An exactly solvable pairing model with p + ip symmetry

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Overview

- Recent interest in p-wave superconductivity is motivated by its applications to He3 films, superconductors as Sr2RuO4, superfluids of fermi cold atoms in optical traps, etc.

- p+ip pairing symmetry in the BCS model gives rise to the pfaffian state, which is closely related to the Moore and Read state for the Fractional Quantum Hall state for the filling fraction 5/2. When considering the vortices in the BCS model one gets non abelians anyons similar to those of the Moore-Read model (Green-Read 2000). Thus the p+ip superconductors allows for Topological Quantum Computation, although non universal.

- So far the studies of the BCS model with p+ip symmetry are based on a mean-field analysis using the BdG Hamiltonian. The corresponding phase diagram contains three regions:

- Weak coupling -> “standard” Cooper pairs (BCS type)
- Weak pairing -> “Moore Read” pairs
- Strong pairing -> localized Cooper pairs (BEC type)

The weak and strong pairing regions are separated by a second order phase transition where the gap vanishes. The mean field wave function also experiences a “topological phase transition” across these two regions (Volovik).

The boundary between the weak pairing and the weak coupling regions has not been well characterized.

It is thus of great interest to have an exactly solvable BCS model with p+ip symmetry to analyze in detail the nature of the Moore-Read Pfaffian state and the different phases boundaries of the model.

This model is the so called “reduced” BCS model with p+ip wave symmetry and it is analogous to the reduced BCS model with s-wave symmetry. The latter model was solved by Richardson in 1963 and it is closely related to the Gaudin spin Hamiltonians.

The Richardson model was extensively used to study ultrasmall superconducting grains made of Al in the 1990s.

The integrability of the Richardson-Gaudin models can be proved using the standard Quantum Inverse Scattering Methods. These are the methods that we apply to the p+ip model.

Outline

- The pfaffian state in the BCS model

- Mean-field approach to the p + i p model

- Exact Bethe ansatz solution

- Numerical solution of the BAEs

- Thermodynamic limit: electrostatic analogy.

- Puzzles in the weak pairing region

The pfaffian state

The BCS state (Proyected BCS state) for p-wave

$$|BCS\rangle \propto \exp\left(\frac{1}{2}\sum_{\vec{k}} g_{\vec{k}}\, c^*_{\vec{k}}\, c^*_{-\vec{k}}\right)|0\rangle \to |PBCS\rangle \propto \left(\sum_{\vec{k}} g_{\vec{k}}\, c^*_{\vec{k}}\, c^*_{-\vec{k}}\right)^N |0\rangle$$

$c^*_{\vec{k}}$ Operator that creates a polarised electron with momenta k

$g_{\vec{k}}$ Wave function of the Cooper pair in momentum space

The proyected state has 2N electrons with wave function

$$\psi(\vec{r}_1,\ldots,\vec{r}_{2N}) = A\left[\phi(\vec{r}_1-\vec{r}_2)\,\phi(\vec{r}_3-\vec{r}_4)\ldots\phi(\vec{r}_{2N-1}-\vec{r}_{2N})\right]$$

$$\phi(\vec{r}) = \sum_{\vec{k}} g(\vec{k})\exp(i\vec{k}\cdot\vec{r})$$

This wave function is a pfaffian = sqrt(determinant)

Moore-Read corresponds to the long distance/small momenta behaviour:

$$\phi(\vec{r}) \propto 1/(x+iy),\ |\vec{r}| \to \infty, \quad g(\vec{k}) \propto 1/(k_x+ik_y),\ |\vec{k}| \to 0$$

BCS mean-field theory

The reduced BCS Hamiltonian with p+ip wave symmetry reads:

$$H = \sum_{\vec{k}} \frac{\vec{k}^2}{2m} c^*_{\vec{k}} c_{\vec{k}} - \frac{G}{4m} \sum_{\vec{k} \neq \vec{k}'} (k_x + i k_y)(k'_x - i k'_y) c^*_{\vec{k}} c^*_{-\vec{k}} c_{-\vec{k}'} c_{\vec{k}'}$$

To be compared with the s-wave symmetry (Richardson model)

$$H = \sum_{\vec{k},\sigma} \frac{\vec{k}^2}{2m} c^*_{\vec{k},\sigma} c_{\vec{k},\sigma} - G \sum_{\vec{k} \neq \vec{k}'} c^*_{\vec{k},\uparrow} c^*_{-\vec{k},\downarrow} c_{-\vec{k}',\downarrow} c_{\vec{k}',\uparrow}$$

In the standard mean-field approximation:

$$\Delta = \sum_{\vec{k}} (k_x - i k_y) \left\langle c_{-\vec{k}} c_{\vec{k}} \right\rangle$$

$$H = \sum_{\vec{k}} \left(\frac{\vec{k}^2}{2m} - \frac{\mu}{2} \right) c^*_{\vec{k}} c_{\vec{k}} - \frac{G}{2m} \sum_{k_x > 0, k_y} \left(\Delta (k_x - i k_y) c_{-\vec{k}'} c_{\vec{k}'} + h.c. \right)$$

Which can be diagonalized by a Bogoliubov transformation

The gap Δ and chemical potential μ are solution of the eqs (m=1)

$$\sum_{k_x \geq 0, k_y} \frac{\vec{k}^2}{\sqrt{\left(\vec{k}^2 - \mu\right)^2 + \vec{k}^2 |\Delta|^2}} = \frac{1}{G}$$

$$\mu \sum_{k_x \geq 0, k_y} \frac{1}{\sqrt{\left(\vec{k}^2 - \mu\right)^2 + \vec{k}^2 |\Delta|^2}} = 2N - L + \frac{1}{G}$$

L is the number of energy levels and N is the number of pairs

The thermodynamic limit is defined by

$$L \to \infty, \quad N \to \infty, \quad G \to 0$$

Such that $g = GL, \quad x = \frac{N}{L} (\mathit{filling\ factor})$ are constant

