V. Quantization of the closed bosonic string

In this lecture we obtain the spectrum of oscillations of the closed bosonic string.

1 Worldsheet action

For this discussion I closely follow section 1.2 of [1]

As a string evolves in time, it sweeps out a two-dimensional surface in spacetime Σ, known as the worldsheet, and which is the analog of the worldline of a point particle in spacetime. Closed string correspond to worldsheets with no boundary, while open string sweep out worldsheets with boundaries. Any point in the worldsheet is labeled by two coordinates, t the 'time' coordinate just as for the pointparticle worldline, and σ, which parametrizes the extended spatial dimension of the string at fixed t. We denote σ, t collectively as ξ^a, $a = 1, 2$.

Our pupose is to write down the action for a string configuration in flat D-dimensional Minkowski space. For the bosonic string, such configurations are in principle described by D embedding functions $X^\mu(\sigma, t)$, with $\mu = 0, \ldots, D-1$, which can be regarded as 2d fields on the worldsheet.

1.1 The Nambu-Goto action

The natural action for a string configuration is the integral of the area element on the worldsheet, in principle measured with the metric inherited from the ambient metric in M_D. The ambient metric is computed as follows

$$ds^a \quad = \quad h_{ab}\, d\xi^a\, d\xi^b$$

$$ds^2 \;=\; \eta_{\mu\nu}\, dX^\mu\, dX^\nu \;=\; \eta_{\mu\nu}\, \frac{\partial X^\mu}{\partial \xi^a}\, \frac{\partial X^\nu}{\partial \xi^b}\, d\xi^a\, d\xi^b \tag{1}$$

hence

$$h_{ab} \;=\; \eta_{\mu\nu}\, \frac{\partial X^\mu}{\partial \xi^a}\, \frac{\partial X^\nu}{\partial \xi^b} \tag{2}$$

The Nambu-Goto action is

$$S_{\rm NG}[X(\xi)] \;=\; -\frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi\, (-h)^{1/2} \tag{3}$$

where $h = det(h_{ab})$ and α' is related to the string tension $T = \frac{1}{2\pi\alpha'}$.

1.2 The Polyakov action

The Nambu-Goto action is not very convenient for quantizing the worldsheet theory. So we are going to replace it by another action, which is classically equivalent, but which is much more convenient for quantization, the Polyakov action.

To do that we introduce another degree of freedom on the worldsheet, a worldsheet metric $g_{ab}(\xi)$ which is in principle independent of the induced metric h_{ab}. The natural action on the worldsheet is then

$$S_{\rm P} \;=\; -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi\, (-g)^{1/2}\, g^{ab}(\sigma,t)\, \partial_a X^\mu\, \partial_b X^\nu \eta_{\mu\nu} \tag{4}$$

with $g = det(g_{ab})$

Classical equivalence with the Nambu-Goto action follows from solving the equations of motion for g_{ab}, namely $\delta S/\delta g_{ab} = 0$. Using

$$\delta g \;=\; -g\, g_{ab}\, \delta g^{ab} \tag{5}$$

one gets

$$\begin{aligned} \delta_g S_P \;&=\; -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi\, (-g)^{1/2}\, \delta g^{ab}\, [-\frac{1}{2}\, g_{ab}\, g^{cd} \partial_c X^\mu\, \partial_d X_\mu \;+ \partial_a X^\mu\, \partial_b X_\mu\,] \;= \\ &=\; -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi\, (-g)^{1/2}\, \delta g^{ab}\, [-\frac{1}{2}\, g_{ab}\, g^{cd}\, h_{cd} \;+\; h_{ab}\,] \end{aligned} \tag{6}$$

The equations of motion read

$$h_{ab} = \frac{1}{2}\, g_{ab}\, g^{cd}\, h_{cd} \tag{7}$$

Taking determinant

$$(-h)^{1/2} = \frac{1}{2}\, (-g)^{1/2}\, g^{cd}\, h_{cd} \tag{8}$$

and replacing into (4) we get

$$S_P[X(\xi), g_{\rm clas}(\xi)] = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi\, (-h)^{1/2} = S_{NG}[X(\xi)] \tag{9}$$

