Toroidal compactification of closed bosonic string theory

1 Motivation

As discussed in the overview lectures, a canonical mechanism to obtain four-dimensional physics at low energies out of a theory with $D > 4$ is to consider the theory in a curved background of the form $M_4 \times X_{D-4}$, with X_{D-4} a $(D-4)$-dimensional *compact* manifold, called the internal space. At energies $E \ll 1/L$, where L is the typical size of the dimensions in X_{D-4}, the physics is essentially 4d, we do not have enough resolution to see the internal space. This is called compactification of the theory.

One of the simplest possibilities is to consider the internal space to be a $(D-4)$-torus. In this section we are interested in exploring this possibility in string theory. Happily, the most interesting phenomena are already present in we compactify just one dimension on a circle, and reduce the 26d bosonic string theory to a 25d theory at low energies.

We start with a discussion of compactification in field theory. As we know,this provides a good approximation to the dynamics of string theory when α' corrections are negligible [1]. That is, when the internal space radius is much larger than the string length scale. Even in this regime there are interesting phenomena, like the Kaluza-Klein mechanism to generate gauge vector bosons out of the higher dimensional metric.

Next we turn to the explicit discussion of compactification in full-fledged string theory. This can be carried out for toroidal compactification because it is described by a free worldsheet theory, which can be quantized exactly

[1] Recall the picture 1.

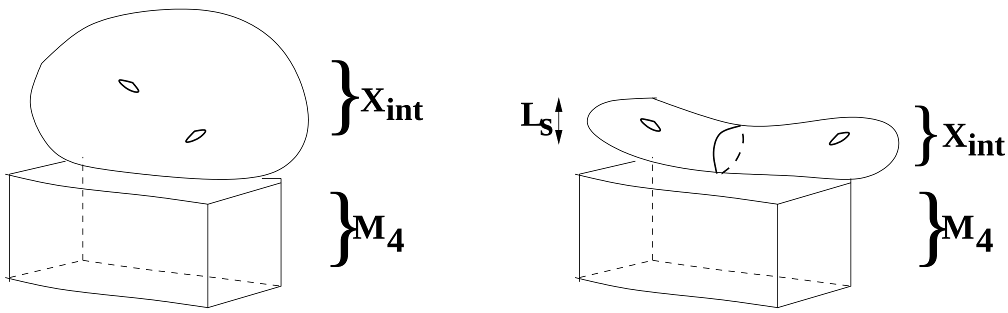

Figure 1: Picture of compactification spacetimes; thick small lines represent string states which are light in the corresponding configuration. When the internal manifold has size of the order of L_s, stringy effects (which do not exist in theories of point particles) become relevant; for instance, string winding modes (where a closed string winds around some internal dimension) may become light.

in the sense of the α' expansion. This means that for compactification on circles of radius comparable or smaller than the string length, string theory may (and does) differ from field theory.

Among the most surprising effects, we will find i) new light (and even massless) particles arising from closed string winding around the internal circle, and ii) T-duality, a complete physical equivalence of two theories living in different spacetimes.

Results in this section are useful in discussing toroidal compactifications in other string theories, like superstrings. Also, they will be useful in the construction of 10d heterotic string theories.

2 Toroidal compactification in field theory

Here we roughly follow ideas in section 8.1 of [1]. Our discussion is sketchy and provides most results without their detailed derivation.

Let us first consider circle compactification in field theory, which is a good approximation to the situation in string theory for circle radius much larger than the string length, so that α' effects (which are the ones related to the fact that the string is an extended object) are negligible.

So we consider field theories in D-dimensions, propagating on a background spacetime of the form $M_d \times \mathbf{S^1}$, with $D = d+1$. To explain why the low-energy physics is d-dimensional, consider first a toy model of a D-dimensional massless scalar field $\varphi(x^0, \ldots, x^{D-1})$ propagates with D-dimensional action

$$S_{5d\varphi} = \int_{M_d \times \mathbf{S^1}} d^D x \, \Lambda^{D-4} \partial_M \varphi \partial^M \varphi \tag{1}$$

with $M = 0, \ldots, D-1$ and where Λ is some scale which we have introduced for dimensional reasons.

Since x^{D-1} parametrizes a circle, it is periodic, and we can expand the x^{D-1} dependence in Fourier modes

$$\varphi(x^0, \ldots, x^{D-1}) = \sum_{k \in \mathbf{Z}} e^{2\pi i k x^{D-1}/L} \, \varphi_k(x^0, \ldots, x^{d-1}) \tag{2}$$

where $L = 2\pi R$ is the length of $\S^1$.

From the d-dimensional viewpoint, we see a bunch of d-dimensional scalar fields $\varphi_k(x^0, \ldots, x^{d-1})$, labeled by the integer index k, which defines the momentum in the extra dimension $p_{D-1} = k/R$. The d-dimensional spacetime mass of those fields increases with k^2. To see that, take the D-dimensional mass-shell condition

$$P^2 = 0 \,, \quad \text{that is } P_{M_d}^2 + p_{D-1}^2 = 0 \tag{3}$$

For the field φ_k, we have

$$P_{M_d}^2 + (k/R)^2 = 0 \tag{4}$$

which means that the d-dimensional mass of the field φ_k is

$$m_k^2 = (k/R)^2 \tag{5}$$

Equivalentely, we may obtain this result from the d-dimensional wave equation for the field φ_k

$$\partial_M \varphi \partial^M \varphi = 0 \quad \rightarrow \quad \partial_\mu \varphi_k \partial^\mu \varphi_k + (k/R)^2 = 0 \tag{6}$$

where $\mu = 0, \ldots, d-1$. And we recover (5).