The solution of the gap and chemical potential eqs yield

$$\mu = \mu(g, x), \qquad \Delta = \Delta(g, x)$$

The mean field wave function:

$$|BCS\rangle \propto \exp\left(\frac{1}{2}\sum_{\vec{k}} g_{\vec{k}}\, c^{*}_{\vec{k}}\, c^{*}_{-\vec{k}}\right)|0\rangle \longrightarrow |PBCS\rangle \propto \left(\sum_{\vec{k}} g_{\vec{k}}\, c^{*}_{\vec{k}}\, c^{*}_{-\vec{k}}\right)^{N} |0\rangle$$

$$g(\vec{k}) = \frac{v_{\vec{k}}}{u_{\vec{k}}} = \frac{E(\vec{k}) - \vec{k}^2 + \mu}{\left(k_x + i\,k_y\right)\Delta^{*}}$$

where $E(\vec{k}) = \sqrt{\left(\vec{k}^2 - \mu\right)^2 + \vec{k}^2 |\Delta|^2}$ is the energy of the quasiparticles

The behaviour as k -> 0 depends crucially on the sign of μ

$$\mu < 0 \longrightarrow g(\vec{k}) \approx k_x - i\,k_y,\ \phi(\vec{r}) \approx e^{-r/r_0}\ : Strong\ pairing\ phase$$

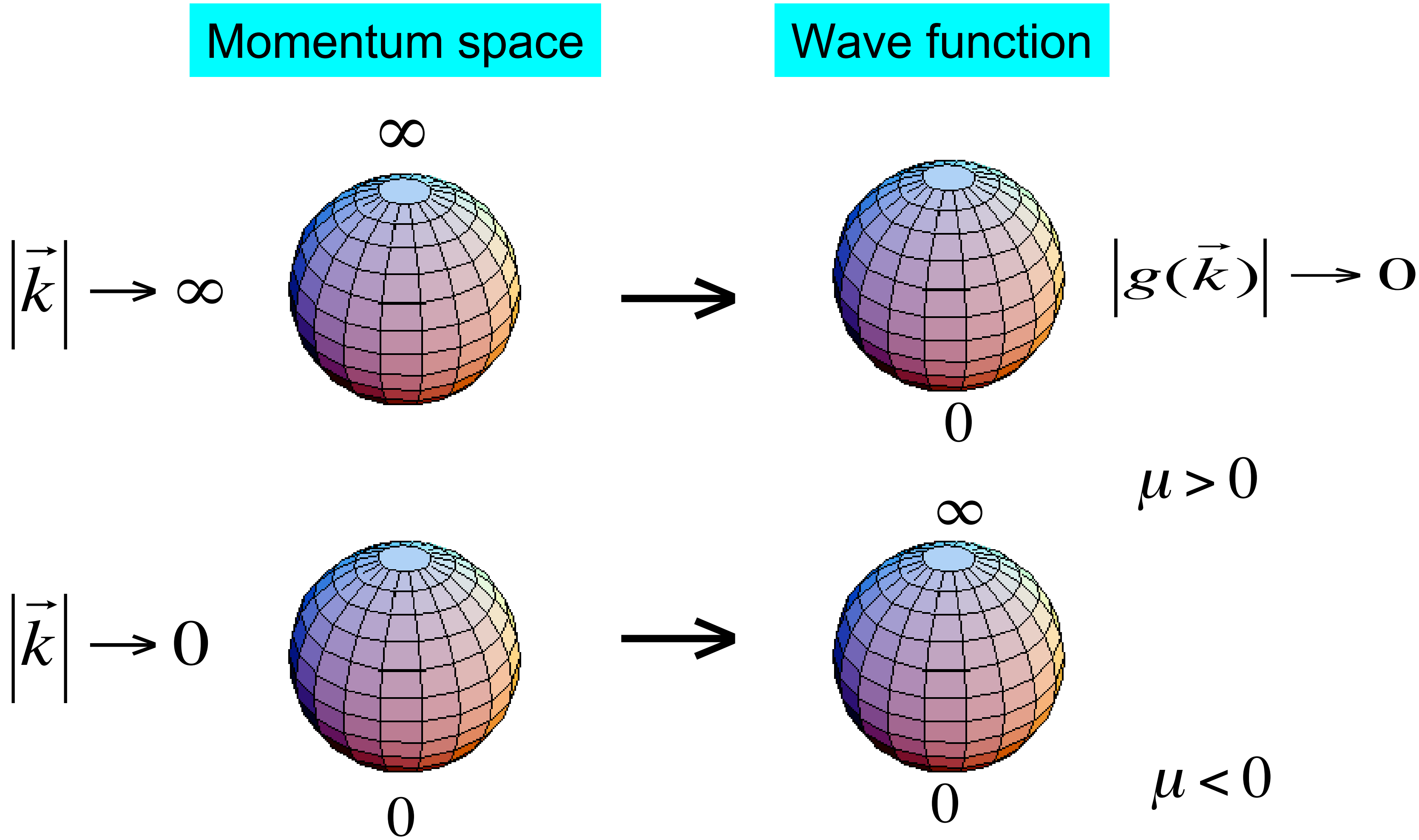
$$\mu > 0 \longrightarrow g(\vec{k}) \approx \frac{1}{k_x + i\,k_y},\ \phi(\vec{r}) \approx \frac{1}{x + i\,y}\ : Weak\ pairing\ phase$$

At $\mu = 0$ there is a second order phase transition (Read-Green line)

$$E(\vec{k}) \propto \left|\vec{k}\right| \longrightarrow 0$$

but there is also a topological transition

Topological nature of the weak-strong transition (Volovik)



Winding number of the map $S_2 \to S_2 : \begin{cases} +1 & \mu > 0 \; (weak \;\; pairing) \\ 0 & \mu < 0 \; (strong \;\; pairing) \end{cases}$

