Modular invariance

1 Generalities

In this Section we mainly follow the line of thought of section 7.3 in [1]. Our computation is however done in the light-cone gauge.

In this lecture we discuss the simplest case where we can witness the remarkable finiteness properties of string theory. The example is provided by the 1-loop vacuum amplitude. It corresponds to a worldsheet diagram for a closed string moving in a circle and closing onto itself, so it has the topology of a two-torus with no insertions of external lines. It represents the 1-loop amplitude of the vacuum going to vacuum process (in spacetime). See figure 1

We know from the overview lectures that the amplitude is obtained by summing over *all possible inequivalent* worldsheet geometries with two-torus topology.

It is crucial to incorporate all possible geometries, and not to double-count equivalent geometries. Concerning this, it is extremely important to realize that a given geometry can receive two different interpretations. A diagram corresponding to a two-torus with circle lengths ℓ_1 and ℓ_2 can be regarded as

1) A closed string of length ℓ_1 propagating over a distance ℓ_2

2) A closed string of length ℓ_2 propagating over a distance ℓ_1

The two processes, although look different, correspond to the same geometry, so should be counted only *once*. This will be crucial later on.

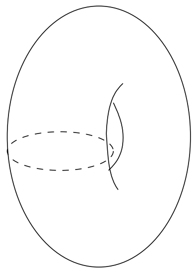

Figure 1: One-loop diagram for the vacuum going to vacuum process.

2 Worldsheet coordinatization in light-cone gauge

Recall our recipe to compute amplitudes. First we sum over geometries of an abstract worldsheet Σ with two-torus topology. Second, for each such geometry we sum over possible configurations of the 2d dynamical fields in Σ (in the light cone gauge, the transverse fluctuations $X^i(\sigma, t)$).

Recall that in the light cone gauge we have **1)** a coordinate σ which parametrizes a direction of fixed length ℓ; **2)** a coordinate t which is locally orthogonal to σ at every point; **3)** a Hamiltonian for the physical degrees of freedom, generating evolution in t for the 2d system. In terms of oscillator and center of mass momentum

$$H = \frac{\sum_i p_i^2}{2p^+} + \frac{1}{\alpha' p^+} [\, L_0 + \tilde{L}_0 \,] \tag{1}$$

with

$$L_0 = \sum_i \left[\sum_{n>0} \alpha^i_{-n} \alpha^i_n + E^i_o \right] \quad , \quad \text{and } E^i_0 = -\frac{1}{24} \tag{2}$$

and similarly for $\tilde{L}_0$.

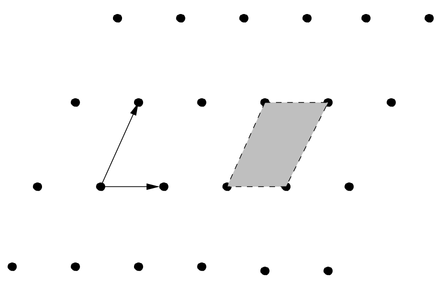

Figure 2: A two-torus can be constructed by modding out the two-dimensional plane by translations in a two-dimensional lattice. The unit cell is a parallelogram with sides identified. Each vector corresponds to a non-contractible cycle in the two-torus

A two-torus can be described as the two-dimensional real plane, modded out by translations by vectors in a two-dimensional lattice, see figure 2

There is a more or less obvious set of worldsheet geometries which we should consider. It is shown in figure 3a), and corresponds to a closed string (of σ-length ℓ) evolving for $t = \tau_2 \ell$ (for $\tau_2 > 00$ and closing back onto itself.

Denoting $z = \sigma + i\,t$, the two-torus is defined by the identifications $z \equiv z + \ell$, $z \equiv z + \tau_2 \ell$.

However, there are more general possibilities, as shown in figure 3b), corresponding to a closed string of length ℓ evolving for $t = \tau_2 \ell$, and gluing back to the original state up to a change in the reference line $\sigma = 0$ (given by a translation by $\tau_1 \ell$ in the σ-direction). Since there is no preferred choice of the reference line, as discussed in the previous lecture, this is an allowed possibility. The geometry corresponds to a two-torus defined by the identifications $z \equiv z + \ell$ and $z \equiv z + \tau \ell$, with $\tau = \tau_1 + i\tau_2$. The parameter τ is called the complex structure of the two-torus, for reasons not very relevant here.