At energies much lower than the compactification scale $M_c = 1/R$, $E \ll 1/R$, the only mode which is observable is the zero mode $\varphi_0(x^0, \ldots, x^{d-1})$. So we see just a single d-dimensional field, with a d-dimensional action, which is obtained by replacing $\varphi(x^0, \ldots, x^{D-1})$ in (1) by the only component we are able to excite $\varphi_0(x^0, \ldots, x^{d-1})$. The x^{D-1} dependence drops and we get

$$S_{eff} = \int_{M_d} d^d x \, \frac{L}{\Lambda^{D-4}} \, \partial_\mu \varphi_0 \partial^\mu \varphi_0 \tag{7}$$

So we recover d-dimensional physics at energies below M_c. This is the Kaluza-Klein mechanism, or Kaluza-Klein reduction. The massive d-dimensional fields φ_k are known as Kaluza-Klein (KK) excitations or KK replicas of φ_0.

Obs: If the higher-dimensional field theory contains massive fields with mass M, the 4d KK tower has masses $m_k^2 = M^2 + (k/R)^2$, so they will not be observable at energies below M.

The Kaluza-Klein reduction works for any higher dimensional field. An important new feature arises when the original higher dimensionl field has non-trivial Lorentz quantum numbers. The procedure is then to first decompose the representation of the $SO(D)$ higher-dimensional Lorentz group with respect to the lower-dimensional one $SO(d)$ (i.e. separate different components according to their behaviour under d-dimensional Lorentz), and finally perform KK reduction for each piece independently. For instance, for a D-dimensional graviton we have the KK reduction on $\mathbf{S^1}$

$$\begin{aligned} G_{MN}(x^0,\ldots,x^{D-1}) \to \quad & G_{\mu\nu}(x^0,\ldots,x^{D-1}) \to & G^{(0)}_{\mu\nu}(x^0,\ldots,x^{d-1}) \\ & G_{\mu,D-1}(x^0,\ldots,x^{D-1}) \to & G^{(0)}_{\mu 4}(x^0,\ldots,x^{d-1}) \\ & G_{D-1,D-1}(x^0,\ldots,x^{D-1}) \to & G^{(0)}_{44}(x^0,\ldots,x^{d-1}) \end{aligned} \tag{8}$$

where the first step is just decomposition in components, and the second is KK reduction. We therefore obtain, at the massless level, a d-dimensional graviton, a d-dimensional $U(1)$ gauge boson, and a d-dimensional scalar.

To be more specific, the only piece of the D-dimensional metric which is visible from the low-energy d-dimensional viewpoint is

$$ds^2 \,=\, G_{\mu\nu}\, dx^\mu\, dx^\nu \,+\, G_{dd}\,(dx^d + A_\mu dx^\mu)^2 \tag{9}$$

where the fields $G_{\mu\nu}$, G_{dd}, A_μ, are already taken to be the zero modes of the KK tower, and so depend only on the non-compact coordinates $x^0,\ldots,x^{d-1}$.

The original D-dimensional invariance under diffeomorphism has a remnant in this truncation of the theory. In particular, it is clear that we have d-dimensional diffeomorphism invariance acting on $x^0,\ldots,x^{d-1}$ (for which $G_{\mu\nu}$ is the graviton). There is an additional freedom to reparametrize the internal coordinate as

$$x'^d \,=\, x^d + \lambda(x^\mu) \tag{10}$$

The effect of this tranformation is to change the d-dimensional vector boson

$$A'_\mu \,=\, A_\mu \,-\, \partial_\mu\lambda \tag{11}$$

So gauge transformations of this vector boson follow from coordinate reparametrization in the internal dimension. This remarkable result (gauge invariance from diffeomorphism invariance in higher dimensions) was the original motivation for the Kaluza-Klein program of unification of interactions, which has motivated much of the modern research in extra dimensions.

Another field whose KK reduction we will be interested in is a D-dimensional 2-form B_{MN}. By an argument similar to the above one for the graviton, the result is a d-dimensional theory with a d-dimensional 2-form $B_{\mu\nu}$ and a $U(1)$ gauge boson $\hat{A}_\mu$. Just as above, gauge invariance of the D-dimensional 2-form inplies invariance of the d-dimensional 2-form under

$$B_{\mu\nu} \to B_{\mu\nu}\,\partial_{[\mu}\,\Lambda_{\nu]}(x^\lambda) \tag{12}$$

We will be interested in performing the KK reduction of the effective field theory for the light modes of the closed bosonic string. This includes a 26d graviton G_{MN}, a 26d scalar dilaton ϕ, and a 26d 2-form field B_{MN}.

As discussed in the overview lectures, the original action is

$$S_{\text{eff.}} = \frac{1}{2k_0^2}\int d^{26}X\,(-G)^{1/2}\,e^{-2\phi}\,\{\,R - \frac{1}{12}\,H_{MNP}\,H^{MNP} + 4\partial_M\phi\partial^M\phi\,\} + \mathcal{O}(\alpha') \tag{13}$$

where $H_{MNP} = \partial_{[M}B_{NP]}$.

Substitution of the 26d fields by the 25d zero modes of the KK tower, leads to the 25d effective action for the latter. Defining $G_{25,25} = e^{2\sigma}$, it is given by [2]

$$\begin{aligned} S_{25d} &= \tfrac{2\pi R}{2k_0^2}\int d^{25}X \quad (-G)^{1/2}\,e^{-2\phi+\sigma}\,[\,R - 4\,\partial_\mu\phi\partial^\mu\sigma + 4\partial_\mu\phi\partial^\mu\phi + \\ &\qquad -\frac{1}{4}\,e^{2\sigma}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{12}\,H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{1}{4}\,e^{2\sigma}\,\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\Big] = \\ &= \tfrac{2\pi R}{2k_0^2}\int d^{25}X \quad (-G)^{1/2}\,e^{-2\phi_{25d}}\,[\,R - 4\,\partial_\mu\sigma\partial^\mu\sigma + 4\partial_\mu\phi\partial^\mu\phi + \\ &\qquad -\frac{1}{4}\,e^{2\sigma}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{12}\,H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{1}{4}\,e^{2\sigma}\,\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\Big] \end{aligned} \tag{14}$$

[2]This combines eqs (8.1.9) and (8.1.13) in [1].

where $H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]} - A_{[\mu} \hat{F}_{\nu\lambda]}$, and where we have defined $\phi_{25d} = \phi - \sigma/2$, the effective 25d dilaton, which fixes the 25d interaction strength.

