# Overview of string theory in perturbation theory

To be honest, we still do not have a complete description of string theory at the non-perturbative level (this will become clear in coming lectures). Still, the perturbative picture is very complete, and is the best starting point to study the theory.

## 1 Basic ideas

### 1.1 What are strings?

String theory proposes that elementary particles are *not* pointlike, but rather they are small 1-dimensional extended objects (strings), of typical size $L_s = 1/M_s$. They can be open or closed strings, as shown in figure 1. At energies well below the string scale $M_s$, there is not enough resolution to see the spatial extension of the objects, so they look like point particles, and usual point particle physics should be recovered as an effective description.

Experimentally, our description of elementary particles as pointlike works nicely up to energies or order 1 TeV, so $M_s >$ TeV. In many string models, however, the string scale turns out to be related to the 4d Planck scale, so we have $M_s \simeq 10^{18}$ GeV. This corresponds to string of typical size of $10^{-33}$ cm, really tiny.

Strings can vibrate. Different oscillation modes of a unique kind of underlying object, the string, are observed as different particles, with different Lorentz (and gauge and global) symmetry quantum numbers. This is schematically shown in figure 2 for closed string states.

The mass of the corresponding particle increases with the number of oscillator modes that we are exciting. So the vibration modes of the string

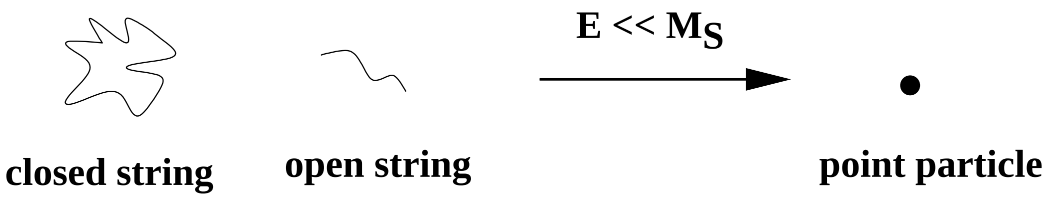


Figure 1: According to string theory, elementary particles are 1-dimensional extended objects (strings).

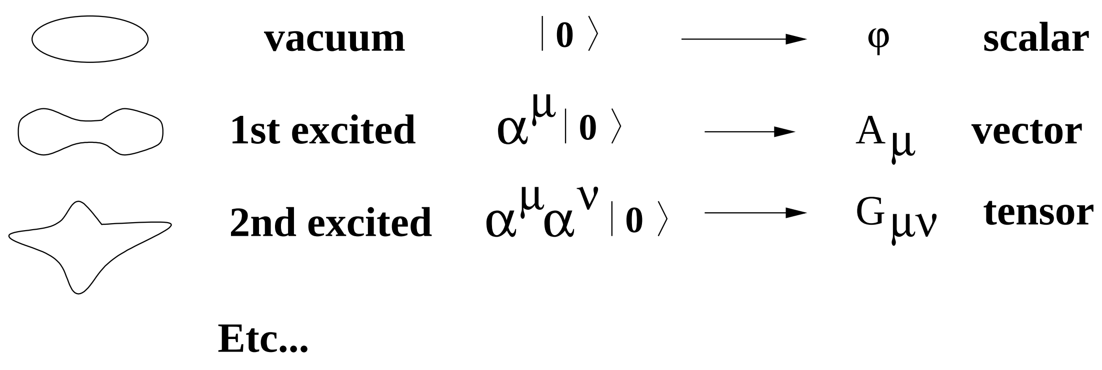


Figure 2: Different oscillation modes of unique type of string correspond to different kinds of particles, with e.g. different Lorentz quantum numbers.

give rise to an infinite tower of particles, with masses increasing in steps of order $M_s$. Since $M_s$ is so large, only the particles with masses of order zero (to leading order) can correspond to the observed ones.

Upon explicit computation of this spectrum of particles, the massless sector always contains a 2-index symmetric tensor $G_{\mu\nu}$. Later on we will see that this field behaves as a graviton, so string theories automatically contain gravity. But before we can explain interactions in string theory we need some further ingredients.

