} < J < \infty$	Dimer spiral phase

Choice of parameters $\alpha = \frac{1}{2},$

$$z_n = \begin{cases} \exp(\delta - i\phi_0)\exp(2\pi i(0,2,4,\ldots)/N) & \textit{even sites} \\ \exp(-\delta + i\phi_0)\exp(2\pi i(0,2,4,\ldots)/N) & \textit{odd sites} \end{cases}$$



$N\phi_0/\pi$
ϱ
$|\langle\Psi_0|\Psi_{ap}\rangle|$
J_2/J_1
0
0.5
1
1.5
2
0.8
0.99

Relation with the Haldane-Shastry model

The wave function for $\alpha = 1/2$ can be written in the hard core boson variables as

$$|\psi\rangle \propto \sum_{x_1,\ldots,x_{N/2}} e^{i\pi\sum_n x_n} \prod_{n<m} \left| \sin\frac{\pi(x_n - x_m)}{N} \right|^2 a^*_{x_1} \ldots a^*_{x_{N/2}} |0\rangle$$

This state is the Gutzwiller projection of the one-band Fermi state

$$|\psi_G\rangle = P_G |FS\rangle = \prod_{i=1}^{N} (1 - n_{i,\uparrow} n_{i,\downarrow}) \prod_{|k|<k_F} c^*_{k,\uparrow} c^*_{k,\downarrow} |0\rangle$$

and it is the GS of the long range AF Hamiltonian

$$H = \frac{J\pi^2}{N^2} \sum_{n<m} \frac{\vec{S}_n \cdot \vec{S}_m}{\sin^2(\pi(n-m)/N)} = -\frac{J(2\pi)^2}{N^2} \sum_{n<m} \frac{z_n z_m}{(z_n - z_m)^2} \vec{S}_n \cdot \vec{S}_m$$

CFT derivation of the HS Hamiltonian

The vertex operators

$$A_z(s) = \chi_s : e^{i s \sqrt{\alpha}\, \varphi(z)}, \quad \alpha = 1/2$$

are the primary fields $\phi_{1/2}$ of spin 1/2 and conformal weight h=1/4 of the $SU(2)_k$ WZW model at level k=1. The fusion rule of this field is

$$\phi_{1/2} \times \phi_{1/2} = \phi_0$$

implies that there is a unique conformal block involving N fields

$$\phi_{1/2} \times \ldots^{N} \ldots \times \phi_{1/2} = \phi_0 \quad (N : even)$$

and it is given by

$$\psi\left(z_1, s_1, ,\ldots, z_{N,} s_N\right) = \left\langle A_{z_1}(s_1)\, A_{z_2}(s_2) \cdots A_{z_N}(s_N) \right\rangle$$

This conformal block satisfies the Knizhnik-Zamolodchikov eq

$$\frac{k+2}{2}\frac{\partial}{\partial z_i}\psi(z_1,\ldots,z_N)=\sum_{j\neq i}^{N}\frac{\vec{S}_i\cdot\vec{S}_j}{z_i-z_j}\psi(z_1,\ldots,z_N)$$

Making a conformal transformation to the cylinder $z=e^w$

$$\psi_{cyl}(w_1,\ldots,w_N)=\prod_{i=1}^{N}z_i^{1/4}\,\psi_{plane}(z_1,\ldots,z_N)$$

The KZ equation becomes

$$\frac{k+2}{2}\,z_i\frac{\partial}{\partial z_i}\psi_{cyl}=\sum_{j\neq i}^{N}\frac{z_i+z_j}{z_i-z_j}\vec{S}_i\cdot\vec{S}_j\,\psi_{cyl}(z_1,\ldots,z_N)$$

From explicit computation one also has the “abelian” KZ eq.

$$4\,z_i\frac{\partial}{\partial z_i}\psi_{cyl}=\sum_{j\neq i}^{N}\frac{z_i+z_j}{z_i-z_j}s_i\,s_j\,\psi_{cyl}(z_1,\ldots,z_N)$$