Notice that the vev for the scalar field $G_{25,25}$ is related to the radius of the internal circle. In fact, only the combination $\rho = Re^{\sigma}$ labels inequivalent theories. Therefore, the radius is not an external parameter, but the vev of a 4d dynamical scalar field. On the other hand, the compactification background is consistent (solves the D-dimensional equations of motion) no matter what circle radius we choose; this implies that in the d-dimensional effective action there is no potential for this scalar, it parametrizes what is called a flat direction of the potential. The field is called a modulus, and its vev parametrizes inequivalent vacua of the theory. The set of vevs for this modulus is called the moduli space (of circle compactifications).

A last important comment. It is interesting to notice that states carrying momentum in the circle direction are charged with respect to A_μ. This is because the global version of the corresponding gauge symmetry is a translation along x^d, hence the corresponding charge is internal momentum. This is a lower-dimensional remnant of the fact that the higher dimensional gravition couples to the energy momentum tensor. On the other hand, the original field theory did not have states charged under the 2-form field, hence the lower-dimensional theory does not have any states charged under the gauge boson $\hat{A}_\mu$. Later on we will see that string theory does contain such charged states.

3 Toroidal compactification in string theory

Let us discuss the circle compactification of the closed bosonic string in string theory language. Naively, to do that, we need to specify the worlsheet action for a string propagating [3] in $M_{25}\times\mathbf{S^1}$, by replacing the Minkowski metric in M_{26} in the Polyakov action by the metric in $M_{25}\times\mathbf{S^1}$. The puzzling feature is that the latter metric is also flat, locally a Minkowski metric as well, so the worldsheet action is still

$$S_{\rm P} \,=\, -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi\,(-g)^{1/2}\,g^{ab}(\sigma,t)\,\partial_a X^\mu\,\partial_b X^\nu \eta_{\mu\nu} \tag{15}$$

The difference between $M_{25}\times\mathbf{S^1}$ and M_{26} is a global effect, they have different topology although the local metric is the same for both. The effects of the compactification will arise not at the level of the local structure of the worldsheet, but in the boundary conditions we have to impose on the 2d worldsheet fields.

3.1 Quantization and spectrum

Indeed, the light-cone quantization can be carried out without change as in the uncompactified theory until we reach the hamiltonian

$$H \,=\, \frac{\ell}{4\pi\alpha' p^+}\int_0^\ell d\sigma\,[\,2\pi\alpha'\,\Pi_i\,\Pi_i \,+\, \frac{1}{2\pi\alpha'}\,\partial_\sigma X^i\,\partial_\sigma X^i\,] \tag{16}$$

In order to rewrite it in terms of oscillator modes, etc, we need to specify the boundary conditions obeyed by the 2d physical fields $X^i(\sigma,t)$. For X^i, $i=1,\ldots,24$, we need to impose

$$X^i(\sigma+\ell,t) \,=\, X^i(\sigma,t) \quad \text{for} \quad i=1,\ldots,24 \tag{17}$$

as usual. However, the fact that X^{25} parametrizes a circle of radius R means that X^{25} and $X^{25}+2\pi R$ correspond to the same point in spacetime. Hence, the following boundary condition defines a consistent closed string

$$X^{25}(\sigma+\ell,t) \,=\, X^{25}(\sigma,t) \,+\, 2\pi R\, w, \quad , \quad w\in\mathbf{Z} \tag{18}$$

[3] It is possible to work in general and finally show that consistency requires the total dimension of spacetime to be $D=26$ so we settle this from the start.

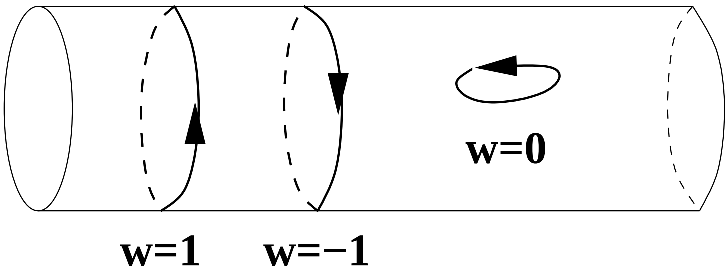

Figure 2: States representing closed strings winding around the compact dimension.

It corresponds to a closed string winding around the internal circle a number of times given by w, which is called the winding number, see fig 2. [4] . Each value of w corresponds to a different closed string sector. The complete spacetime 25d spectrum is given by the set of states of closed string in all possible winding sectors.

The existence of winding is possible only because strings are extended objects. The sector $w = 0$ corresponds to taking strings which are already closed without the compactification. These are the fields that appear in the approximation of compactifying the effective 26d field theory. We will see that for large radius states in non-zero winding sectors are very heavy, and

[4]It is amusing to notice that, from the viewpoint of the 2d theory, configurations of fields $X^i(\sigma,t)$ satisfying boundary conditions with non-zero winding correspond to solitonic states of the 2d field theory. The topological quantity associated to these solitons is the spatial integral of the derivative of the 2d field, namely $\int_0^\ell \partial_\sigma X^{25} = 2\pi R w$. As usual, solitons of a field theory are associated to non-trivial topology of the target space where the fields take values (recall that in the 't Hooft-Polyakov monopole, the existence of a soliton in the 4d theory was associated to the non-trivial topology of the space of vacua, namely the space where the Higgs field takes values). Please recall that here we are talking about solitons on the worldsheet, and have no relation at all with spacetime solitons.