## 1.2 The worldsheet

As a string evolves in time, it sweeps out a two-dimensional surface in spacetime $\Sigma$, known as the worldsheet, and which is the analog of the worldline of a point particle in spacetime. Closed string correspond to worldsheets with no boundary, while open string sweep out worldsheets with boundaries. Any point in the worldsheet is labeled by two coordinates, $t$ the 'time' coordinate just as for the point particle worldline, and $\sigma$, which parametrizes the extended spatial dimension of the string at fixed $t$.

A classical string configuration in $d$-dimensional Minkowski space $M_d$ is given by a set of functions $X^\mu(\sigma,t)$ with $\mu = 0,\ldots,d-1$, which specify the coordinates in $M_d$ of the point corresponding to the string worldsheet point $(\sigma,t)$.

This can be expressed by saying that the functions $X^\mu(\sigma,t)$ provide a map from a two-dimensional surface (the abstract worldsheet), parametrized by $(\sigma,t)$ to a $d$-dimensional space $M_d$ (spacetime, also known as target space of the embedding functions).

$$\begin{array}{cccc} X^\mu : & \Sigma & \rightarrow & M_d \\ & (\sigma,t) & \rightarrow & X^\mu(\sigma,t) \end{array} \tag{1}$$

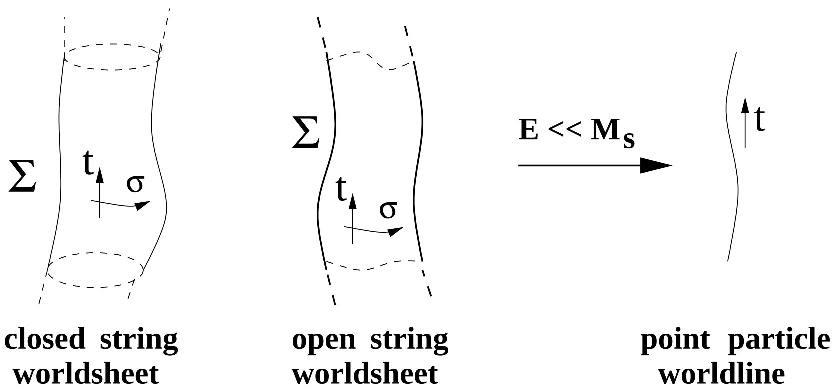


Figure 3: Worldsheets for closed and open strings. They reduce to worldlines in the point particle (low energies) limit.

This is pictorially shown in figure 4.

A natural definition for the classical action for a string configuration is given by the total area spanned by the worldsheet (in analogy with the worldline interval length as action for a point particle).

$$S_{\rm NG} \;=\; -T \int_{\Sigma} dA \tag{2}$$

where $T$ is the string tension, related to $M_s$ by $T = M_s^2$. One also often introduces the quantity $\alpha'$, with dimensions of length squared, defined by $T = M_s^2 = \frac{1}{2\pi\alpha'}$.

In terms of the embedding functions $X^\mu(\sigma,t)$, the action (2) can be written as

$$S_{\rm NG} \;=\; -T \int_{\Sigma} (\, \partial_\tau X^\mu \, \partial_\tau X_\mu \;-\; \partial_\sigma X^\mu \, \partial_\sigma X_\mu \,)^{1/2} \, d\sigma \, dt \tag{3}$$

This is the so-called Nambu-Goto action. It is difficult to quantize, so quantization is simpler if carried out starting with a different, but classically

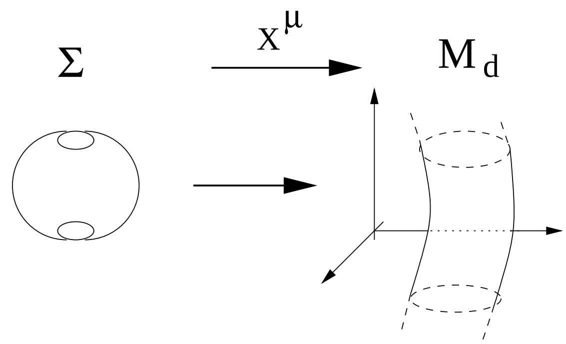


Figure 4: The functions $X^\mu(\sigma, t)$ define a map, an embedding, of a 2-dimensional surface into the target space $M_d$.

equivalent action, known as the Polyakov action

$$S_{\text{Polyakov}} = -T/2 \int_\Sigma \sqrt{-g}\, g^{\alpha\beta}(\sigma,t)\, \partial_\alpha X^\mu\, \partial_\beta X^\nu \eta_{\mu\nu}\, d\sigma\, dt \tag{4}$$

where we have introduced an additional function $g(\sigma, t)$. It does not have interpretation as an embedding. The most geometrical interpretation it receives is that it is a metric in the abstract worldsheet $\Sigma$. At this point it is useful to imagine the worldsheet as an abstract two-dimensional world which is embedded in physical spacetime $M_d$ via the functions $X^\mu$. But which to some extent makes sense by itself.