Solution of the gap and chemical potential equations

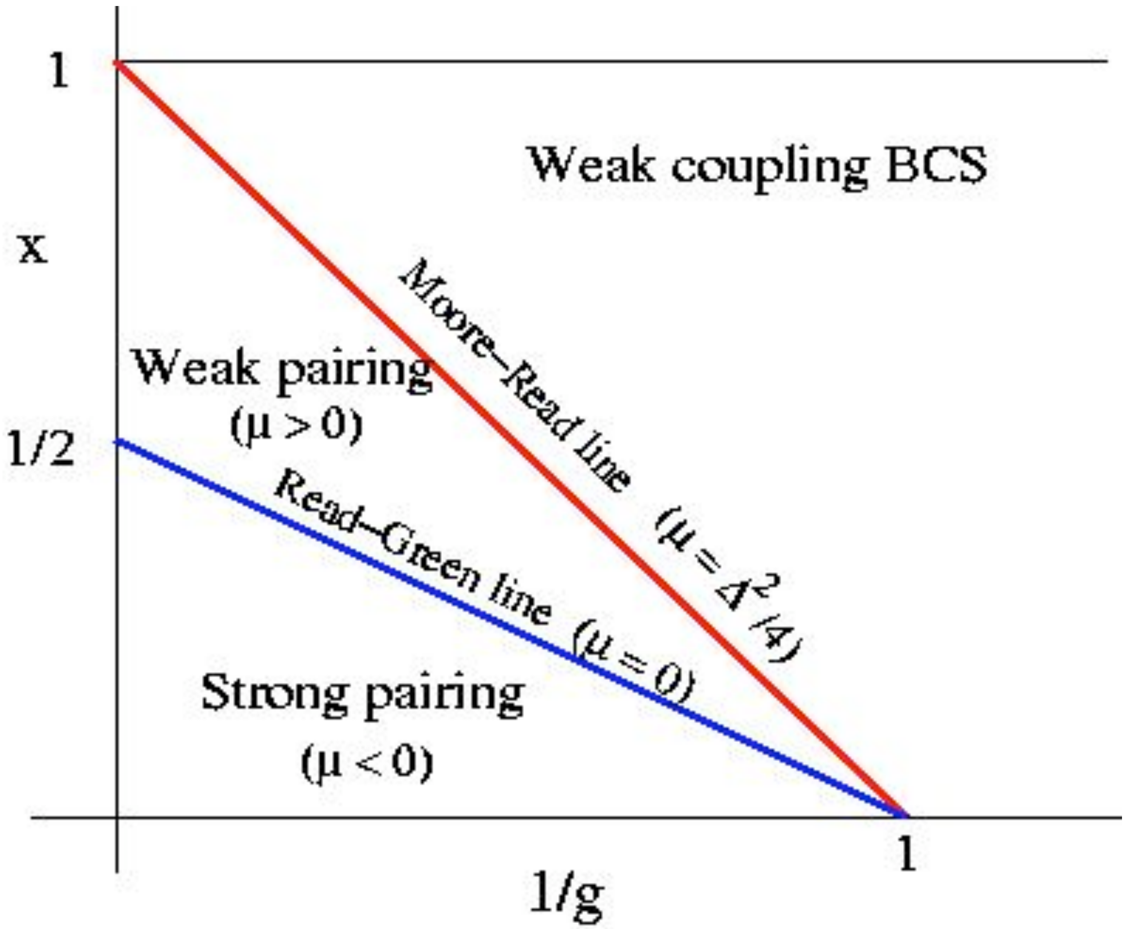
Parameterize the dispersion relation as

$$E(\vec{k}) = \sqrt{\left(\vec{k}^2 - \mu\right)^2 + \vec{k}^2 |\Delta|^2} = \sqrt{\left(\vec{k}^2 - a\right)\left(\vec{k}^2 - b\right)}$$

The parameters a and b have a meaning in the electrostatic solution of the exact model (see later)

Weak coupling	$a,b = \varepsilon_0 \pm i\Delta_0$	$\mu > \Delta^2/4$	$x > x_{MR}$
Moore – Read line	$a = b = -\mu$	$\mu = \Delta^2/4$	$x_{MR} = \left(1 - \frac{1}{g}\right)$
Weak pairing	$a < b < 0$	$0 < \mu < \Delta^2/4$	$x_{RG} < x < x_{MR}$
Read – Green line	$a < b = 0$	$\mu = 0$	$x_{RG} = \frac{1}{2}\left(1 - \frac{1}{g}\right)$
Strong pairing	$a < b < 0$	$\mu < 0$	$x < x_{RG}$

Phase diagram of the p + ip wave model



Duality between weak and strong pairing phase

Given two points in the phase diagram

$$(g, x_I) \in \textit{weak pairing phase} \; (\mu > 0)$$
$$(g, x_{II}) \in \textit{strong pairing phase} \; (\mu < 0)$$

$$If \quad x_I + x_{II} = 1 - \frac{1}{g} \Rightarrow E_I = E_{II}, \; \Delta_I = \Delta_{II}, \mu_I = -\mu_{II}$$

The Read-Green line is selfdual

The Moore-Read line is dual to the “empty” state x = 0
(in particular the GS energy on this line is zero)

This duality also appears in the exact solution and plays an important role.

The Bethe ansatz solution

Recall the Hamiltonian:

$$H = \sum_{\vec{k}} \frac{\vec{k}^2}{2m} c^*_{\vec{k}} c_{\vec{k}} - \frac{G}{4m} \sum_{\vec{k} \neq \vec{k}'} (k_x + i k_y)(k'_x - i k'_y) c^*_{\vec{k}} c^*_{-\vec{k}} c_{-\vec{k}'} c_{\vec{k}'}$$

Setting $z^2_{\vec{k}} = \vec{k}^2 / m$

Define the hard core boson operators: $b^*_{\vec{k}} = \frac{k_x - i k_y}{|\vec{k}|} c^*_{\vec{k}} c^*_{-\vec{k}}$

Then the Hamiltonian can be brought into the form

$$H = \sum_{k_x \geq 0, k_y} z^2_{\vec{k}} N_{\vec{k}} - G \sum_{\vec{k} \neq \vec{k}'} z_{\vec{k}} z_{\vec{k}'} b^*_{\vec{k}} b_{\vec{k}'}$$

And can be solved using the Quantum Inverse Scattering Method starting from the XXZ R-matrix and taking a quasi-classical limit

The Schroedinger equation: $H|\psi\rangle = E|\psi\rangle$

is solved exactly by the states (m=1)

$$|\psi\rangle = \prod_{m=1}^{N} C(y_m)|0\rangle, \qquad C(y) = \sum_{k_x \geq 0, k_y} \frac{k_x - i k}{\vec{k}^2 - y} c^*_{\vec{k}} c^*_{-\vec{k}}$$

where the "rapidities" y_m satisfy the Bethe ansatz eqs

$$-\frac{G^{-1} - L + 2N - 1}{y_m} - \sum_{k=1}^{L} \frac{1/2}{y_m - z_k^2} + \sum_{j \neq m}^{N} \frac{1}{y_m - y_j} = 0 \quad (m = 1, \ldots, N)$$