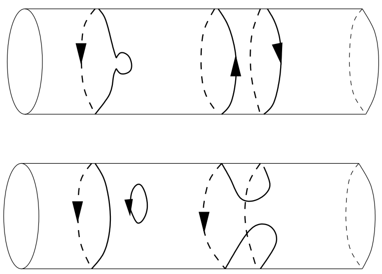

Figure 3: String interactions conserve winding number.

this is a good approximation. For small radius, non-zero winding state lead to very interesting surprises!

Winding number is conserved in string interactions, see figure 3

Since the X^i, $i = 2, \ldots, 24$ behave as usual, we only center on the analysis of X^{25}. The mode expansion for the boundary conditions (2) are

$$X^{25}(\sigma, t) \;=\; x^{25} \;+\; \frac{p_{25}}{p^+}\, t \;+\; \frac{2\pi R\, w}{\ell}\, \sigma \;+\; i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbf{Z} - \{\mathbf{0}\}} \left[\frac{\alpha_n^i}{n}\, e^{-2\pi i\, n\, (\sigma + t)/\ell} \;+\; \frac{\tilde{\alpha}_n^i}{n}\, e^{2\pi i\, n\, (\sigma - t)/\ell} \right] \quad (19)$$

Notice that the momentum must be quantized $p_{25} = k/R$, with $k \in \mathbf{Z}$ just like in the field theory discussion.

For future convenience, we may recast the expansion in terms of left and right movers $X^{25}(\sigma, t) = X_L^{25}(\sigma + t) + X_R^{25}(\sigma - t)$

$$\begin{aligned} X_L^{25}(\sigma + t) \;&=\; \tfrac{x^{25}}{2} \;+\; \tfrac{p_L}{2p^+}\,(t + \sigma) \;+\; i\sqrt{\tfrac{\alpha'}{2}}\, \textstyle\sum_{n \in \mathbf{Z} - \{\mathbf{0}\}} \tfrac{\alpha_n^i}{n}\, e^{-2\pi i\, n\, (\sigma + t)/\ell} \\ X_R^{25}(\sigma - t) \;&=\; \tfrac{x^{25}}{2} \;+\; \tfrac{p_R}{2p^+}\,(t - \sigma) \;+\; i\sqrt{\tfrac{\alpha'}{2}}\, \textstyle\sum_{n \in \mathbf{Z} - \{\mathbf{0}\}} \tfrac{\tilde{\alpha}_n^i}{n}\, e^{2\pi i\, n\, (\sigma - t)/\ell} \end{aligned} \quad (20)$$

with

$$p_L \;=\; \frac{k}{R} \;+\; \frac{wR}{\alpha'} \quad ; \quad p_R \;=\; \frac{k}{R} \;-\; \frac{wR}{\alpha'} \quad (21)$$

These will be called left and right moving momenta (although notice that each is a combination of the real spacetime momentum and winding).

The hamiltonian differs from the one in the non-compact situation only in the new contributions of winding terms to $\partial_\sigma X^{25}$. In terms of modes, etc, we obtain

$$
\begin{aligned}
H &= H_{w=0} + \frac{\ell}{4\pi\alpha' p^+} \int_0^\ell d\sigma \, \frac{1}{2\pi\alpha'} \, (\frac{2\pi R \, w}{\ell})^2 = \\
&= \sum_{i=2}^{24} \frac{p_i^2}{2p^+} + \frac{(k/R)^2}{2p^+} + \frac{R^2 w^2}{2\alpha'^2 p^+} + \frac{1}{\alpha' p^+} (N + \tilde{N} - 2)
\end{aligned}
\tag{22}
$$

where $H_{w=0}$ is the usual hamiltonian in the non-compact case. As usual, we build the Hilbert space of the theory by taking oscillator groundstates (each one labeled by a 25d momentum, a quantized momentum $k \in \mathbf{Z}$ in the circle, and a winding number) and applying oscillator creation operators to it.

The level matching constraint is $P = 0$ with

$$
\begin{aligned}
P &= \int_0^\ell d\sigma \, \Pi_i \, \partial_\sigma X^i = \frac{p^+}{\ell} \int_0^\ell d\sigma \, \partial_t X^i \, \partial_\sigma X^i = \\
&= P_{w=0} + \frac{p^+}{\ell} \, \ell \, \frac{k/R}{p^+} \, \frac{2\pi R w}{\ell} = \frac{2\pi}{\ell} (N - \tilde{N} + kw)
\end{aligned}
\tag{23}
$$

Each state corresponds to a particle in 25d spacetime. The 25d mass of the corresponding state is given by

$$
M_{25d}^2 = 2p^+ H - \sum_{i=2}^{24} p_i^2 \tag{24}
$$

We obtain

$$
M_{25d}^2 = \frac{k^2}{R^2} + \frac{R^2}{\alpha'^2} w^2 + \frac{2}{\alpha'} (N + \tilde{N} - 2) \tag{25}
$$

As mentioned above, for large R^2/α', the states with non-zero winding have large $\alpha' M^2$ and decouple. For not so large R^2/α', effects of winding states are very relevant and we cannot trust results obtained from the field theory approximation (namely, the physics obtained only from the $w = 0$

sector). Winding states, equivalentely α' effects, lead to important modifications of the physics, which can be regarded as important modification to how string theory feels the geometry when curvature lengths are as small as the string length (this is called stringy geometry for instance in the book by B. Greene).