1.3 Symmetries of Polyakov action

The action (4) has some important symmetries which we now discuss

1. D-dimensional Poincaré invariance.

$$\begin{aligned} X'^\mu(\xi) &= \Lambda^\mu_\nu\, X^\nu(\xi) + a^\mu \\ g'_{ab}(\xi) &= g_{ab}(\xi) \end{aligned} \tag{10}$$

   It is a global symmetry from the worldsheet viewpoint.

2. Two-dimensional diffeomorphism invariance, namele coordinate reparametrization of the worldsheet.

$$\begin{aligned} \xi'^a &= \xi'^a(\xi) \\ X'^\mu(\xi') &= X^\mu(\xi) \\ g'_{ab}(\xi') &= \frac{\partial \xi^c}{\partial \xi'^a}\, \frac{\partial \xi^d}{\partial \xi'^b}\, g_{cd}(\xi) \end{aligned} \tag{11}$$

   It is a local (i.e. ξ dependent) symmetry. The 2d fields $X^\mu(\xi)$ behave as scalars while $g_{ab}(\xi)$ is a 2-index tensor (metric).

3. Two-dimensional Weyl invariance

$$\begin{aligned} X'^{\mu}(\xi) &= X^{\mu}(\xi) \\ g'_{ab}(\xi) &= \Omega(\xi)\, g_{ab}(\xi) \end{aligned} \tag{12}$$

It is a local symmetry.

Weyl-related string configurations correspond to the *same* embedding of the world-sheet in spacetime. So this is an extra redundancy in the Polyakov description, not present in the Nambu-Goto description.

It is convenient to emphasize at this point that a commonly mentioned symmetry, conformal invariance, is a subset of these symmetries. In particular, in covariant quantization one fixes the so-called conformal gauge, which amounts to using diff and Weyl invariances to set $g_{ab} = \eta_{ab}$. There is then a left-over local symmetry which is the set of coordinate transformations, whose effect on the metric can be undone with a Weyl transformation (so that the gauge fixed flat metric is preserved). This set of transformations is the 2d conformal group, which is extremely important in string theory. However, we will quantize the string in a different gauge, and conformal symmetry will not be manifest.

2 Light-cone quantization

For this section, we follow the computations in sections 1.3 and 1.4 of [1]. A more detailed treatment, using the formalism of quantization of constrained systems can be found in [2].

In quantizing the 2d field theory, we need to fix the gauge freedom. The light-cone gauge is the simplest one, and the most convenient to obtain the spectrum. This is because the final states will be the physical states of the theory, and in particular spacetime gauge particles will arise in the unitary gauge (namely, we will obtain only the two physical polarization modes of massless gravitons or gauge particles, and no spacetime spurious gauge degrees of freedom).

2.1 Light-cone gauge fixing

Define the light-cone coordinates

$$
\begin{aligned}
X^{\pm} &= \frac{1}{\sqrt{2}}\,(X^0 \pm X^1) \\
X^i & \qquad i = 2, \ldots, D-1
\end{aligned}
\tag{13}
$$

The metric (scalar product) in M_D then reads

$$A^\mu B_\mu = -A^+ B^- - A^- B^+ + A^i B^i \tag{14}$$

so

$$A_- = -A^+ \quad , \quad A_+ = -A^- \quad , \quad A_i = A^i \tag{15}$$

The gauge fixing proceeds through several steps

1. Reparametrization of t

Fix the t reparametrization freedom by setting the so-called light-cone condition

$$X^+(\sigma, t) = t \tag{16}$$

see figure 1. So X^+ will play the role of worldsheet time, and its conjugate variable $P_+ = -P^-$ will play the role of worlsheet energy (2d hamiltonian).