The total energy is $E = (1+G)\sum_{m=1}^{N} y_m$

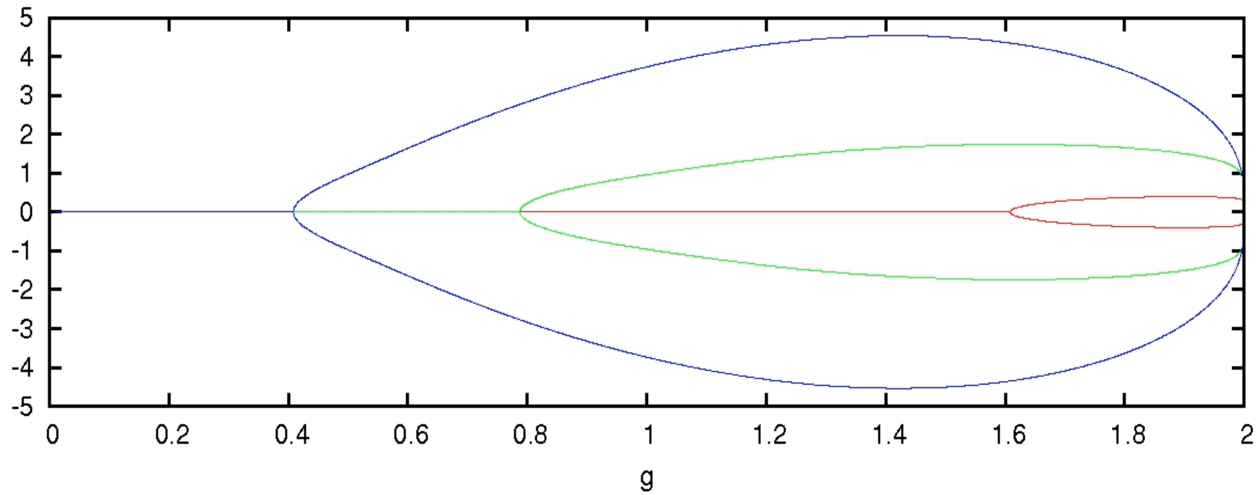
$$\lim_{G \to 0} y_m = z_k^2$$

$$For\ G \neq 0 \to y_m : real\ or\ complex$$

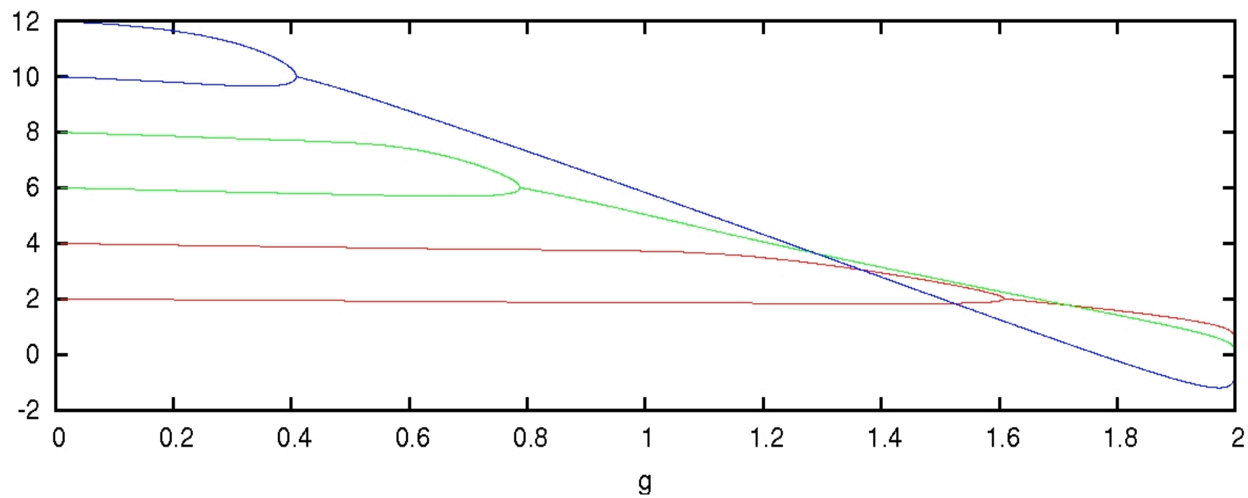
Complex solutions always appear in conjugate pairs

Roots of the p + i p model (numerical solution with L=12,N=6, x=1/2)