For future convenience, we split the hamiltonian and mass in left and right handed pieces. We have $H = H_L + H_R$ with

$$\begin{aligned} H_L &= \frac{1}{4p^+}\left[\sum_{i=1}^{24} p_i^2 + p_L^2\right] + \frac{1}{\alpha' p^+}(N + E_0) \\ H_R &= \frac{1}{4p^+}\left[\sum_{i=1}^{24} p_i^2 + p_R^2\right] + \frac{1}{\alpha' p^+}(\tilde{N} + \tilde{E}_0) \end{aligned} \tag{26}$$

and $M^2 = M_L^2 + M_R^2$ with

$$\begin{aligned} M_L^2 &= \frac{p_L^2}{2} + \frac{2}{\alpha'}(N-1) \\ M_R^2 &= \frac{p_R^2}{2} + \frac{2}{\alpha'}(\tilde{N}-1) \end{aligned} \tag{27}$$

We see that one may carry out the quantization of the left and right moving coordinates independently, reach a mass formular for each side, and finally combine things together (satisfying the level matching constraint) at the end. This is only to re-emphasize the fact that in 2d the field theory of purely left-moving and purely right-moving fields make sense independently [5]. At a last stage, states of both theories are combined together to give physical states.

The level-matching constraint is

$$M_L^2 = M_R^2 \tag{28}$$

It is an easy exercise to obtain the one-loop partition function for this theory. For a two-torus worldsheet with geometry specified by τ_1, τ_2, we have

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}_{\mathrm{closed}}}\left[\, e^{-\tau_2 \ell H}\, e^{i\tau_1 \ell P}\,\right] =$$

[5] This observation will be crucial in the construction of heterotic string theorires.

$$= \sum_{k,w=-\infty}^{\infty} \mathrm{tr}_{\mathcal{H}_{k,w}} \left[e^{-\tau_2 \pi \alpha' \sum_{i=1}^{24} p_i^2} \, e^{-\tau_2 \pi \alpha' (k/R)^2} \, e^{-\tau_2 \pi R^2 w^2/\alpha'} \, e^{-2\pi\tau_2 (N+\tilde{N}-2)} \, e^{2\pi i \tau_1 (N-\tilde{N})} \, e^{2\pi i \tau_1 \, kw} \right]$$

Here $\mathcal{H}_{k,w}$ is the closed string sector with momentum k and winding number w. Most of this computation is already familiar, the only new piece is the contribution over discrete momenta and the windings. We get

$$Z(\tau) \;=\; |\eta(\tau)|^{-48} \, (2\pi\alpha'\tau_2)^{-23/2} \sum_{k,w=-\infty}^{\infty} \exp\left[-\pi\tau_2 \left(\frac{\alpha' k^2}{R^2} + \frac{R^2 w^2}{\alpha'} + 2\pi i \tau_1 \, kw \right) \right] \tag{29}$$

This expression is modular invariant. Invariance under $\tau \to \tau + 1$ is obvious, whereas invariance under $\tau \to -1/\tau$ can be shown by using Poisson resummation formula

$$\sum_{n\in\mathbf{Z}} \exp\left[-\pi A (n+\theta)^2 + 2\pi i \, (n+\theta)\phi \right] \;=\; A^{-1/2} \sum_{k\in\mathbf{Z}} \exp\left[-\pi A^{-1} (k+\phi)^2 - 2\pi i k \phi \right] \tag{30}$$

on both sums over k and w. It is interesting to point out that the sum over winding and momenta is almost invariant under $\tau \to -1/\tau$, except for picking up a factor of $(\tau\overline{\tau})^{1/2}$ which compensates for the lack of invariance of $|\eta(\tau)|^{-48} (\tau_2)^{-23/2}$.

It is important to point out that in string theory compactified on a circle, winding states are crucial in obtaining a modular invariant partition partition function. One intuitive way to argue about this is as follows. Consider starting with the partition function of the uncompactified theory

$$Z_{\text{uncomp.}} \;=\; \mathrm{tr}_{\mathcal{H}_{\text{uncomp.}}} \left[e^{-\tau_2 \ell H} \, e^{i\tau_1 \ell P} \right] \tag{31}$$

In order to describe the theory compactified on a circle, we may do by explicitly forcing that the only states that propagate are those invariant under translations of $2\pi R$ in X^{25}, by inserting the projector

$$\Pi \;=\; \sum_{w\in\mathbf{Z}} e^{iw 2\pi R \Pi_{25}} \tag{32}$$

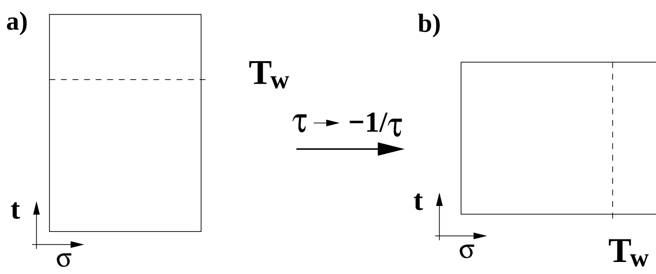

Figure 4: Under the modular transformation $\tau \to -1/\tau$, the roles of σ and t are exchanged. An insertion of T_w in the t (appearing from the insertion of the projector onto states invariant under discrete X^{25} translations) is mapped to an insertion of T_w in the σ direction, implying that we obtaine string closed up to translation in X^{25}, namely strings with winding w. Recall that sides of the rectangle are identified to make the worldsheet a two-torus.

in the trace. Here Π_{25} is the momentum operator, and $T_w = e^{i2\pi wR\Pi_{25}}$ translates X^{25} by $2\pi Rw$. The partition function is

$$Z_{\text{comp.}} = \sum_{w\in\mathbf{Z}} \text{tr}_{\,\mathcal{H}_{\text{uncomp.}}} [\, e^{-\tau_2 \ell H}\, e^{i\tau_1 \ell P}\, T_w\,] \tag{33}$$

This can be shown pictorially as in figure 4a. As the closed string propagates along the t direction, it crosses a cut along which the field $X^{25}(\sigma,t)$ jumps an amount $2\pi Rw$.