2. Reference line in σ

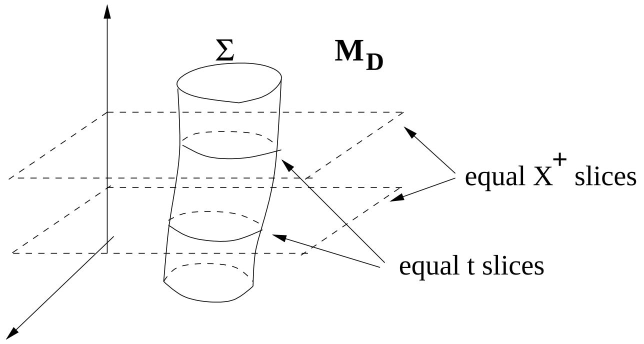

Figure 1: The light cone condition defines equat t slices on the worldsheet in terms of equal X^+ slices on spacetime.

Choose a line on the worldsheet $\sigma_0(t)$ intersecting all constant t slices orthogonally (w.r.t. the 2d metric g). Namely

$$g_{t\sigma}(\sigma, t) \;=\; 0 \qquad \text{at } \sigma = \sigma_0(t) \tag{17}$$

Notice that this still leaves the freedom of an overall motion of the reference line. This will be important as an additional constraint on the final spectrum (see (43)).

3. Reparametrization of σ

For slices of constant t, define a new spatial coordinate σ' for each point of the slice. σ' is defined as the (diffeomorphism and Weyl) invariant distance to the reference line along the slice

$$\sigma' \;=\; c(t) \int_{\sigma_0}^{\sigma} f(\sigma, t)\, d\sigma \tag{18}$$

where

$$f(\sigma) \;=\; (-g)^{-1/2}\, g_{\sigma\sigma}(\sigma, t) \tag{19}$$

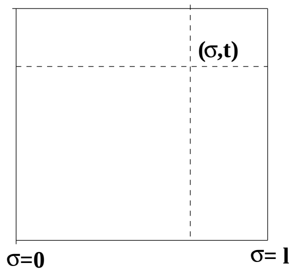

Figure 2: The coordinate t on the worldsheet corresponds to the coordinate X^+ of the spacetime point where it is embedded. The coordinate σ is defined as the invariant distance, to a reference line $\sigma = 0$, along fixed t slices. The total string length is fixed to be ℓ.

and $c(t)$ is a σ independent coefficient used to impose that the total length of the string is fixed, a constant in t which we call ℓ. The situation is shown in figure 2.

In the new coordinates, $f(\sigma')$ is σ' independent. In the following we will only use this coordinatization, and we drop the prime. So we write

$$\partial_\sigma f(\sigma, t) \; = \; 0 \qquad (20)$$

4. Weyl invariance

Now we use Weyl invariance to impose that

$$g = -1 \qquad \forall \sigma, t \qquad (21)$$

Since $f(\sigma)$ is Weyl-invariant, it still satisfies $\partial_\sigma f(\sigma, t) = 0$. Using the definition of f, we get

$$\partial_\sigma g_{\sigma\sigma} = 0 \qquad (22)$$

This concludes the gauge fixing. The metric and inverse metric read

$$(g_{ab}) \; = \; \begin{pmatrix} g_{\sigma\sigma}(t)^{-1}[-1 + g_{t\sigma}(\sigma,t)^2] & g_{t\sigma}(\sigma,t) \\ g_{t\sigma}(\sigma,t) & g_{\sigma\sigma}(t) \end{pmatrix} \; ; \; (g^{ab}) \; = \; \begin{pmatrix} -g_{\sigma\sigma}(t) & g_{t\sigma}(\sigma,t) \\ g_{t\sigma}(\sigma,t) & g_{\sigma\sigma}(t)^{-1}[1 - g_{t\sigma}(\sigma,t)^2] \end{pmatrix}$$