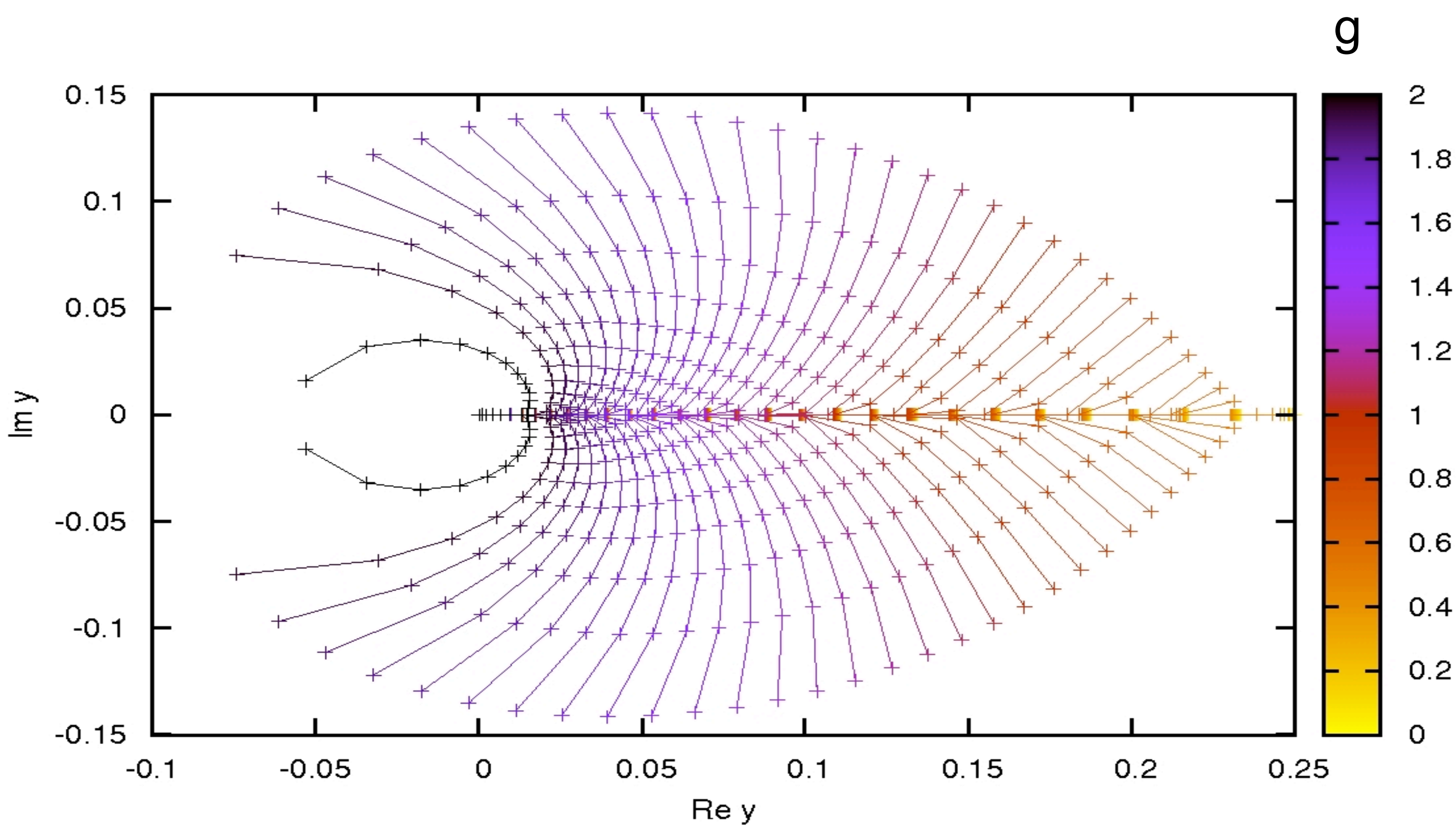
Im y_m



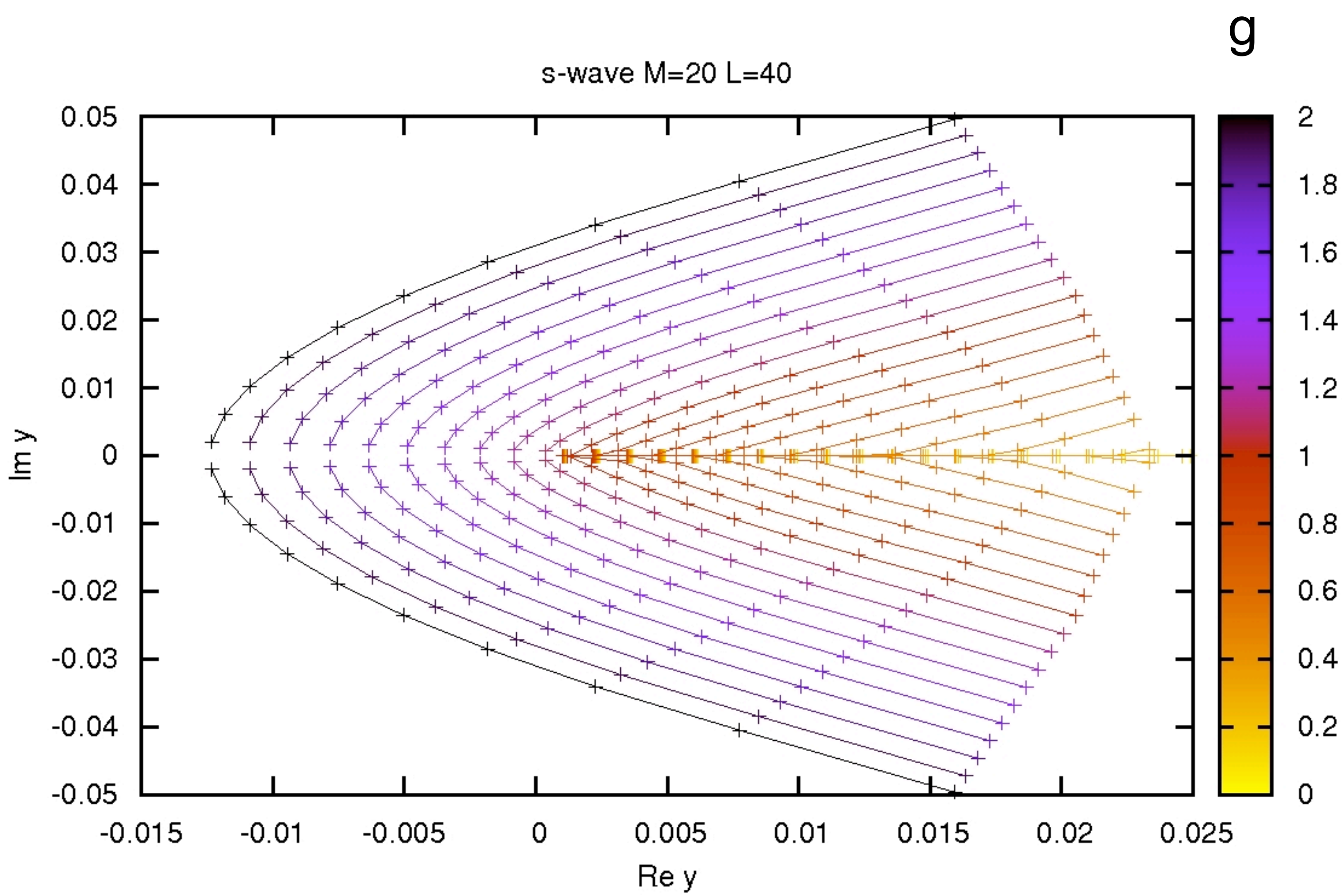
Re y_m



Roots in the complex y-plane (p + i p model)