Under the modular transformation $\tau \to -1/\tau$, the roles of σ and t are exchanged, so the cut is found in the σ direction, as in figure 4b. Such picture represents a 1-loop diagram for a closed string which is closed up to a translation of the coordinate X^{25} by $2\pi Rw$, namely a closed string satisfying the boundary conditions (18). This means that to achieve a modular invariant partition function it is absolutely essential to add sectors with non-zero winding; namely, we have additional pieces

$$\sum_{w\in\mathbf{Z}} \mathrm{tr}\,_{\mathcal{H}_w}\left[\, e^{-\tau_2 \ell H_w}\, e^{i\tau_1 \ell P}\right] \tag{34}$$

where the trace is taken over the Hilbert space of string states in the sector of winding w.

Subsequently, we would have to enforce that in these new sectors the propagating modes are also invariant under translations of X_{25}, by introducing a projector. The total result is the double sum in k, w in (29). Sum in w sums over different sectors, whereas the sum in k projects onto states invariant under X^{25} translations.

3.2 α' effects I: Enhanced gauge symmetries

At large values of R, one easily recovers that the string spectrum reproduces the spectrum obtained using the field theory approximation. Indeed, winding states are very heavy, so only the $w = 0$ sector has a chance of being light. States with different k are merely KK replicas of the basic fields that exist in the 26d theory.

Forgetting the tachyon and its KK replicas (which can be lighter than $M^2 = 0$ for large enough R), the massless modes are $\alpha_{-1}^M \tilde{\alpha}_{-1}^N|0\rangle$, suitably decomposed according to whether $M, N = 25$, or $M, N = \mu$. Explicitly, we get

$$\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0\rangle \tag{35}$$

which are the 25d graviton, 2-form, and a scalar (from the trace). We also have

$$\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{25}|0\rangle \quad , \quad \alpha_{-1}^{25}\tilde{\alpha}_{-1}^{\mu}|0\rangle \tag{36}$$

two 25d gauge bosons. Taking symmetric and antisymmetric combinations, they are easily seen to arise from the 26d metric and 2-form, respectively. Hence the generic gauge symmetry in 25d is $U(1) \times U(1)$.

Finally we also have

$$\alpha_{-1}^{25} \tilde{\alpha}_{-1}^{25} |0\rangle \tag{37}$$

which is an additional scalar. This and the trace of (35) are the 25d dilaton and geometric moduli.

As in field theory, the charge of states under the gauge boson arising from the 26d graviton is given by their internal momentum, k. It is also easy to argue that the charge of states under the gauge boson arising from the 26d 2-form is given by their winding number w. Namely, starting from the coupling of a string to the 2-form field in 26d

$$\int_\Sigma B_M N \, \partial_a X^M \, \partial_b X^N \, \epsilon^{ab} \tag{38}$$

It is clear that we obtain a coupling of a string wrapped on $\mathbf{S^1}$ to the mixed component $B_{\mu,25}$,

$$\int dt \int_0^\ell d\sigma \, B_{\mu,25} \, \partial_\sigma X^{25} \, \partial_t X^\mu \simeq w \int dt \, \hat{A}_\mu \partial_t X^\mu \tag{39}$$

the state behaves as a 25d point particle coupling to $\hat{A}_\mu$ with charge w.

As announced before, as we let R approach the string length scale $L_s = \sqrt{\alpha'}$ new surprising features arise. In fact we can check that at $R = \sqrt{\alpha'}$ there appear new massless states from sectors of non-zero winding. The mass formulae in this point in moduli space are

$$\begin{aligned} \alpha' M_L^2 &= \frac{1}{2}(k+w)^2 + 2(N-1) \\ \alpha' M_R^2 &= \frac{1}{2}(k-w)^2 + 2(\tilde{N}-1) \end{aligned} \tag{40}$$

Denoting $|k,w\rangle$ the vacuum in the sector of momentum k and winding w, there are additional massless states, satifying the level matching condition (28).

We obtain four additional gauge bosons

$$\begin{array}{ccc} \alpha_{-1}^{\mu}|1,-1\rangle & , & \alpha_{-1}^{\mu}|-1,1\rangle \\ \tilde{\alpha}_{-1}^{\mu}|1,1\rangle & , & \tilde{\alpha}_{-1}^{\mu}|-1,-1\rangle \end{array}$$

One should recall that they are charged under the generic $U(1)\times U(1)$ gauge symmetry, with charges given precisely by the pairs (k,w). The total gauge group is non-abelian and it is in fact $SU(2)^2$.

We also obtain eight new additional massless scalars

$$\begin{array}{c} \alpha_{-1}^{25}|1,-1\rangle \quad , \quad \alpha_{-1}^{25}|-1,1\rangle \\ \tilde{\alpha}_{-1}^{25}|1,1\rangle \quad , \quad \tilde{\alpha}_{-1}^{25}|-1,-1\rangle \\ |2,0\rangle \quad , \quad |-2,0\rangle \quad , \quad |0,2\rangle \quad , \quad |0,-2\rangle \end{array} \tag{41}$$

Checking the charges under the generic $U(1)^2$ symmetry, it is possible to see that these scalars, along with the radial modulus (37) transform in the representation $(\mathbf{3},\mathbf{3})$ of $SU(2)\times SU(2)$. The set of charges for the gauge bosons, and the scalars are shown in figure 5, and can be seen to correspon to roots of $SU(2)^2$ and weights of $(3,3)$.