Computing the Laplacian in two different ways one gets

$$H\psi_{cyl} = E\psi_{cyl}$$

$$\boxed{H = -\sum_{n\neq m}\left(\frac{z_n z_m}{(z_n - z_m)^2} + \frac{1}{12} w_{n,m}(c_n - c_m)\right)\vec{S}_n \cdot \vec{S}_m}$$

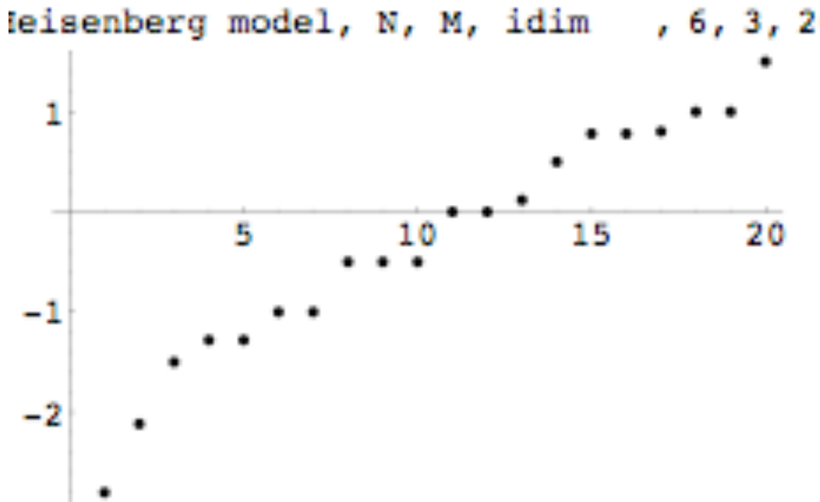
$$w_{n,m} = \frac{z_n + z_m}{z_n - z_m}, \quad c_n = \sum_{n\neq m} w_{n,m}, \qquad E = \frac{1}{16}\sum_{n\neq m} w_{n,m}^2 - \frac{N(N+1)}{16}$$

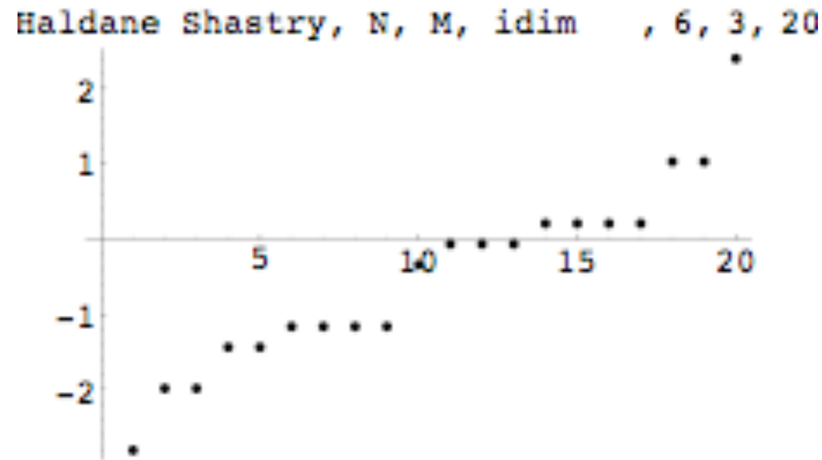
In the uniform case $\qquad z_n = e^{2\pi i n/N} \rightarrow c_n = 0 \quad \forall n$

and we recover the usual HS Hamiltonian. For other values of z_n we obtain an inhomogenous version of it.