Roots for the exactly solvable s-wave model



Thermodynamic limit

$$L, N \to \infty, \quad G \to 0, \quad with \quad x = \frac{N}{L}, \quad g = GL \quad finite$$

Let us assume that the roots y_m form an arc Γ in the complex plane with a density $r(y)$

The energies $\varepsilon = z_k^2$ form another arc Ω with density $\rho(\varepsilon)$

The BAEs become

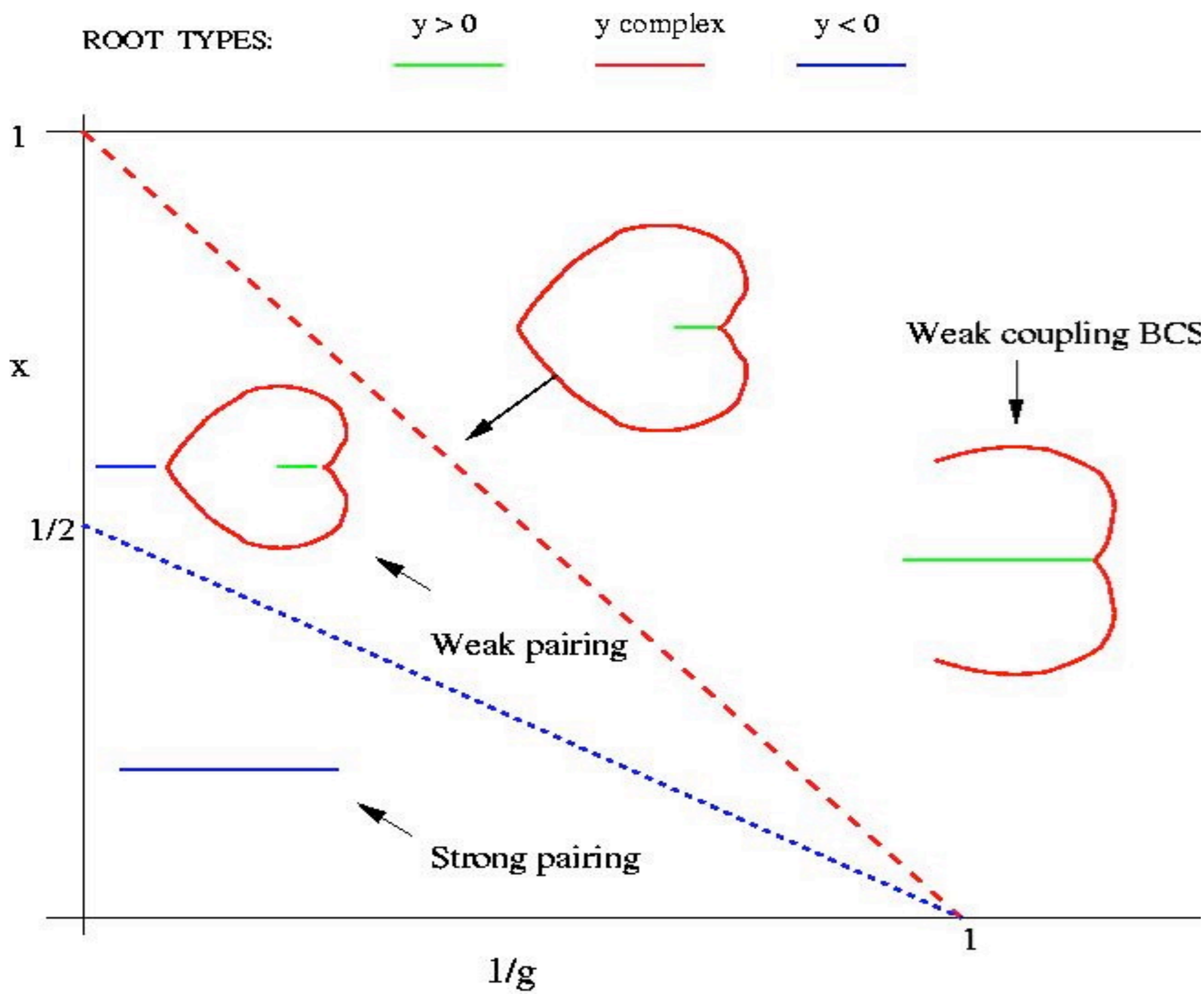
$$\int_\Omega d\varepsilon \frac{\rho(\varepsilon)}{\varepsilon - y} - \frac{q_0}{y} - P\int_\Gamma |dy'| \frac{r(y')}{y' - y} = 0, \quad y \in \Gamma$$