2.2 Gauge-fixed Polyakov action, Hamiltonian

The Polyakov lagrangian in light-cone coordinates reads

$$\begin{aligned} L &= -\tfrac{1}{4\pi\alpha'} \textstyle\int_0^\ell d\sigma \quad [-2\, g^{tt}\partial_t X^+\, \partial_t X^- \,+ g^{tt}\partial_t X^i\, \partial_t X^i \,-\, 2\, g^{\sigma t}\, \partial_t X^+\, \partial_\sigma X^- \,+ \\ &\qquad +2\, g^{\sigma t}\, \partial_\sigma X^i\, \partial_t X^i \,+ g^{\sigma\sigma}\, \partial_\sigma X^i\, \partial_\sigma X^i\,] = \\ &= -\tfrac{1}{4\pi\alpha'} \textstyle\int_0^\ell d\sigma \quad [\, g_{\sigma\sigma}\, (2\, \partial_t X^- - \partial_t X^i\, \partial_t X^i) \,-\, 2\, g_{\sigma t}\, (\partial_\sigma X^- - \partial_\sigma X^i\, \partial_t X^i) \,+ \\ &\qquad g_{\sigma\sigma}^{-1}\, (1 - g_{\sigma t}^2)\, \partial_\sigma X^i\, \partial_\sigma X^i\,] \end{aligned} \tag{23}$$

Defining the center of mass and relative coordinates $x^-(t)$, $Y^-(\sigma, t)$

$$\begin{aligned} x^-(t) &= \frac{1}{\ell} \int_0^\ell d\sigma\, X^-(\sigma, t) \\ X^-(\sigma, t) &= x^-(t) \,+\, Y^-(\sigma, t) \end{aligned} \tag{24}$$

we obtain

$$\begin{aligned} L \;=\; & -\frac{\ell}{2\pi\alpha'}\, g_{\sigma\sigma}\, \partial_t x^-(t) \,-\, \frac{1}{4\pi\alpha'} \int_0^\ell d\sigma\, [\, - g_{\sigma\sigma}\, \partial_t X^i\, \partial_t X^i \,+ \\ & -\, 2\, g^{\sigma t}\, (\partial_\sigma Y^- - \partial_\sigma X^i\, \partial_t X^i) \,+\, g_{\sigma\sigma}^{-1}\, (1 - g_{\sigma t}^2)\, \partial_\sigma X^i\, \partial_\sigma X^i\,] \end{aligned} \tag{25}$$

The $Y^-(\sigma, t)$ does not have time derivatives in this lagrangian, so it acts as a Lagrange multiplier imposing

$$\partial_\sigma g_{\sigma,t}(\sigma, t) \;=\; 0 \quad \forall \sigma, t \tag{26}$$

Since we have $g_{\sigma t}(\sigma = 0, t) = 0$ due to (17), we get

$$g_{\sigma,t}(\sigma, t) = 0 \quad \forall \sigma, t \tag{27}$$

The lagrangian becomes

$$L \;=\; -\frac{\ell}{2\pi\alpha'}\, g_{\sigma\sigma}\, \partial_t x^-(t) \,+\, \frac{1}{4\pi\alpha'} \int_0^\ell d\sigma\, [\, g_{\sigma\sigma}\, \partial_t X^i\, \partial_t X^i \,-\, g_{\sigma\sigma}^{-1}\, \partial_\sigma X^i\, \partial_\sigma X^i\,]$$

The momentum conjugate to $x^-(t)$ is

$$p_- \;=\; -p^+ \,=\, \frac{\partial L}{\partial(\partial_t x^-)} \;=\; -\frac{\ell}{2\pi\alpha'}\, g_{\sigma\sigma} \tag{28}$$

so $g_{\sigma\sigma}$ is not really an independent coordinate variable, but a momentum variable.