The important fact we would like to emphasize is that this looks like the action for a two-dimensional field theory coupled to two-dimensional gravity. Many of the wonderful properties of string theory arise from subtle relation between the 'physics' of this two-dimensional world and the physics of spacetime.

The two-dimensional field theory has a lot of gauge and global symmetries, which will be studied later on. For the moment let us simply say that

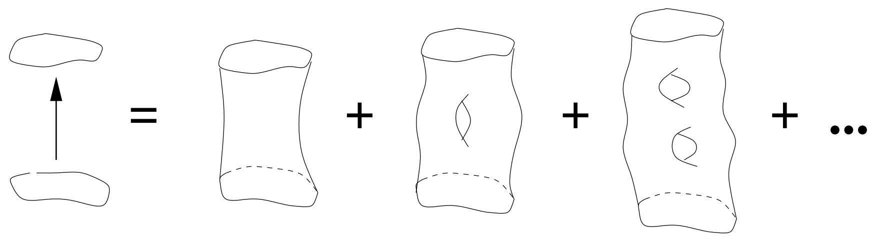

Figure 5: The genus expansion for closed string theories .

after fixing the gauge the 2d action becomes

$$S_P[X(\sigma,t)] \;=\; -T/2\int_\Sigma \partial_\alpha X^i\,\partial_\alpha X^i, \qquad i=2,\ldots,d-1 \tag{5}$$

It is just a two-dimensional quantum field theory of $d-2$ free scalar fields. This is easy to quantize, and gives just a bunch of decoupled harmonic oscillators, which are the string oscillation modes mentioned before. It is important to notice that the fact that the worldsheet theory is a free theory does *not* imply that there are no interactions between strings in spacetime. There are interactions, as we discuss in the following.

Before concluding, let us emphasize a crucial property of the worldsheet field theory, its conformal invariance. This property is at the heart of the finiteness of string theory, as we discuss below.

## 1.3 String interactions

A nice discussion is in section 3.1. of [1]

The quantum amplitudes between string configurations are obtained by performing a path integral, namely summing over all possible worldsheets which interpolate between the configurations, see figures 5, 6.

The sum organizes into a sum over worldsheet topologies, with increasing number of handles and of boundaries (for theories with open strings) This

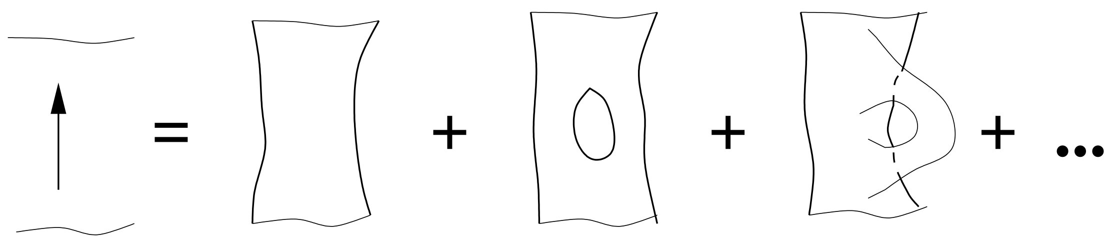

Figure 6: The genus expansion for theories with open strings. Notice that one must include handles and boundaries .

is the so-called genus expansion (the genus of a closed Riemann surface is the number of handles. In general it is more useful to classify 2d surfaces (possibly with boundaries) by their Euler number, defined by $\xi = 2-2g-n_b$, with $g$ and $n_b$ the numbers of handles and boundaries, respectively).

Formally, the amplitude is given by

$$\langle b|\text{evolution}|a\rangle = \sum_{\text{worldsheets}} \int [\mathcal{D}X]\, e^{-S_P[X]}\, \mathcal{O}_a[X]\, \mathcal{O}_b[X] \tag{6}$$

where $\mathcal{O}_i[X]$ are the so-called vertex operators, which put in the information about the incoming and outgoing state. They are very important in tring theory and conformal field theory but we will not discuss them much in these lectures.