This is a very surprising effect. For a particular value of the compactification radius $R=\sqrt{\alpha'}$, stringy effects (namely the existence of winding) generate an enhanced gauge symmetry in spacetime (enhanced as compared with the symmetry at a generic value of R). Indeed a dramatic effect! This mechanism of generating gauge bosons goes well beyond what was achievable from the field theory KK mechanism.

Of course it is possible to cook up a new 25d effective field theory by including by hand the new massless modes. So this effective field theory

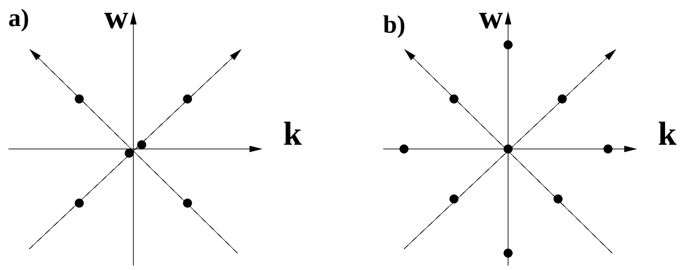

Figure 5: Charges of gauge bosons (a) and scalars (b) at the enhanced symmetry point $R = \sqrt{\alpha'}$. The charges tell us that the gauge bosons fill out a $SU(2) \times SU(2)$ group (the roots of each $SU(2)$ factor point along the dashed lines), whereas the scalars fill out a representation $(3,3)$ of $SU(2)^2$.

would contain gravity and non-abelian $SU(2)^2$ gauge interactions, and a bunch of 9 scalars transforming in the representation $(3,3)$ coupled to these gauge bosons. It is important to understand two facts:

• This effective field theory is *not* derived from the 26d effective field theory by compactification; we know that the latter missed the crucial issue of winding states, and is a good approximation at large R, and not at $R = \sqrt{\alpha'}$

• This effective field theory is a good approximation to the 25d physics for R close to $\sqrt{\alpha'}$. As we will see shortly, going away from $R = \sqrt{\alpha'}$ makes some fields massive, so for R very different from $\sqrt{\alpha'}$ these masses are too large and it is not a good idea to include the corresponding fields in the effective field theory.

It is interesting to understand what happens when we vary slightly the value of R away form the value $\sqrt{\alpha'}$. Since we have solved the string states for all values of R, we simply read off the mass formulae and see that the additional gauge bosons, as well as the additional scalars get masses (proportional to the deviation of R and $\sqrt{\alpha'}$.

This sounds very much like a Higgs mechanism, with gauge bosons becoming massive and some scalars being eaten and becoming the longitudinal components of the massive vector bosons. Indeed this is correct: for small departures from $R = \sqrt{\alpha'}$ the 25d effective field theory language should be appropriate and the breaking of the gauge group is just a Higgs mechanism triggered by the scalars in the $(3,3)$.

A finer point is that the number of scalars that disappears is larger than the number of gauge bosons becoming massive. This is however consistent. Out of the original 9 massless scalars, 4 of them are eaten by the 4 gauge bosons associated to the broken generators, 1 of the remaining remains massless (and is interpreted as the geometric modulus parametrizing R), and the 4 remaining become massive due to couplings between them and the scalars picking up a vev.

As discussed by Polchinski (around eq (8.3.22), organizing the 9 scalars in a 3×3 matrix M_{ij}, the scalar potential for the theory at $R = \sqrt{\alpha'}$ includes an $SU(2)^2$ invariant term

$$V(M) = \epsilon^{ijk}\, \epsilon^{i'j'k'}\, M_{ii'}\, M_{jj'}\, M_{kk'} \tag{42}$$

Giving a vev to one of the scalars, say M_{33}, we generate mass terms

$$\epsilon^{ij} \epsilon^{i'j'}\, M_{ii'}\, M_{jj'} \tag{43}$$

for $i, i', j, j' = 1, 2$. Namely four fields become massive due to the scalar potential.

A tantalizing (but more advanced) comment is that the field that has received the vev has flat potential, so it is a modulus, and parametrizes the deviation of R from $\sqrt{\alpha'}$. So it is what we have called the geometric modulus. Increasing the vev for this field would eventually lead us into the large volume regime.

However notice that in principle any of the 9 fields in M_{ij} can be the one in getting the vev. They are in the same $SU(2)^2$ multiplet, so gauge invariance tells us that none of these fields is priviledged. Therefore, starting from the enhanced symmetry point, there seem to exist different regimes which can be interpreted as large volume regimes in suitable variables. This will become clearer after we study next section.

3.3 α' effects II: T-duality

The existence of winding states in string theory leads to another amazing surprise. Recall the mass formula (44)

$$M_{25d}^2 = \frac{k^2}{R^2} + \frac{R^2}{\alpha'^2} w^2 + \frac{2}{\alpha'} (N + \tilde{N} - 2) \tag{44}$$

It is invariant under the so-called T-duality transformation

$$R \to \frac{\alpha'}{R} \quad ; \quad k \leftrightarrow w \tag{45}$$

Namely the complete spectrum of the theory at radius R is the same as the spectrum of the theory at radius α'/R, up to a relabeling of k and w.

This is extremely striking. If we are 25d observers and measure the spectrum of states, we would be unable to distinguish whether it is coming from a string theory compactified on a circle of radius R or α'/R.

Striking again! The theory at large $R \to \infty$ has infinite towers of momentum states becoming massless (the KK step $1/R$ is very small); this is a typical signal of a decompactification limit. On the other hand, in the T-dual theory the radius is going to zero $R' = \alpha'/R \to 0$, and we still recover infinite towers of states becoming massless, but now they are coming from string with winding number w (since the T-dual circle is small, it costs almost no energy to increase the winding number). So the small R limit looks also as a decompactification limit, and it *is* a decompactification limit in T-dual language!