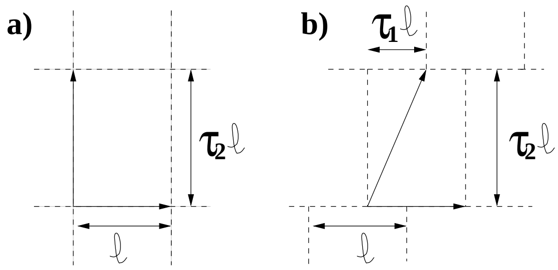

Figure 3: Figure a) shows an obvious class of worldsheet geometries with two-torus geometries, a closed string of length ℓ evolves for some time $t = \tau_2 \ell$ and closes back to the initial state. Figure b) shows the more general class, where the closed string is glued back to the original state modulo a change in the reference line in σ.

3 The computation

3.1 Structure of the amplitude in operator formalism

We have to sum over all possible configurations of 2d physical fields $X^i(\sigma, t)$ for a given 2d geometry. In operator formalism, this amounts to considering the complete set of quantum 2d states at a given time (i.e. the Hilbert space of the 2d theory), apply evolution in t for a total time of $t = \tau_2 \ell$ and glue the resulting state to the initial one (modulo a σ-translation by $\tau_1 \ell$). The amplitude for two-torus geometry corresponding to τ is therefore

$$Z(\tau) \;=\; \sum_{\text{states}} \langle \text{st.}|\, e^{-\tau_2 \ell H}\, e^{i\tau_1 \ell P}\, |\text{st.}\rangle \tag{3}$$

where P is the generator of translations along σ

$$P \;=\; \int_0^\ell d\sigma\, \Pi_i\, \partial_\sigma X^i \;=\; \frac{2\pi}{\ell}\,(L_0 - \tilde{L}_0) \tag{4}$$

(namely $\partial_\sigma X^i$ gives the amoung of X shift induced by the σ-translation, and Π_i implements the effect of the X shift on the Hilbert space).

The amplitude hence corresponds to taking a trace over the Hilbert space $\mathcal{H}_{\rm cl.}$ of the closed string 2d theory

$$\begin{aligned} Z(\tau) &= \operatorname{tr}_{\mathcal{H}_{\rm cl.}} \left(e^{-\tau_2 \ell H} e^{i\tau_1 \ell P}\right) = \\ &= \operatorname{tr}_{\mathcal{H}_{\rm cl.}} \left(\exp[-\tau_2 2\pi\alpha' p^+ [\, \frac{\sum p_i^2}{2p^+} + \frac{1}{\alpha' p^+}(L_0 + \tilde{L}_0)\,]\, \exp[2\pi i \tau_1 (L_0 - \tilde{L}_0)]\right) = \\ &= \operatorname{tr}_{\mathcal{H}_{\rm cl.}} \left(\exp[-\tau_2 \pi\alpha' \sum p_i^2]\, \exp[2\pi i(\tau_1 + i\tau_2) L_0]\, \exp[2\pi i (\tau_1 - i\tau_2)\tilde{L}_0]\right) = \end{aligned} \tag{5}$$

Defining $q = e^{2\pi i \tau}$, we have

$$Z(\tau) = \operatorname{tr}_{\mathcal{H}_{\rm cl.}} \left(\exp[-\tau_2 \pi\alpha' \sum p_i^2]\, q^{L_0}\, \overline{q}^{\tilde{L}_0}\right) \tag{6}$$

Then we should sum over geometries, i.e. integrate over τ. Notice that when we integrate over τ_1 the level-matching constraint $L_0 = \tilde{L}_0$ is automatically implemented

$$\int d\tau_1\, e^{2\pi i \tau_1 (L_0 - \tilde{L}_0)} \simeq \delta_{L_0, \tilde{L}_0} \tag{7}$$

Hence, we can take the trace over the unconstrained set states constructed by applying arbitrary numbers of all possible left and right oscillators to the vacuum. Subsequently the sum over geometries will implement that only physical states, satisfying the level matching constraint, propagate.

Hence the general structure of the states we are tracing over is

$$\prod_{n,i} (\alpha_{-n}^i)^{K_{n,i}} \prod_{m,j} (\tilde{\alpha}_{-m}^j)^{\tilde{K}_{m,j}}\, |p_-, p_i\rangle \tag{8}$$

That is, the Hilbert space is given by a set of momentum states, on which we apply an arbitrary number of times K, $\tilde{K}$ oscillator creation operators out of an infinite set labeled by n, i, m, j.