Notice that the quantity (6) is basically a quantum correlation function between two operators in the 2d field theory. However, notice the striking fact that (6) is in fact a sum of such correlators for 2d field theories living in 2d spaces with different topologies. Certainly it is a strange prescription, a strange quantity, in the language of 2d field theory. However, it is the prescription that arises naturally from the spacetime point of view.

The basic string interaction processes and their strengths are shown in figure 7. It is important to notice that these vertices are delocalized in a

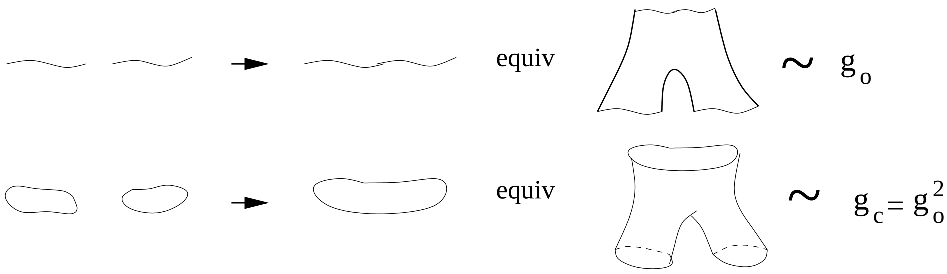


Figure 7: Basic interaction vertices in string theory.

spacetime region of typical size $L_s$. At low energies $E \ll M_s$ they reduce to usual point particle interaction vertices.

There is also one vertex, shown in figure 8. It couples two open strings with one closed string. It is important to notice that the process that turns the closed strings into a closed one corresponds locally on the worldsheet exactly to joining two open string endpoints (twice). This coupling cannot be forbidden in a theory of interacting open strings (since this process also mediates the coupling of three open strings), so it implies that any theory of interacting open strings necessarily contains closed strings. (The reverse statement is not valid, it is possible to have interacting theories of closed strings without open strings).

A fundamental property of string theory is that the amplitudes of the theory are finite order by order in perturbation theory. This, along with other nice properties of string interactions (like unitarity, etc) implies that string theory provides a theory which is consistent at the quantum level, it is well defined in the ultraviolet. There are several ways to understand why string theory if free from the ultraviolet divergences of quantum field theory:

**a)** In quantum field theory, ultraviolet divergences occur when two interaction vertices coincide at the same point in spacetime. In string theory,

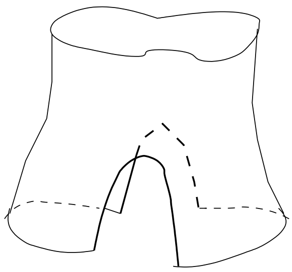

Figure 8: String vertex coupling open strings to closed strings. It implies that theories with open strings necessarily contain closed strings.

vertices are delocalized in a region of size $L_s$, so $L_s$ acts as a cutoff for the would-be divergences.

**b)** As is pictorially shown in figure 9, going to very high energies in some loop, the ultraviolet behaviour starts differing from the quantum field theory behaviour as soon as energies of order $M_s$ are reached. This is so because longer and longer string states start being exchanged, and this leads to a limit which corresponds not to a ultraviolet divergence, but to an infrared limit in a dual channel.

**c)** More formally, using conformal invariance on the worldsheet, any limit in which a string diagram contains coincident or very close interaction vertices can be mapped to a diagram with well-separated vertices and an infinitely long dual channel. This is a formalization of the above pictorial argument.

Using the above rules for amplitudes, it is possible to compute interactions between the massless oscillation modes of string theory. These interactions turn out to be invariant under gauge and diffeomorphism transformations for spacetime fields. This means that the massless 2-index tensor $G_{\mu\nu}$ contains only two physical polarization states, and that it indeed interacts as a graviton. Also, massless vector bosons $A_\mu$ have only two physical polarizations,