$$q_0 = \frac{1}{2G} - \frac{L}{2} + N$$

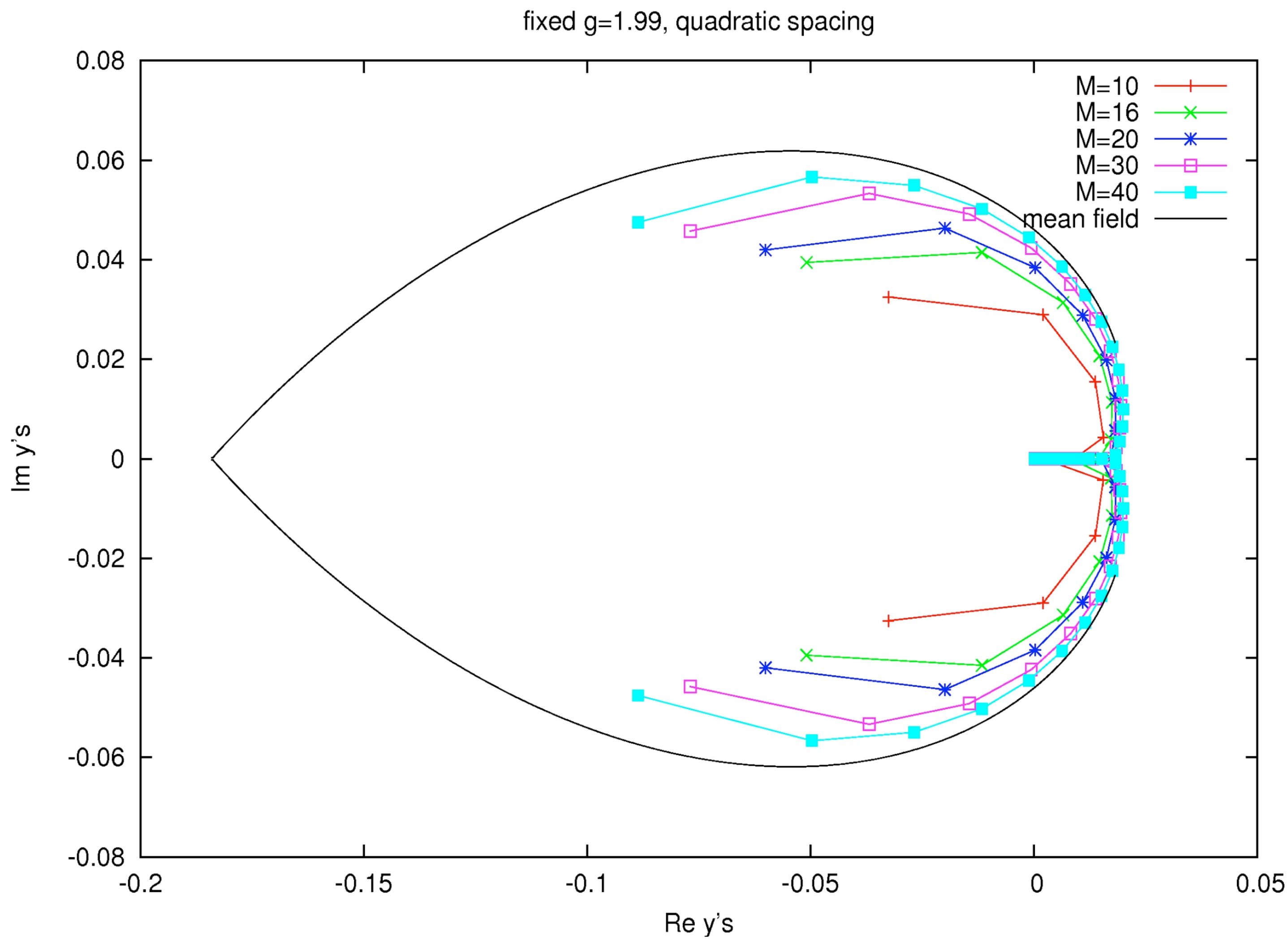
$$N = \int_\Gamma |dy| r(y), \quad E = \int_\Gamma |dy| y\, r(y)$$

There is a analytic solution of these equations which agree with the mean-field solution to leading order in L and N

Structure of the arcs formed by the roots y

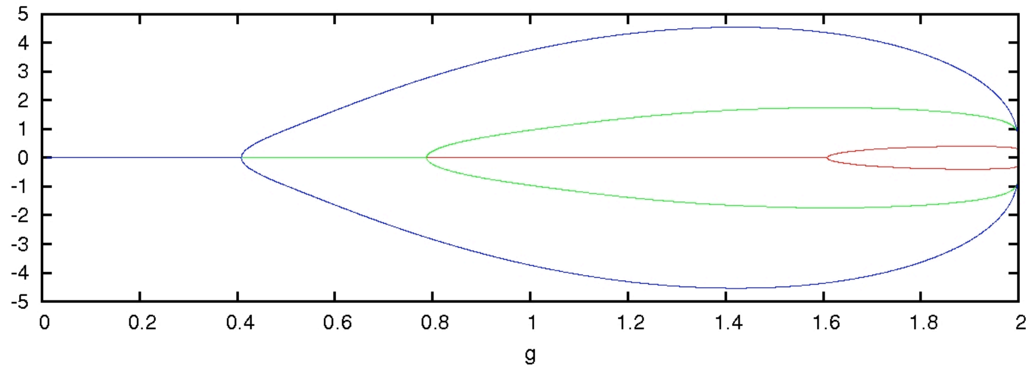


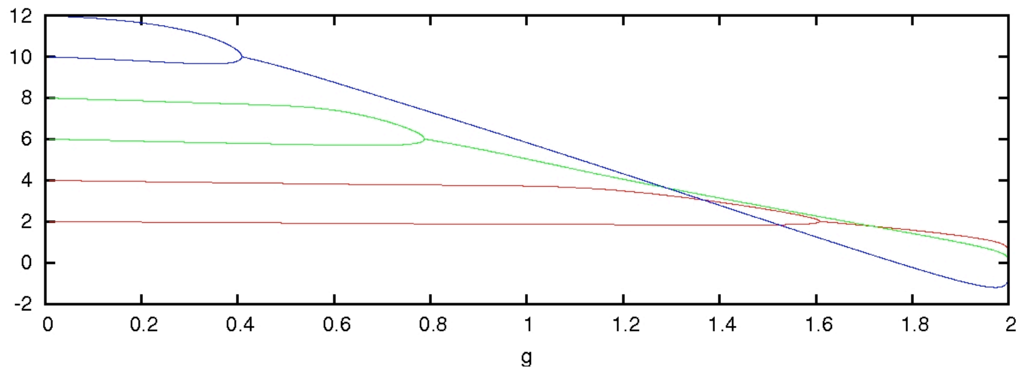
The equation of the complex arc can be determined and compared with the numerical results:
Example with x = 1/2 and g = 1.99 (weak coupling region)



The Moore-Read line

$$x \to x_{MR} = 1 - \frac{1}{g} \quad \Rightarrow \quad y_m \to 0 \quad \forall m$$





Recall $|\psi\rangle = \prod_{m=1}^{N} C(y_m)|0\rangle,$ $\quad C(y) = \sum_{k_x \geq 0, k_y} \frac{k_x - i k_y}{\vec{k}^2 - y} c^*_{\vec{k}} c^*_{-\vec{k}}$