The momenta conjugate to $X^i(\sigma,t)$ are

$$\Pi^i(\sigma,t) \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_t X^i)} \;=\; \frac{1}{2\pi\alpha'}\, g_{\sigma\sigma}\,\partial_t X^i(\sigma,t) \;=\; \frac{p^+}{\ell}\,\partial_t X^i(\sigma,t) \tag{29}$$

We can construct the Hamiltonian

$$\begin{aligned}
H \;&=\; p_-\partial_t x^-(t) \,+\, \int_0^\ell d\sigma\,\Pi_i(\sigma,t)\,\partial_t X^i(\sigma,t) \,-\, L \;= \\
&=\; -\frac{\ell}{2\pi\alpha'}\, g_{\sigma\sigma}\,\partial_t x^-(t) \,+\, \int_0^\ell d\sigma\,\frac{1}{2\pi\alpha'}\, g_{\sigma\sigma}\,\partial_t X^i(\sigma,t)\,\partial_t X^i(\sigma,t) \,+ \\
&\quad +\frac{\ell}{2\pi\alpha'}\, g_{\sigma\sigma}\,\partial_t x^-(t) \,-\, \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\,[\,g_{\sigma\sigma}\,\partial_t X^i\,\partial_t X^i \,-\, g_{\sigma\sigma}^{-1}\,\partial_\sigma X^i\,\partial_\sigma X^i\,] \;= \\
&=\; \frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\,[\,g_{\sigma\sigma}\,\partial_t X^i\,\partial_t X^i \,+\, g_{\sigma\sigma}^{-1}\,\partial_\sigma X^i\,\partial_\sigma X^i\,] \;=
\end{aligned} \tag{30}$$

In terms of momenta

$$H \;=\; \frac{\ell}{4\pi\alpha' p^+}\int_0^\ell d\sigma\,[\,2\pi\alpha'\,\Pi_i\,\Pi_i \,+\, \frac{1}{2\pi\alpha'}\,\partial_\sigma X^i\,\partial_\sigma X^i\,] \tag{31}$$

The equations of motion for x^-, $p_- = p^+$ are

$$\begin{aligned}
\partial_t x^-(t) \;&=\; \frac{\partial H}{\partial p_-} \;=\; -\frac{\partial H}{\partial p^+} \;=\; \frac{H}{p^+} \\
\partial_t p^+(t) \;&=\; -\frac{\partial H}{\partial x^-} \;=\; 0
\end{aligned} \tag{32}$$

so p^+ is conserved, and x^- is linear in t and has trivial dynamics.

The equations of motion for X^i, Π_i are

$$\begin{aligned}
\partial_t X^i(\sigma,t) \;&=\; \frac{\delta H}{\delta \Pi_i} \;=\; c\,2\pi\alpha'\,\Pi_i \\
\partial_t \Pi_i(\sigma,t) \;&=\; -\frac{\delta H}{\delta X^i} \;=\; \frac{c}{2\pi\alpha'}\,\partial_\sigma X^i
\end{aligned} \tag{33}$$

with $c = \ell/(2\pi\alpha' p^+)$ So we get

$$\partial_t^2 X^i = c^2\,\partial_\sigma^2 X^i \tag{34}$$

the wave equation for two-dimensional fields $X^i(\sigma,t)$. Indeed, for fixed (because it is conserved) p^+, we see that H is the hamiltonian for $D-2$ *free* bosons in 2d [1].

It is useful to set $\ell = 2\pi\alpha' p^+$, and so $c=1$.