One might thing that this puzzling feature is not a property of full-fledged string theory, but just an accidental property of the spectrum. This is not correct, and one can show that string interactions also respect T-duality. T-duality is the complete physical equivalence of the theories compactified on circles of radius R and α'/R.

In other words, both theories are described by exactly the same worldsheet theory, and differ on how the spacetime coordinates (the spacetime geometry) is recovered from the 2d worldsheet theory.

To be more specific, it is convenient to describe our worldsheet theory as given by two sets of 2d fields $X_L^i(\sigma+t)$ and $X_R^i(\sigma-t)$, which are decoupled. Now there are two ways to construct the true spacetime coordinates $X^i(\sigma,t)$ out of them. One possibility is

$$\begin{aligned} X^i(\sigma,t) &= X_L^i(\sigma+t)+X_R^i(\sigma-t) \quad ; \quad i=2,\ldots,24 \\ X^{25}(\sigma,t) &= X_L^{25}(\sigma+t)+X_R^{25}(\sigma-t) \end{aligned} \tag{46}$$

whereas there is another

$$\begin{aligned} X^i(\sigma,t) &= X_L^i(\sigma+t)+X_R^i(\sigma-t) \quad ; \quad i=2,\ldots,24 \\ X^{25}(\sigma,t) &= X_L^{25}(\sigma+t)-X_R^{25}(\sigma-t) \end{aligned} \tag{47}$$

The relation between one and the other is

$$p_L^{25} \to p_L^{25} \quad ; \quad p_R^{25} \to -p_R^{25} \quad ; \tag{48}$$

which corresponds to the T-duality transformation (45).

The implications of this are difficult to overemphasize. It certainly suggests that spacetime is a secondary concept in string theory, and that it is derived from more fundamental concepts like the worldsheet theory. What this means for our understanding of the nature of spacetime in string theory is still unclear.

A final comment we would like to make in this respect is that T-duality is in fact a $\mathbf{Z_2}$ remnant of a gauge symmetry. Indeed, there is a value of R for which the theory is self-dual, this is our old friend $R = \sqrt{\alpha'}$. At this point, the complete spectrum is invariant under $k \leftrightarrow w$.

It is also easy to see that the effect of this transformation is nothing but a gauge transformation within the enhanced gauge group $SU(2)^2$. Finally, it is possible to see that two T-dual deviations from $R = \sqrt{\alpha'}$ are mapped to each other by a relabeling transformation which is a subroup of this group: indeed, regarding $SU(2)$ as $SO(3)$ (the rotation group in 3d) a rotation of π around the axis distinguished by the field getting a vev (the direction 3 if $M_{3,i'}$ gets the vev) in the first $SO(3)$ has the effect of mapping the vev for one of the modulus to its negative. Hence maps a deformation toward $R > \sqrt{\alpha'}$ to a deformation towards $R < \sqrt{\alpha'}$.

This means that two T-dual theories are identified by a gauge transformation, so should not be considered as really different. Hence the moduli space of compactification is not really parametrized by the real line (i.e. possible values of R) but rather by the real line modulo $R \to 1/R$. The moduli space can therefore be described (with no redundancy) by the set of points $R > \sqrt{\alpha'}$.

Again this has amazing implications, since it suggests the existence of a minimum distance in string theory. These issues must be taken with a grain of salt, however, since in the study of D-branes the community has realized that there exist other objets in string theory which are able to probe distances much shorter than L_s [2].

We see that even the simplest compactification is rich enough to illustrate the amazing features of string theory regarding the nature of spacetime.

3.4 Additional comments

Let us conclude by pointing out some generalizations of the concepts we have studied in toroidal compactifications

- Toroidal compactification of more than one dimension

  This is studied nicely enough in section 8.4 in [1]. One can proceed in analogy with the circle case. Some of the new features of this situation are the appearance of scalars from the KK reduction of the 26d 2-form. They have flat potential and are new moduli from the viewpoint of the lower-dimensional theory, characterizing the background B-field in the internal space. The complete moduli space (without taking into account dualities) is called Narain moduli space and is described as a coset

  $$\frac{O(k,k,\mathbf{R})}{O(k,\mathbf{R})\times \mathbf{O(k,R)}} \qquad (49)$$

  The set of T-dualities is larger, and is given by the group $O(k,k,\mathbf{Z})$, so the true moduli space is

  $$\frac{O(k,k,\mathbf{R})}{O(k,\mathbf{R})\times \mathbf{O(k,R)}\times \mathbf{O(k,k,Z)}} \qquad (50)$$

  A standard reference on all these issues is [3].

- Buscher's T-duality

  The existence of T-dual configuration does not require spacetime to be a cartesian product with one factor given by a circle. In fact, T-duality can be extended to geometries with one Killing vector with compact orbits (with finite length, at least asymptotically). Buscher's formulae provide the backgroud obtained by applying T-duality along the orbits of this Killing vector. Surprisingly T-duality is even able to relate geometries with different topology.

- Compactification on non-toroidal geometries

  Although this can be considered in bosonic string theory, it has found more applications in the supertring context. We will discuss some of this for heterotic string theories in later lectures.

References

[1] J. Polchinski, 'String theory', Vol 1.

[2] S. H. Shenker, 'Another length scale in string theory?', hep-th/9509132; see also the D-brane scattering discussion in 'D-brane dynamics', by C. Bachas, Phys. Lett. B374 (1996) 37, hep-th/9511043.

[3] A. Giveon, M. Porrati, E. Rabinovici, 'Target space duality in string theory', Phys. Rept. 244 (1994) 77, hep-th/9401139.