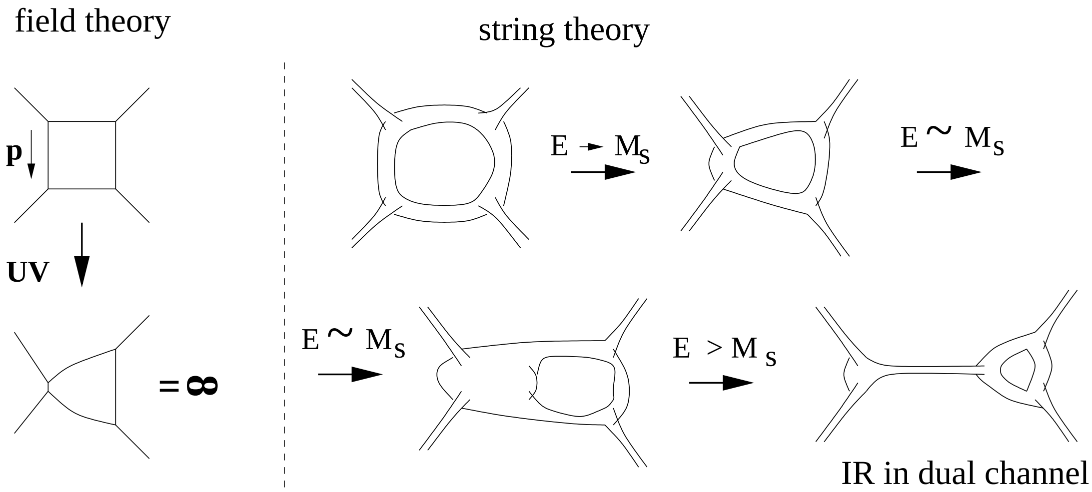


Figure 9: Different ultraviolet behaviours in quantum field theory and in string theory. When high energy modes exchanged in the loop reach energies of order $M_s$, long strings start being exchanged and dominate the amplitude. So at those energies the behaviour differs from the quantum field theory divergence, which is effectively cut-off by $M_s$. The ultra-high energy regime corresponds to exchange of very long strings, which can be interpreted as the infrared regime of a 'dual channel diagram'. .

and interact exactly as gauge bosons. We will not discuss these issues in the present lectures, but a good description can be found in [2] or [1].

Hence, string theory provides a unified description of gauge and gravitational interactions, which is consistent at the quantum level. It provides a unified ultraviolet completion for these theories. This is why we love string theory!

## 1.4 Critical dimension

Conformal invariance in the 2d worldsheet theory is a crucial property for the consistency of the theory. However, this symmetry of the classical 2d field theory on the worldsheet may in principle not be preserved in the 2d quantum field theory, it may suffer what is called an anomaly (a classical symmetry which is not preserved at the quantum level), see discussion in chapter 3 of [1].

As is usual in quantum field theories with potential anomalies, the anomaly disappear for very specific choices of the field content of the theory. In the case of the conformal anomaly of the 2d worldsheet field theory, the field content is given by $d$ bosonic fields, the fields $X^\mu(\sigma, t)$. In order to cancel the conformal anomaly, it is possible to show that the number of fields in the 2d theory must be 26 bosonic fields, so this is the number of $X^\mu$ fields that we need to consider to have a consistent string theory.

Notice that this is very striking, because the number of fields $X^\mu$ is precisely the number of spacetime dimensions where the string propagates. The self-consistency of the theory forces us to admit that the spacetime for this string theory has 26 dimensions. This is the first situation where we see that properties of spacetime are constrained from properties of the worldsheet theory. In a sense, in string perturbation theory the worldsheet theory is more fundamental than physical spacetime, the latter being a derived concept.

Finally, let us point out that there exist other string theories where the worldsheet theory contains other fields which are not just bosons (superstring theories, to be studied later on). In those theories the anomaly is different and the number of spacetime dimensions is fixed to be 10.

## 1.5 Overview of closed bosonic string theory

In this section we review the low-lying states of the bosonic string theory introduced above (defined by 26 bosonic degrees of freedom in the worldsheet, with Polyakov action), and their interactions.

The lightest states in the theory are

- the string goundstate, which is a spacetime scalar field $T(X)$, with tachyonic mass $\alpha' M^2 = -2$. This tachyon indicates that bosonic string theory is unstable, it is sitting at the top of some potential. The theory will tend to generate a vacuum expectation value for this tachyon field and roll down the slope of the potential. It is not know whether there is a minimum for this potential or not; if there is, it is not know what kind of theory corresponds to the configuration at the potential minimum. The theories we will center on in later lectures, superstrings, do not have such tachyonic fields, so they are under better control.