The model seems integrable: additional conserved quantities

$$\boxed{H_3 = -i\sum_{n\neq m\neq l} \frac{z_n z_m z_l}{z_{n,m}^2 z_{m,l}^2 z_{l,n}^2}\vec{S}_n \cdot (\vec{S}_m \times \vec{S}_l) + \sum_{n\neq m}\left(-\frac{1}{12}c_n + \frac{17}{8}c_n^{(2)} w_{n,m}\right)\vec{S}_n \cdot \vec{S}_m}$$


eisenberg model, N, M, idim , 6, 3, 2
1
-1
-2
5
10
15
20


Haldane Shastry, N, M, idim , 6, 3, 20
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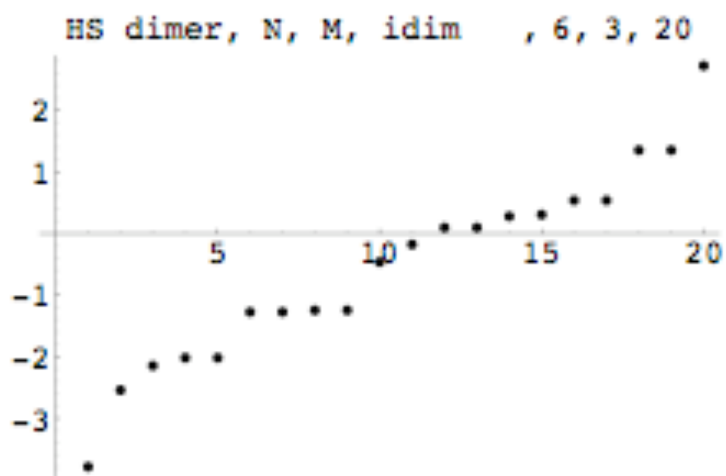
HS dimer, N, M, idim , 6, 3, 20
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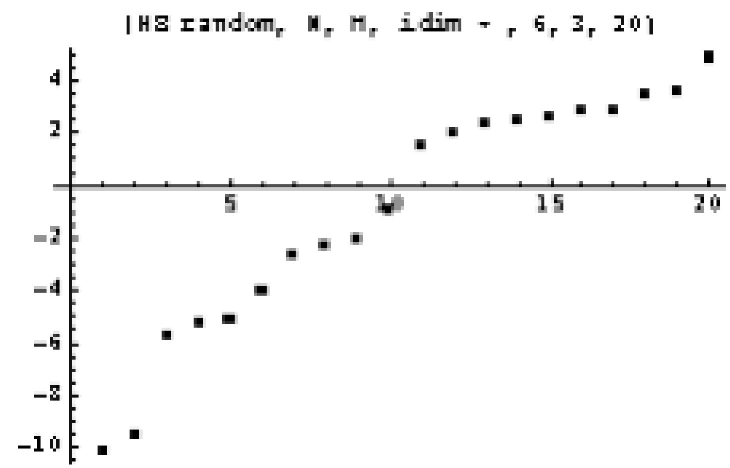
4
2
-2
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-6
-8
-10
5
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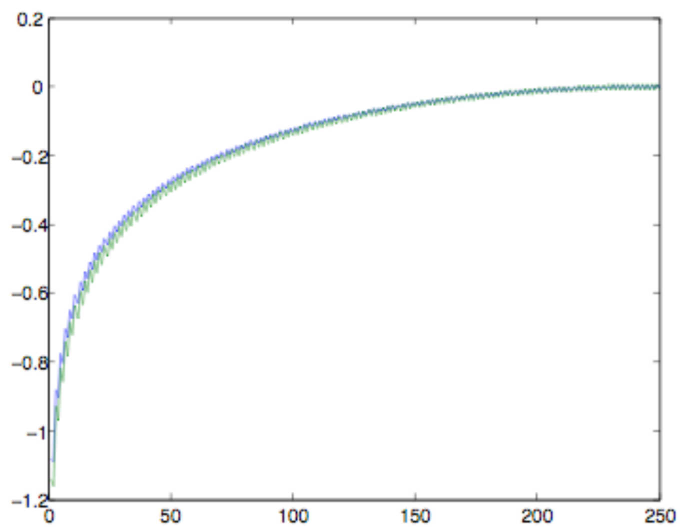
20


0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
0
50
100
150
200
250

Conclusions

-Infinite dimensional version of the MPS using CFT

-Description of critical and non critical systems in various phases

-Inhomogenous version of the Haldane-Shastry model

Prospects

-Generalizations of the HS model to $SU(2)_k$ with k>1

-Infinite dimensional version of PEPS