2.3 Oscillator expansions

The general solution to the equations of motion is a superposition of left- and right-moving waves

$$X^i(\sigma,t) \;=\; X^i_L(\sigma+t) \;+\; X^i_R(\sigma-t) \tag{35}$$

For closed strings, we need to impose boundary conditions, periodicity in σ

$$X^i(\sigma+\ell,t) \;=\; X^i(\sigma,t) \tag{36}$$

The general form of X_L, X_R with those boundary conditions is

$$\begin{aligned} X^i_L(\sigma+t) &= \frac{x^i}{2} + \frac{p_i}{2p^+}(t+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbf{Z}-\{\mathbf{0}\}} \frac{\alpha^i_n}{n}\, e^{-2\pi i\, n\,(\sigma+t)/\ell} \\ X^i_R(\sigma-t) &= \frac{x^i}{2} + \frac{p_i}{2p^+}(t-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbf{Z}-\{\mathbf{0}\}} \frac{\tilde{\alpha}^i_n}{n}\, e^{2\pi i\, n\,(\sigma-t)/\ell} \end{aligned} \tag{37}$$

The coefficients x^i, p_i denote the center of mass coordinate and momentum, while the two infinite sets of coeffients α^i_n, $\tilde{\alpha}^i_n$ denote the amplitudes of the momentum n mode for left and right movers.

Promoting the worldsheet degrees of freedom $x^-(t)$, p^+, X^i, Π^i to operators, with canonical commutators, we obtain the commutation relations

$$\left[x^-,p^+\right] \;=\; -i$$

[1] Recalling our discussion about the α' expansion, this means that we can quantize the theory exactly in α'.

$$\begin{aligned}
\left[x^i, p_j\right] &= i\delta_{ij} \\
\left[\alpha^i_m, \alpha^j_n\right] &= \left[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n\right] = m\,\delta_{ij}\,\delta_{m,-n} \\
\left[\alpha^i_m, \tilde{\alpha}^j_n\right] &= 0
\end{aligned} \tag{38}$$

We can obtain the hamiltonian in terms of these

$$\begin{aligned}
H &= \frac{1}{2}\int_0^\ell d\sigma\,[\,2\pi\alpha'\,\Pi^i\,\Pi^i + \frac{1}{2\pi\alpha'}\,\partial_\sigma X^i\,\partial_\sigma X^i\,] = \\
&= \frac{p_i p_i}{2p^+} + \frac{1}{2\alpha' p^+}\left[\sum_{n>0}[\,\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n\,] + E_0 + \tilde{E}_0\right]
\end{aligned} \tag{39}$$

We get the quantum mechanics of the center of mass motion and two infinite sets of decoupled harmonic oscillators. Here we have normal-ordered the creation and annihilation modes and E_0, $\tilde{E}_0$ are the corresponding zero point energies, to be discussed below.

The Hilbert space of string states is obtained by defining a vacuum $|k\rangle = |k_-, k_i\rangle$ by

$$p^+|k\rangle = k_-|k\rangle \quad , \quad p_i|k\rangle = k_i|k\rangle \quad , \quad \alpha^i_n|k\rangle = \tilde{\alpha}^i_n|k\rangle = 0 \quad \forall n > 0 \tag{40}$$

and acting on it with the creation ladder operators α^i_{-n}, $\tilde{\alpha}^i_{-n}$, with $n > 0$, in an arbitrary way (almost, see later for an additional constraint).

As discussed in the overview lectures, each oscillation state of the string is observed as a particle from the spacetime viewpoint, with spacetime mass

$$M^2 = -p^2 = 2p^+p^- - p_i p_i \tag{41}$$

Notice that p^- corresponds to ∂_{x^+}, which in light cone gauge is ∂_t, which corresponds to the 2d hamiltonian H, so $p^- = H$, and $M^2 = 2p^+H - p_ip_i$. We have

$$\alpha'\,M^2 = N + \tilde{N} + E_0 + \tilde{E}_0 \tag{42}$$

with $N = \sum_{n>0} \alpha_{-n}^i \alpha_n^i$ the total left oscillator number (analogously for $\tilde{N}$). It is important to recall from the commutation relations, that a single mode α_n^i or $\tilde{\alpha}_n^i$ contributes n to the oscillator number.

Hence the masses of spacetime particles increase with the number of oscillators in the corresponding string state.