- a two-index tensor field, which can be decomposed in its symmetric (traceless) part, its antisymmetric part, and its trace. All these fields are massless, and correspond to a 26d graviton $G_{MN}(X)$, a 26d 2-form $B_{MN}(X)$ and a 26d massless scalar $\phi(X)$, known as the dilaton. These fields are also present in other string theories.

Forgetting the tachyon for the moment, it is possible to compute scattering amplitudes. It is possible to define a spacetime action for these fields, whose tree-level amplitudes reproduce the string theory amplitudes in the low energy limit $E \ll M_s$, usually denoted point particle limit or $\alpha' \to 0$.

This action should therefore be regarded as an effective action for the dynamics of the theory at energies below $M_s$. Clearly, the theory has a cutoff $M_s$ where the effective theory ceases to be a good approximation. At that scale, full-fledged string theory takes over and softens the UV behaviour of the effective field theory.

The spacetime effective theory for the string massless modes is

$$S_{\text{eff.}} = \frac{1}{2k_0^2} \int d^{26}X \, (-G)^{1/2} \, e^{-2\phi} \{ R - \frac{1}{12} H_{MNP} H^{MNP} + 4\partial_M\phi\partial^M\phi \} + \mathcal{O}(\alpha') \quad (7)$$

where $M, N, P = 0, \ldots, 25$, and $H_{MNP} = \partial_{[M} B_{NP]}$. Notice that very remarkably this effective action is invariant under coordinate transformations in 26d, and under the gauge invariance (with 1-form gauge parameter $\Lambda_M(X)$)

$$B_{MN}(X) \to B_{MN}(X) + \partial_{[M}\Lambda_{N]}(X) \quad (8)$$

(which in the language of differential forms reads $B \to B + d\Lambda$).

Notice that the coupling constant of the theory $k_0$ can be changed if the scalar field $\phi$ acquires a vacuum expectation value $\phi_0$. Hence, the spacetime string coupling strength (the $g_c$ in our interaction vertices) is not an arbitrary external parameter, but it is a vacuum expection value for a dynamical spacetime field of the theory. In many other situations, string models contain this kind of 'parameters' which are actually not external parameters, but vevs for dynamical fields of the theory. This is the familiar statement that string theory does not contain external dimensionless parameters.

These fields, like the dilaton and others, are known as moduli, and typically have no potential in their effective action (so they can take any vev, in principle). This also leads to phenomenological problems, because we do not observe such kind of massless scalars in the real world, whereas they are ubiquitous in string theory.

The above action is said to be written in the string frame (which means that the field variables we are using are those naturally associated with the vertex operators one constructs from the 2d conformal field theory viewpoint. From the specetime viewpoint, it is most convenient to redefine the fields as

$$\tilde{G} = e^{\phi_0 - \phi} \quad ; \quad \tilde{\phi} = \phi - \phi_0 \tag{9}$$

to obtain the action

$$S_{\rm eff.} = \frac{1}{2k^2} \int d^{26}X \, (-\tilde{G})^{1/2} \, \{ \, \tilde{R} + \frac{1}{12} e^{-\tilde{\phi}/12} \, H_{MNP} \, H^{MNP} \, - \, \frac{1}{6} \partial_M \tilde{\phi} \partial^M \tilde{\phi} \, \} \, + \mathcal{O}(\alpha') \tag{10}$$

with indices raised by $\tilde{G}$. This action is said to be written in the Einstein frame, because it contains the gravity action in the canoncial Einstein form. Notice that the change between frames is just a relabeling of fields, not a coordinate change or anything like that.

So we have obtained an effective action which reduces basically to Einstein gravity (plus some additional fields). The 26d Planck mass is given by $M_{26d}^{24} = M_s^{24}/g_c^2$. This effective theory is not renormalizable, and is valid only up to energies $M_s$, which is the physical cutoff of the effective theory; there is however an underlying theory which is well defined at the quantum level, valid at all energies (UV finite) and which reduces to the effective theory below $M_s$. String theory has succeeded in providing a consistent UV completion of Einstein theory.

It is also important to point out that this version of quantum gravity is also consistent with gauge invariance, for instance with the gauge invariance of the 2-form fields. Other string theories (with open strings, or some superstrings) also contain vector gauge bosons, with effective action given by Yang-Mills. So the theory contains gauge and gravitational interactions consistently at the quantum level.

Let us conclude by mentioning some of the not-so-nice features of the theory at hand.