3.2 The momentum piece

The trace over center of mass degrees of freedom give an overall factor independent of the oscillator ocuppation numbers $K_{n,i}$, $K_{m,j}$. Moreover, the center of mass trace factorizes as product of traces over different directions

$$tr_{\text{c.m.}}\, e^{-\tau_2 \pi \alpha' \sum_i p_i^2} \;=\; (\text{tr}_{\text{ c.m.1d}}\, e^{-\tau_2 \pi \alpha' p^2})^{24} \tag{9}$$

For each direction, we can take the trace by summing over (center of mass) position eigenstates

$$\begin{aligned} \text{tr}_{\text{ c.m.1d}}\, e^{-\tau_2 \pi \alpha' p^2} \;&=\; \textstyle\int dx\, \langle x|\, e^{-\tau_2 \pi \alpha' p^2}\, |x\rangle \;= \\ \textstyle\int dx \int \frac{dp}{2\pi} \langle x|p\rangle\, \langle p|\, e^{-\tau_2 \pi \alpha' p^2}\, |p\rangle\, \langle p|x\rangle \;&=\; (\textstyle\int dx)\, (4\pi^2 \alpha' \tau_2)^{-1/2} \end{aligned} \tag{10}$$

Hence

$$tr_{\text{c.m.}}\, e^{-\tau_2 \pi \alpha' \sum_i p_i^2} \;=\; V_{24}\, (4\pi^2 \alpha' \tau_2)^{-12} \tag{11}$$

where V_{24} is a regularized volume of the 24d transverse space.

3.3 The oscillator piece

The oscillator creation operators can be applied independently, so the trace factorizes in traces over the Hilbert space of each independent oscillator.

For a single oscillator, the trace over states $(\alpha^i_{-n})^K|0\rangle$ goes like

$$\begin{aligned} \text{tr}\, q^{\hat{N}+E_0} \;&=\; q^{E_0} \textstyle\sum_{K=0}^{\infty} \langle 0|\, (\alpha^i_n)^K\, q^{\hat{N}} (\alpha^i_{-n})^K\, |0\rangle \;= \\ &= q^{E_0}\, \textstyle\sum_{K=0}^{\infty}\, q^{Kn} \;=\; q^{-1/24}\, \frac{1}{1-q^n} \end{aligned} \tag{12}$$

For two oscillators, the trace over states $(\alpha_{-n_1})^{K_1}(\alpha_{-n_2})^{K_2}|0\rangle$ is

$$\begin{aligned} \text{tr}\; q^{\hat{N}+E_0} \;&=\; q^{-2/24} \sum_{K_1,K_2=0}^{\infty} \langle 0|\, (\alpha_{n_1})^{K_1}\, (\alpha_{n_2})^{K_2}\, q^{\hat{N}_1+\hat{N}_2}\, (\alpha_{-n_1})^{K_1}\, (\alpha_{-n_2})^{K_2}\, |0\rangle \;= \\ &=\; q^{-2/24} \sum_{K_1,K_2=0}^{\infty} \langle 0|\, (\alpha_{n_1})^{K_1}\, q^{\hat{N}_1}\, (\alpha_{-n_1})^{K_1}\; (\alpha_{n_2})^{K_2}\, q^{\hat{N}_2}\, (\alpha_{-n_2})^{K_2}\, |0\rangle \;= \\ &=\; q^{-2/24}\, (1-q^{n_1})^{-1}\, (1-q^{n_2})^{-1} \end{aligned} \tag{13}$$

So for the infinite set of left and right oscillators

$$\mathrm{Tr}\; q^{L_0}\,\overline{q}^{\tilde{L}_0} \;=\; q^{E_0}\,\overline{q}^{\tilde{E}_0} \prod_{i=2}^{26}\prod_{n=1}^{\infty}(1-q^n)^{-1} \;\prod_{j=2}^{26}\prod_{m=1}^{\infty}(1-q^m)^{-1} \;=\; \left| q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\right|^{-48} \qquad (14)$$

Using the definition of the Dedekind eta function (30)

$$\eta(\tau) \;=\; q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad (15)$$

the complete partition function, for fixed τ, is

$$Z(\tau) \;=\; V_{24}\,(4\pi^2\alpha'\tau_2)^{-12}\,|\eta(\tau)|^{-48} \qquad (16)$$

4 Modular invariance

4.1 Modular group of $\mathbf{T}^2$

To obtain the complete partition function we should sum over all inequivalent geometries. As we have discussed, it is crucial not to overcount geometries. Since we have characterized the worldsheet geometry in terms of τ, it is crucial to realize that there exist different values of τ which nevertheless correspond to the *same* geometry.

i) For instance, as shown in figure 4, two two-tori corresponding to τ and $\tau+1$ are defined by the same lattice on the 2-plane, hence correspond to the same two-torus geometry.

ii) A slightly trickier equivalence is that of two two-tori with complex structure parameters τ and $-1/\tau$. Let us verify this in the simpler case of $\tau_1 = 0$; in this case we have the equivalence of τ_2 and $1/\tau_2$. This is shown in figure 5: the two-torus with parameter i/τ_2 is equivalent to that with parameter $i\tau_2$, up to an exchange of the roles of σ and t, and a rescaling to ensure that the total length of the new σ coordinate is ℓ.