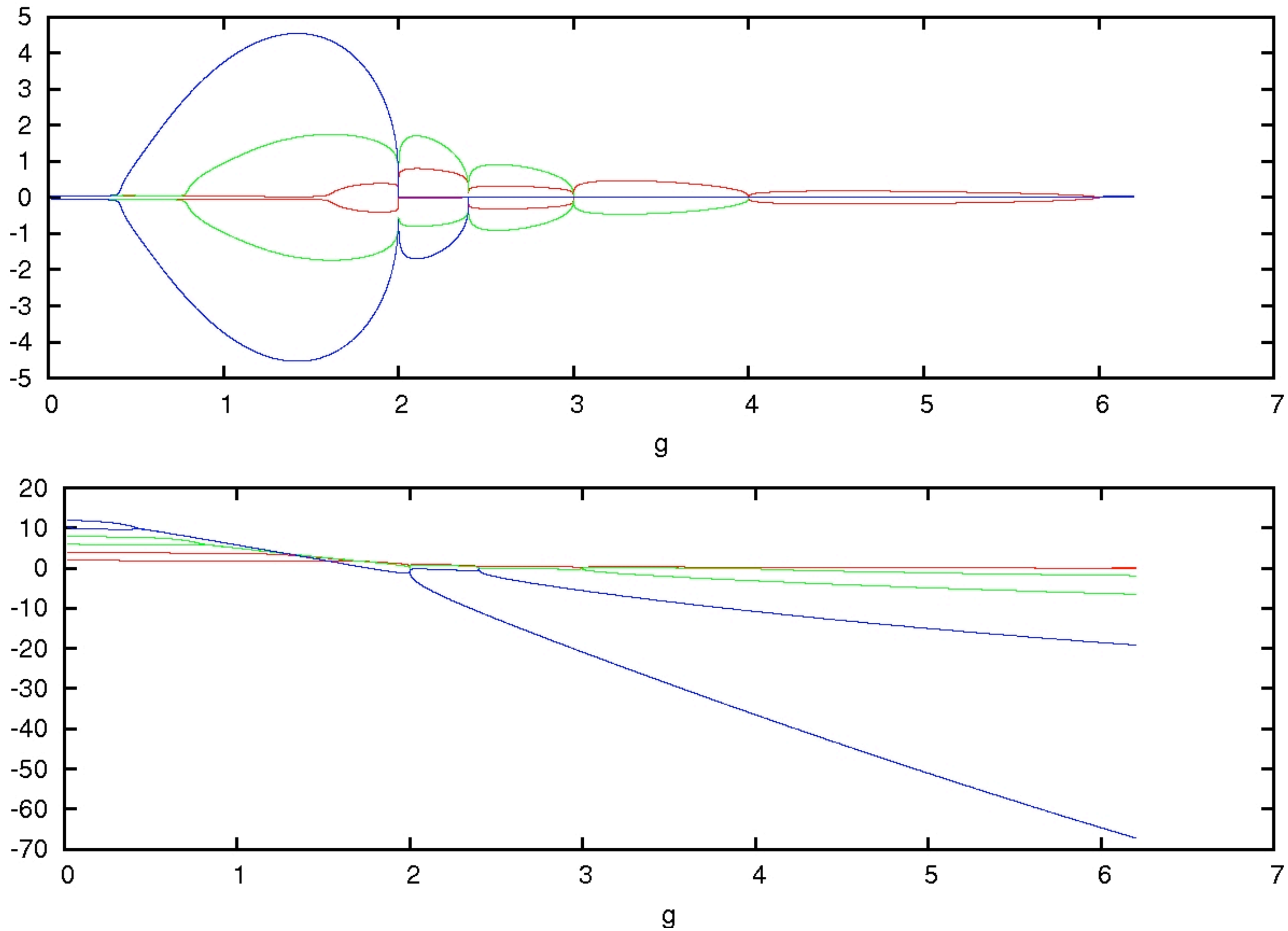
$$C(0) = \sum_{k_x \geq 0, k_y} \frac{1}{k_x + i k_y} c^*_{\vec{k}} c^*_{-\vec{k}} \to |\psi\rangle = \prod_{m=1}^{N} C(0)|0\rangle,$$ MR state

At $x_{MR} = 1 - \frac{1}{g}$ all the roots collapse to y = 0 and there is no arc as we assumed above. This implies that the mean field approximation is not valid on the Moore-Read line

$$\lim_{x \to x_{MR}} \lim_{L \to \infty} \neq \lim_{L \to \infty} \lim_{x \to x_{MR}}$$

This suggests the existence of a phase transition on the MR-line, whose nature is not clear

The collapse of roots to zero occurs in the whole weak pairing phase for fixed values of the coupling



Weak-strong pairing duality: dressing operation

N_W : number of roots in the weak pairing phase

N_0 : number of zero roots (y=0)

N_S : number of non zero roots

$$N_W = N_0 + N_S$$

The collapse of roots happens iff

$$\frac{N_0}{L} + 2\frac{N_S}{L} = 1 - \frac{1}{g}$$

Moreover the non zero roots satisfy the BAE in the strong pairing phase

$$\frac{N_0 + N_S}{L} + \frac{N_S}{L} = 1 - \frac{1}{g} \Leftrightarrow \quad x_I + x_{II} = 1 - \frac{1}{g}$$

Weak-strong duality

An eigenstate |S> in the strong pairing phase can be dressed by N_0 Moore-Read pairs obtaining an eigenstate |W> in the weak pairing phase with the same energy

$$DRESSING: \quad H|S\rangle = E|S\rangle \Rightarrow H|W\rangle = H\left[C(0)\right]^{N_0}|S\rangle = E|W\rangle$$

Discontinuity of the GS energy on the MR line

Take one pair for g >1. Its energy in the L>> 1 limit is finite

$$\frac{|y_1|}{\omega}\log\left(1+\frac{\omega}{|y_1|}\right)=1-\frac{1}{g}$$

Dress this pair with N_0 MR pairs

$$x_I + x_{II} = x_I + \frac{1}{L} = 1-\frac{1}{g} \rightarrow x_I = 1-\frac{1}{g}-\frac{1}{L} \approx 1-\frac{1}{g} = x_{MR}$$

$$\lim_{x\rightarrow x_{MR}} E_0(g,x) = |y_1| \neq E_0(g,x_{MR}) = 0$$

One may call this a zero order phase transition

Questions and suggestions:

- What´s the thermodynamic limit of this model in the weak pairing phase?

It seems that one have to distinguish between

rational and irrational values of the coupling g.

Rational g´s -> collapse of roots (non mean-field description)

Irrational g´s -> smooth arcs (mean-field description)

- What are the elementary excitations:

is there a gap in the weak pairing phase?

is there any signature of non abelian anyons?

- The Moore-Read line is a crossover or a true phase transition (perhaps topological) ?