- First, it lives in 26 dimensions. We will solve this issue in subsequent lectures by the process known as compactifications

- The theory does not contain fermions. This will be solved by introducing a more interesting kind of string theory (by modifying the worldsheet field content), the superstrings. These theories still live in 10 dimensions so they need to be compactified as well

- The theory does not contatin non-abelian vector gauge bosons. Such gauge bosons are however present in some superstring theories (heterotic and type I, and in type II theories in the presence of topological defects).

- Other questions which remain unsolved (like supersymmetry and supersymmetry breaking, or the moduli and vacuum degeneracy problems) will also appear along the way.

One issue that can be addressed at this point is to obtain four-dimensional physics (at low energies) from a theory originally with more dimensions. The standard technique to do so is known as compactification, and can be applied not only to reduce the closed bosonic string theory to four-dimensions, but also to other more interesting string theories. For this reason, it is interesting to study compactification right now. However, before that, we need to take a small detour and learn how to formulate string theory in spacetimes more complicated than Minkowski space.

## 1.6 String theory in curved spaces

See for instance sect. 3.7 in [1].

We have obtained an effective action for the low-lying modes of string theory. In principle, configurations of these fields which satisfy the corresponding (classical) equations of motion should correspond to classical backgrounds where strings can propagate.

However, the worldsheet description we provided is only valid when the background is trivial (26d Minkowski space). It is a natural question to ask how the worldsheet theory is modified so that it describes propagation of a string in a spacetime with non-trivial metric $G_{MN}(X)$, and non-trivial background for the two-index antisymmetric tensor field $B_{MN}(X)$ and the dilaton $\phi$.

The effect of the metric is relatively simple: The string action is still the worldsheet area, now computed using the new metric in spacetime. Using the Polyakov version of the worldsheet action, eq (4) generalizes to

$$S_P^G\,[X(\sigma,t),g(\sigma,t)] \;=\; \frac{1}{4\pi\alpha'}\int_\Sigma d\sigma\,dt\,\sqrt{-g}G_{MN}[X(\sigma,t]]\,g^{\alpha,\beta}\,\partial_\alpha X^M(\sigma,t)\,\partial_\beta X^N(\sigma,t) \quad (11)$$

Where $G_{MN}(X)$ is a function(al) of $X(\sigma,t)$. This action is also known as non-linear sigma model, for historical reasons not to be discussed here.

One may wonder about the double role played that the spacetime graviton in string theory. On one hand, we have claimed that the graviton arises as one of the states in the string spectrum in flat space. On the other, a background metric, made out of gravitons, appears explicitly in the worldsheet action of a string propagating in curved space. (This issue is related to the discussion on how to split a field configuration as a background plus a fluctuation around it.)

This dicotomy can be understood in detail for metrics which are small perturbations of flat space metric

$$G_{MN} = \eta_{MN} + \delta G_{MN} \quad (12)$$

Replacing this into the worldsheet action (11), we obtain an expansion around the flat space action. In a path integral, expanding the exponential as well one gets that amplitudes in curved space can be regarded as amplitudes in flat space with corrections due to graviton insertions

$$\begin{aligned}\int [\mathcal{D}X]\,e^{-S_P^G} \;=\; & \int [\mathcal{D}X]\,e^{-S_P^\eta} + \int [\mathcal{D}X]\,e^{-S_P^\eta}\,\mathcal{O}_\mathcal{G}[\mathcal{X}] + \\ & \int [\mathcal{D}X]\,e^{-S_P^\eta}\,\mathcal{O}_\mathcal{G}[\mathcal{X}]\,\mathcal{O}_\mathcal{G}[\mathcal{X}] + \ldots \end{aligned} \quad (13)$$

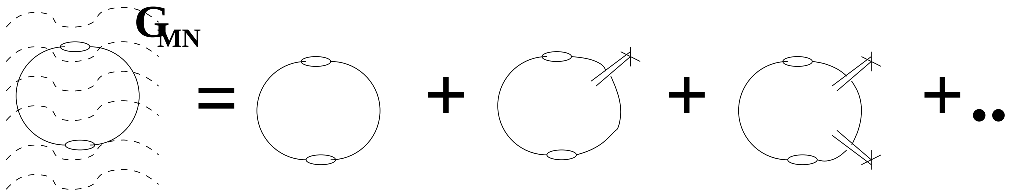


Figure 10: Amplitudes in curved space can be regarded as a resummation of amplitudes of amplitudes in flat space, with increasing number of graviton insertions. Hence the curved background can be regarded as built out of gravitons, in quite an explicit way.

where $\mathcal{O}_{\mathcal{G}}[\mathcal{X}]$ is the vertex operator for the graviton, as a state in the string spectrum. Recalling that a path integral with a vertex operator insertion corresponds to addint an external leg, the situation is pictorially shown in figure 10

Even for metrics which cannot be regarded as deformations of flat space (for instance, if the corresponding manifolds are topologically different from flat space), then (11) is the natural prescription.