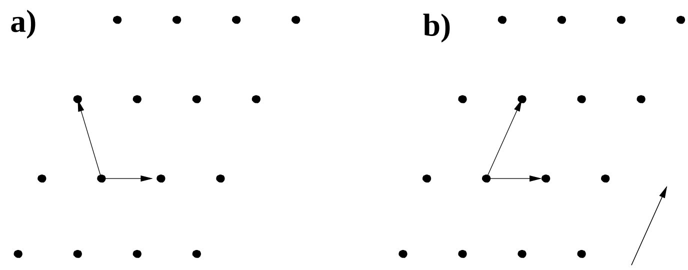

Figure 4: The two-tori corresponding to τ and $\tau + 1$ correspond to the same two-dimensional lattice of translation, hence are the same two-torus.

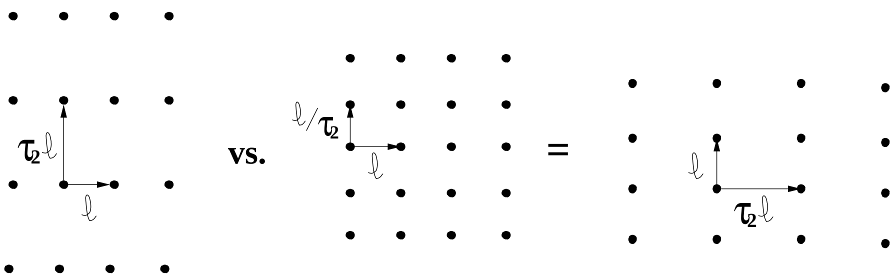

Figure 5: The geometry of two two-tori with parameters $i\tau_2$ and i/τ_2 is the same, as can be seen by exchanging the roles of σ and t and performing a rescaling of coordinates.

Two two-tori with parameters τ and $-1/\tau$ are simply related by the exchange of the roles of the two basis vectors generating the two-dimensional lattice.

In other words, there exist different choices of τ which lead to the same geometry, namely two two-tori which can be related by coordinate changes on the worldsheet

Denoting $z = \sigma + i\,t$, the two torus geometrical parameter τ is specified by the periodic identifications

a) $\sigma \to \sigma + \ell$, $t \to t$ which gives $z \to z + \ell$

b) $\sigma \to \sigma + \tau_1 \ell$, $t \to t + \tau_2 \ell$ which gives $z \to z + \tau \ell$

Performing a change of variables

$$\sigma' = \sigma + t/\tau_2 \quad ; \quad t' = t \tag{17}$$

The two-torus is defined in terms of the identifications

a) $\sigma \to \sigma + \ell$, $t \to t$, which gives $\sigma' \to \sigma' + \ell$, $t' \to t'$, namely $z' \to z' + \ell$

b) $\sigma \to \sigma + \tau_1 \ell$, $t \to t + \tau_2 \ell$, which gives $\sigma' \to \sigma' + (\tau_1 + 1)\ell$, $t' \to t' + \tau_2 \ell$, namely $z' \to z' + (\tau + 1)$

So in these coordinates the two-torus has parameter $\tau + 1$.

Performing instead a change of variables

$$\sigma' = \frac{\tau_2 t + \tau_1 \sigma}{\tau_1^2 + \tau_2^2} \quad ; \quad t' = \frac{\tau_1 t - \tau_2 \sigma}{\tau_1^2 + \tau_2^2} \tag{18}$$

the two-torus is defined in terms of the identifications

a) $\sigma \to \sigma - \ell$, $t \to t$, which gives $\sigma' \to \sigma' + \tau_1' \ell$, $t' \to t' + \tau_2' \ell$, namely $z' \to z' + \tau' \ell$ with $\tau' = -1/\tau$

b) $\sigma \to \sigma + \tau_1 \ell$, $t \to t + \tau_2 \ell$, which gives $\sigma' \to \sigma' + \ell$, $t' \to t'$, namely $z' \to z' + \ell$.