There is one further constraint we must impose on the spectrum. Recall that after gauge fixing we still had the freedom to perform an overall translation of the reference line $\sigma = 0$ by a t independent amount. This forces to restrict the spectrum to the subsector invariant under translations in σ. This amounts to requiring the net 2d momentum along σ to vanish, namely the left- and right-moving operators in a state should carry the same total momentum. Recalling that a mode n carries momentum n, the constraint is

$$N = \tilde{N} \tag{43}$$

the so-called level matching constraint. It is an important fact that the quantization procedure can be performed independently for left- and right- movers (e.g. defining left- and right-moving hamiltonians, and mass operators, etc) and they only talk to each other at the level of building the physical spectrum via the constraint (43).

Finally, we need to compute the zero point energies $E_0 = \tilde{E}_0$. Formally, for each i

$$E_0^i = \frac{1}{2} \sum_{n=1}^{\infty} n \tag{44}$$

This is infinite so we compute it with a regularization prescription, i.e. as the limit $\epsilon \to 0$ of the non-singular part of

$$Z(\epsilon) = \frac{1}{2} \sum_{n=1}^{\infty} n\, e^{-n\epsilon} \tag{45}$$

After some massage

$$Z(\epsilon) \;=\; \frac{1}{2}\sum_{n=1}^{\infty} n\, e^{-n\epsilon} \;=\; -\frac{1}{2}\frac{d}{d\epsilon}\sum_{n=1}^{\infty} e^{-n\epsilon} \;=\; -\frac{1}{2}\frac{d}{d\epsilon}\frac{1}{1-e^{-\epsilon}} \tag{46}$$

Since

$$\begin{aligned}\frac{1}{1-e^{-\epsilon}} \;&=\; \frac{1}{\epsilon}\,\frac{1}{1-\epsilon/2+\epsilon^2/6+\mathcal{O}(\epsilon^3)} \;=\; \frac{1}{\epsilon}\left[1+\epsilon/2-\epsilon^2/6+\epsilon^2/4+\mathcal{O}(\epsilon^3)\right] \;= \\ &=\; \frac{1}{\epsilon}+\frac{1}{2}+\frac{1}{12}\epsilon+\mathcal{O}(\epsilon^2)\end{aligned} \tag{47}$$

we get

$$Z(\epsilon) \;=\; -\frac{1}{2}\left[-\frac{1}{\epsilon^2}+\frac{1}{12}+\mathcal{O}(\epsilon)\right] \tag{48}$$

Dropping the infinite part and letting $\epsilon \to 0$, the zero point energy for a single 2d free boson is

$$E_0^i = \tilde{E}_0^i = -\frac{1}{24} \tag{49}$$

So for $D-2$ we have $E_0 = \tilde{E}_0 = -(D-2)/24$

$$\alpha' M^2 \;=\; N + \tilde{N} - 2\,\frac{D-2}{24} \tag{50}$$

Dropping the infinity amounts to redefining the vacuum energy. One might think that this is not possible because the Polyakov action includes a worldsheet metric (i.e. gravity). However, this is not present in our gauge fixing and the problem is avoided. It is important to emphasize that this infinity is not present in other gauge fixings (like the conformal gauge), so the infinity is an artifact of our gauge fixing. However, the zero point energy we have computed has physical consequences, like fixing the dimension of spacetime to be 26. In the light-cone gauge, which is not manifestly Lorentz invariant, it appears when we require the spectrum to be Lorentz invariant, as we motivate below. In other gauges, the condition appears in other ways. For instance, in the conformal gauge fixing, as the cancellation of the conformal anomaly.

For $D = 26$ we have

$$\alpha' M^2 = N + \tilde{N} - 2 \tag{51}$$

2.4 Light spectrum

It is now time to obtain the lightest particles in the spectrum of the string. The states with smallest number of oscillators that we can construct satisfying (43) are

$$\begin{array}{lcl} N = \tilde{N} = 0 & |k\rangle & \alpha' M^2 = -2 \\ N = \tilde{N} = 1 & \alpha^i_{-1} \alpha^j_{-1} |k\rangle & \alpha' M^2 = 0 \end{array} \tag{52}$$

The closed string groundstate is a spacetime tachyon. This field is troublesome, and it is thought to signal an instability of the theory. The result of this instability is not known.