So in these coordinates the two-torus has parameter $-1/\tau$.

This shows that the geometries corresponding to values of τ related by the transformations $\tau \to \tau + 1$, $\tau \to -1/\tau$ are equivalent up to coordinate changes, diffeomorphisms. It is important to notice that the diffeormorphisms involved are 'large', that is they are not continuously connected to the identity (they involve drastic things like exchanging the roles of σ, t; however, they are simply coordinate changes).

The set of transformations of τ which leaves the geometry invariant has the structure of a group, called the modular group of the two-torus. By composing the transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$, the most general tranformation is of the form

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \text{with } a, b, c, d \in \mathbf{Z} \quad \text{and } \mathbf{ad} - \mathbf{bc} = \mathbf{1} \tag{19}$$

The parameters a, b, c, d can be written as a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of integer entries and unit determinant. The group is therefore $SL(2, \mathbf{Z})$.

The set of inequivalent geometries is therefore characterized by the parameter τ in the upper half complex plane (recall we had $\tau_2 > 0$, modulo $SL(2, \mathbf{Z})$ transformations. A choice of fundamental domain of τ is shown in figure 6

$$-1/2 \leq \tau_1 < 1/2 \quad , \quad |\tau| \leq 1 \tag{20}$$

The set of points in F_0 correspond to the set of all possible two-torus geometries. Integrating τ over F_0 corresponds to summing over two-torus geometries with no overcounting.

4.2 Modular invariance of the partition function

The closed bosonic string partition function $Z(\tau)$ should be the same for equivalent tori, since it should be invariant under reparametrizations of the

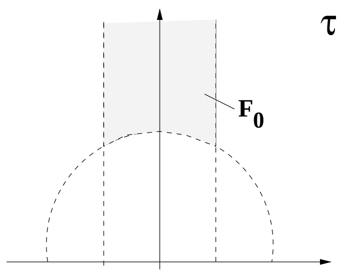

Figure 6: Fundamental domain of τ. Any point in the upper half plane can be mapped to some point in F_0 using the basic modular transformations $\tau \to \tau + 1$, $\tau \to -1/\tau$.

worldsheet. So $Z(\tau)$ should be modular invariant, i.e. $SL(2, \mathbf{Z})$ invariant. This is not completely obviour, since the diffeomorphisms involved in reparametrizations changing τ by modular transformations are not small, so in principle our gauge fixing procedure (good for 'small' diffeomorphism, continuously connected to the identity) is not good enough to take care of them [1].

Happily, using the modular tranformation properties of Dedekind's eta function (31), we find that

$$Z(\tau) \simeq \tau_2^{-12} |\eta(\tau)|^{-48} \stackrel{\tau \to \tau+1}{\longrightarrow} \tau_2^{-12} |\eta(\tau)|^{-48} \tag{21}$$

$$Z(\tau) \simeq \tau_2^{-12} |\eta(\tau)|^{-48} \stackrel{\tau \to -1/\tau}{\longrightarrow} \frac{(\tau_1^2 + \tau_2^2)^{12}}{\tau_2^{12}} \frac{1}{|\tau|^{24} |\eta(\tau)|^{48}} = \tau_2^{-12} |\eta(\tau)|^{-48}$$

It is modular invariant! From the viewpoint of the way we computed $Z(\tau)$, invariance under e.g. $\tau \to -1/\tau$ is remarkable: The sum over all states of a

[1] We may say that, since even within our gauge fixing we still encounter the same geometry for different values of τ, our gauge fixing slices are passing through each gauge orbit more than once. If the value of Z is the same in each such point, we may by hand just keep one of them. If not, then the theory is not invariant under large diffeomorphisms, it does not have a consistent worldsheet geometry.

string along σ propagating in t is the same as the sum over all states of the string in the dual channel, a string along t and propagating in σ. Strinking conspiracy of the sum over the string tower... From another viewpoint, it is just a simple consequence of the geometry of the worldsheet. The amplitude is a function of the worldsheet geometry, and gives the same number for different values of τ that correspond to the same intrinsic geometry.

The complete vacuum amplitude is obtained by summing over inequivalent geometries, that is restricting to integrating τ over F_0

$$Z \;=\; \int_{F_0} \frac{d^2\tau}{4\tau_2}\, (4\pi^2\alpha'\tau_2)^{-12}\, |\eta(\tau)|^{-48} \tag{22}$$

where $d^2\tau/(4\tau_2)$ is an $SL(2,\mathbf{Z})$ invariant measure in the space of two-tori geometries (the so-called Teichmuller space). It is easy to check this invariance by hand.

4.3 UV behaviour of the string amplitude

It is now time to study the UV behaviour of this amplitude. To understand better the nice UV properties of string theory, it is useful to obtain the vacuum to vacuum amplitude in a theory of point particles. In a theory of one point particle of mass m in D dimensions, the amplitude of a diagram given by a circular worldline of length l is

$$Z_m \;=\; V_d \int \frac{d^D k}{(2\pi)^D} \int_0^\infty \frac{dl}{2l}\, e^{-(k^2+m^2)l/2} \tag{23}$$

with $(k^2+m^2)/2$ the worldline hamiltonian, and $dl/(2l)$ the measure in the space of circle geometries, with the denominator $2l$ removing the freedom of translation plus inversions of the circle. We have

$$Z_m \;=\; iV_d \int_0^\infty \frac{dl}{2l}\, (2\pi l)^{-D/2}\, e^{-m^2 l/2} \tag{24}$$

For any $D > 0$ this amplitude is divergent in the UV, as $l \to 0$. On the other hand, it is IR convergent if $m^2 > 0$.

One could imagine that string theory is just a theory with an infinite number of particles in spacetime. That is not really true, in a very subtle way which we will see below. If that were true, then the vacuum to vacuum amplitude in string theory would be just the sum of contributions like (24) for all particles in the string tower. Using that the mass of a string state is given by $m^2 = 2/\alpha'(L_0 + \tilde{L}_0)$ we have

$$Z' = iV_d \int_0^\infty \frac{dl}{2l} (2\pi l)^{-D/2} \, \mathrm{tr}_{\mathcal{H}} \, e^{-l/\alpha'(L_0+\tilde{L}_0)} \tag{25}$$

We prefer to sum over the extended Hilbert space of the theory by not requiring directly $L_0 = \tilde{L}_0$, and rather imposing this constraint by hand via a delta function

$$\delta_{L_0,\tilde{L}_0} = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{2\pi} \, e^{i(L_0-\tilde{L}_0)\theta} \tag{26}$$

to get

$$Z' = iV_d \int_0^\infty \frac{dl}{2l} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{2\pi} (2\pi l)^{-D/2} \, \mathrm{tr}_{\mathcal{H}} \, e^{-l/\alpha'(L_0+\tilde{L}_0)} \, e^{i(L_0-\tilde{L}_0)\theta} \tag{27}$$

and introducing $\tau = \frac{\theta}{2\pi} + i\frac{l}{\alpha'}$

$$Z' = iV_d \int_R \frac{d^2\tau}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-D/2} \, \mathrm{tr}_{\mathcal{H}} \, q^{L_0} \, \bar{q}^{\tilde{L}_0} \tag{28}$$

with R the region $\tau_2 > 0$, $-1/2 \leq \tau_1 < 1/2$.

This is the same as the true string amplitude, except for the crucial difference of the intergration region, $R \neq F_0$. Indeed if (28) were the true string amplitude we would obtain the same UV divergences at $\tau_2 \to 0$ as for a theory of point particles. On the other hand, in the true string amplitude (22), the UV divergent region $\tau_2 \to 0$ is simply *absent*!

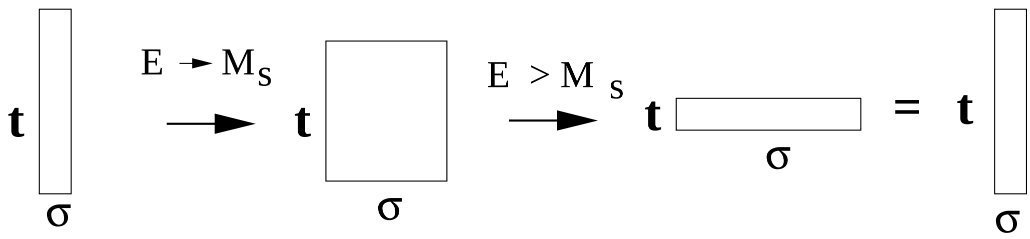

Figure 7: As the energy in the internal loop increases, longer strings run through it. The UV limit is geometrically equivalent to some infrared contribution, which has been already counted.