The second states transform as a two-index tensor with respect to the $SO(D-2)$ subgroup of the Lorentz group manifest in the light-cone gauge.

One should recall that in a Lorentz invariant theory in D dimensions, physical states of fields belong to representations of the so-called little group (subgroup of Lorentz group which leaves invariant the D-momentum of the particle). For massive particles, the D-momentum can be brought to the form $P = (M, 0, \ldots, 0)$ in the particle's rest frame, so the little group is $SO(D-1)$. For massless particles, the D-momentum can be brought to the form $(M, M, 0, \ldots)$, so the little group is $SO(D-2)$.

Our particles in the first excited sector are clearly not enough to fill out a representation of $SO(D-1)$, so to have Lorentz invariance it is crucial that they are massless. Notice that this is so only because we have imposed $D = 26$, so this is a derivation of the dimension of spacetime in which string theory can propagate consistently. Indeed, it is possible to construct the Lorentz generators in terms of the oscillator numbers etc and check that the Lorentz algebra is recovered only if $D = 26$. We skip this computation which can however be found in standard textbooks, like [2]

Let us also point out that massive states in the theory *do* fill out representations of $SO(D-1) = SO(25)$, altough only $SO(24)$ is manifest.

The massless two-index tensor can be split in irreducible representations of $SO(24)$, by taking its trace (which is a 26d scalar particle, the dilaton ϕ), its antisymmetric part (which is a 26d 2-form field $B_{\mu\nu}$) and its symmetric traceless part (which is a 26d symmetric tensor field $G_{\mu\nu}$).

2.5 Lessons

The result of light cone quantization for the bosonic string can be phrased in terms of the following recipe, which will be valid for other string theories as well

- The only relevant degrees of freedom left are the center of mass and $D-2$ transverse coordinates $X^i(\sigma, t)$, $i = 2\ldots, D-1$

- For closed string theories the 2d theory splits into two sectors, left- and right-movers, which can be quantized independently. The only relation between them appears at the final stage, when imposing the level matching condition on the physical spectrum.

- The spacetime $(\text{mass})^2$ operator (on each sector) is given by the oscillator numbers plus the zero point energy, which should be computed using the $e^{-\epsilon n}$ regularization.

2.6 Final comments

Upon studying interactions of these 26d fields one concludes that $G_{\mu\nu}$ is a 26d graviton and $B_{\mu\nu}$ is a gauge potential. So 26d interactions between these fields are invariant under 26d coordinate reparametrization and gauge transformations for B

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]}(X) \tag{53}$$

The 26d low energy effective action for these modes was described in the overview lessons. In the string frame

$$S_{\text{eff.}} = \frac{1}{2k^2} \int d^{26}X \, (-\tilde{G})^{1/2} \, \{ \, \tilde{R} + \frac{1}{12} e^{-\tilde{\phi}/12} \, H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{6}\partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \, \} + \mathcal{O}(\alpha') \tag{54}$$

We emphasize again that the dilaton vev fixes the string interaction coupling constant in the 26d theory. So the string interaction coupling constant is not an arbitrary external parameter, but the vacuum expectation value of a spacetime dynamical scalar field in the theory. Instead of a continuum of different string theories, labeled by the value of the coupling constant, we have a unique string theory with a continuous set of vacua parametrized by the vev for a scalar field with flat potential $V(\phi) \equiv 0$. Fields with flat potential are called moduli, and the set of vacua is called the moduli space of the theory.

References

[1] J. Polchinski, 'String theory', Vol 1.

[2] D. Lust, S. Theisen, 'Lectures on string theory', Lect. Notes Phys. 346 (1989)1