To understand a bit better where the UV region has gone, let us consider summing over two-torus worldsheets as the energy of the intermediate states increases, see figure 7. As the energy increases, longer and longer strings are exchanged for a shorter and shorter time. For $E \gg M_s$ the diagram of very long strings propagating over a very short time has the same geometry as and IR contribution (by exchange of the roles of σ, t), so it has been already counted. Notice that very remarkably the sum of the UV behaviours of all the states in the string tower resums into an infrared behaviour, which is typically convergent [2]

Notice that to get this result it was crucial not to overcount the worldsheet geometries. Worldsheet geometry provides an extremely clever cutoff, which makes string theory quite different from just a field theory with an infinite number of fields.

Let us comment that this feature that any UV divergent region is absent in string theory is completely general, and valid for other diagrams, with

[2] In the closed bosonic string theory, the IR is divergent due to the existence of a tachyonic state. The IR is well-behaved in other theories with no spacetime tachyons, like the superstrings.

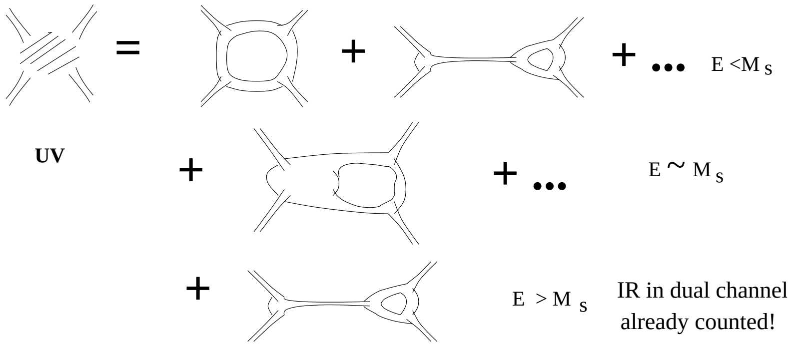

Figure 8: The contribution to a 1-loop four-string scattering amplitudes. The first line shows some low-energy contributions; the second line shows the first contributions for higher energy, with longer strings being exchanged in one internal leg. The third line shows the same diagram for energies much larger than M_S; this seemingly UV regime in geometrically the same as one of the IR contributions, so it has been already counted and should not be included again.

more handles and with external insertions. For instance see figure 8. Just as above, the UV behaviour of the complete tower of string states resums into and IR contribution in a dual channel, which is a non-divergent contribution.

Let us conclude by pointing out that the low energy contribution to the partition function, the vacuum to vacuum amplitude is divergent in the bosonic string theory. This is because the IR contribution is dominated by the lightest mode, which is a tachyon with $m^2 = -4/\alpha'$. In the IR $\tau_2 \to i\infty$ the string partition function reduces to the point particle one with m given by the lightest state mass; one clearly gets an exponential $e^{+\tau_2}$ which diverges. In theories with no spacetime tachyon, the IR limits are however well-behaved, so the finiteness of string theory works as discussed above.

Concerning the IR divergence found above, one may wonder whether it is a physical infinity. It is easy to show that the vacuum to vacuum amplitude is related to the vacuum energy density, namely to the cosmological constant in spacetime. Since the spacetime theory is coupled to gravity, it is indeed a physical observable, and the infinity is physical. So the theory is to some extent sick.

There is a lot of speculation about the meaning of the tachyon in bosonic string theory. Our present idea is that it signals an instability of the vacuum of the theory, rather than an essential inconsistency of the theory; the problem is that we have no idea which is the correct vacuum, around which there would be not spacetime tachyons.

A Modular functions

There is a lot of mathematical literature on modular functions, namely functions of the parameter τ which have nice transformation properties under the $SL(2,\mathbf{Z})$ modular group. A useful reference for them is [2].

Recall that the modular group is the set of transformations

$$\tau \to \frac{a\tau+b}{c\tau+d} \quad \text{with } a,b,c,d \in \mathbf{Z} \quad \text{and } \mathbf{ad}-\mathbf{bc}=\mathbf{1} \tag{29}$$

and is generated by $\tau \to \tau+1$, $\tau \to -1/\tau$

The Dedekind eta function

Introduce $q = e^{2\pi i\tau}$.The Dedekind eta function is defined by

$$\eta(\tau) \;=\; q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \tag{30}$$

Under modular transformations

$$\begin{aligned} \eta(\tau+1) &= e^{\pi i/12}\,\eta(\tau) \\ \eta(-1/\tau) &= (-i\tau)^{1/2}\,\eta(\tau) \end{aligned} \tag{31}$$

(The first is trivial to show, while the second is tricky and one should consult the literature).

The theta functions

For future use it is useful to introduce the theta function with characteristics θ, ϕ

$$\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) \;=\; \eta(\tau)\, e^{2\pi i\,\theta\phi}\, q^{\frac{1}{2}\theta^2 - \frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n+\theta-1/2}\, e^{2\pi i \phi})\, (1 + q^{n-\theta-1/2}\, e^{-2\pi i \phi}) \quad (32)$$

These functions also have an expression as infinite sums

$$\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) \;=\; \sum_{n\in \mathbf{Z}} q^{(n+\theta)^2/2}\, e^{2\pi i\,(n+\theta)\phi} \quad (33)$$

The fact that (32) and (33) are equal is related to *bosonization*, namely the fact that in two dimensions a theory of free fermions can be rewritten as a theory of free bosons (with a compact target space). The two expressions for the theta functions correspond to the partition functions of the same theory in terms of different field variables. This will be understood better when we study 2d theories with fermions in the superstring.

Some useful and often appearing values of the characteristics are 0, 1/2. For future convenience, we list the product form of the corresponding theta functions

$$\begin{aligned}
\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) &= \prod_{n=1}^{\infty} (1 - q^n)\, (1 + q^{n-1/2})^2 \\
\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (\tau) &= \prod_{n=1}^{\infty} (1 - q^n)\, (1 - q^{n-1/2})^2 \\
\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (\tau) &= q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)\, (1 + q^n)\, (1 + q^{n-1}) \;=
\end{aligned}$$

$$
\begin{aligned}
&= \; 2\, q^{1/8} \prod_{n=1}^{\infty} (1-q^n)\,(1+q^n)^2 \\
\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (\tau) &= \; i\, q^{1/8} \prod_{n=1}^{\infty} (1-q^n)^2\,(1-q^{n-1}) \;=\; 0
\end{aligned}
\tag{34}
$$

Finally, we list some useful properties of theta functions. Under integer shifts of the characteristics

$$
\vartheta \begin{bmatrix} \theta + m \\ \phi + n \end{bmatrix} (\tau) \;=\; e^{2\pi i \theta n}\, \vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau) \tag{35}
$$

This can be shown very easily using the infinite sum form (33).

Under modular transformations

$$
\begin{aligned}
\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (\tau+1) &= e^{-\pi i(\theta^2-\theta)}\, \vartheta \begin{bmatrix} \theta \\ \theta+\phi-1/2 \end{bmatrix} (\tau) \\
\vartheta \begin{bmatrix} \theta \\ \phi \end{bmatrix} (-1/\tau) &= (-i\tau)^{1/2}\, \vartheta \begin{bmatrix} \phi \\ -\theta \end{bmatrix} (\tau)
\end{aligned}
\tag{36}
$$

The first is very easy to show, using the infinite sum form (33) and using the trick that $e^{\pi i n^2} = e^{\pi i n}$ (since $n^2 = n \bmod 2$). The second is also easy in the infinite sum form using the Poisson resummation formula

$$
\sum_{n\in\mathbf{Z}} \exp\left[-\pi A(n+\theta)^2 + 2\pi i\,(n+\theta)\phi\right] \;=\; A^{-1/2} \sum_{k\in\mathbf{Z}} \exp\left[-\pi A^{-1}(k+\phi)^2 - 2\pi i k\theta\right] \tag{37}
$$

This is a particular case of a more general Poisson resummation formula, which we will need in later lectures. Let Λ, Λ^* be two dual lattices [3], and let $|\Lambda^*/\Lambda|$ be their index. We have

$$
\begin{aligned}
&\textstyle\sum_{v\in\Lambda} \exp\left[-\pi(v+\theta)\cdot A\cdot(v+\theta) + 2\pi i\,(v+\theta)\cdot\phi\right] = \\
&= \frac{1}{|\Lambda^*/\Lambda|\sqrt{\det A}} \textstyle\sum_{k\in\Lambda^*} \exp\left[-\pi(k+\phi)\cdot A^{-1}\cdot(k+\phi) - 2\pi i k\theta\right]
\end{aligned}
\tag{38}
$$

[3]Recall that given an n-dimensinoal lattice Λ in $\mathbf{R^n}$, the dual lattice Λ^* is defined as the set of vectors k in $\mathbf{R^n}$ such that $k\cdot v\in\mathbf{Z}$ for any vector $v\in\Lambda$.

This general formula can be shown by repeatedly using the one-dimensional one (37).

References

[1] J. Polchinski, ‘String theory’, Vol 1.

[2] D. Mumford, ‘Tata lectures on theta’, vols 1, 2, Birkhauser, Basel 1983