Since there are also other massless fields in the spectrum of the string, it is natural to couple them to the worldsheet, so as to obtain a worldsheet action for strings propagating on non-trivial backgrounds. The resulting action is

$$S_P^G\,[X(\sigma,t),g(\sigma,t)] = \frac{1}{4\pi\alpha'}\int_\Sigma d\sigma\,dt\,\sqrt{-g}\,[\,G_{MN}[X(\sigma,t]\,g^{\alpha\beta}\,\partial_\alpha X^M(\sigma,t)\,\partial_\beta X^N(\sigma,t)\;+ \\ +\,B_{MN}[X(\sigma,t]]\,\epsilon^{\alpha,\beta}\,\partial_\alpha X^M(\sigma,t)\,\partial_\beta X^N(\sigma,t)\;+\;\alpha' R[g]\phi\,] \quad (14)$$

It satisfies the criterion that for backgrounds near the trivial one it expands as resummation over insertions of the corresponding vertex operators. Moreover, the different terms have a nice interpretation also in the form (14).

- We have already explained that the piece depending on $G_{MN}$ is simply the area of the worldsheet as measured with the curved spacetime metric. That is, the natural generalization of the Nambu-Goto idea.

• The term that depends on $B_{MN}$ is exactly the result of interpreting the two-index tensor as a 2-form $B_2 = B_{MN} dX^M \wedge dX^N$ in spacetime, and integrating it over the 2-dimensional surface given by the world-sheet. In the language of differential forms

$$S_B = \frac{1}{4\pi\alpha'} \int_\Sigma B_2 \tag{15}$$

Notice that the term is purely topological in spacetime, it does not depend on the spacetime metric.

The physical interpretation of this term is that strings are charged objects with respect to $B_2$, when the latter is regarded as a gauge potential (recall the gauge invariance $B_2 \to B_2 + d\Lambda$). It is the analog of the minimal coupling of a point particle to a vector gauge potential $A_1$, given by integrating $A_1$ over the particle worldline).

• The term that depends on $\phi$ is very special. In principle it corresponds to an Einstein term for the 2-dimensional worldsheet metric $g_{\alpha\beta}(\sigma, t)$. However, 2d gravity is very special, is almost topological. This means that in 2 dimensions, the integral of the curvature scalar over a surface is, by Gauss theorem, just a number, determined by the topology of the surface. This number is simply the Euler number of the surface, given by

$$\xi = 2 - 2g - n_b \tag{16}$$

where $g$ is the number of handles and $n_b$ is the number of boundaries.

Insertion of this term in an amplitude corresponds exactly to weighting it by a factor $e^{-\phi\xi}$. It is possible to check that the power of $e^{-2\phi}$ appearing in the amplitude for a given diagram (worldsheet topology) is exactly the same power as for the closed string coupling $g_c$ (in theories with open strings, the same is true for powers of $e^{-\phi}$ and of the open string coupling $g_o$ (recall $g_c = g_o^2$). This is an alternative way of rediscovering that the vev for the dilaton plays the role of the string coupling constant.

Again we see that string theory does not contain external adimensional parameters. All parameters are in fact vevs for dynamical fields.

It is important to realize that in the presence of non-trivial backgrounds the worldsheet action, regarded as a 2d field theory, is no longer a free field theory. From this viewpoint, it is natural to study it in perturbation theory around the free theory. The expansion parameter is $\alpha'/R^2$, where $R$ is the typical length scale of variation of any spacetime field, so this is known as the $\alpha'$ expansion.

It is important to realize that string theory in a general background has therefore a double expansion. First, there is the loop expansion in the string coupling constant, which corresponds to the genus expansion summing over worldsheet topologies. Second, for any given worldsheet topology, the computation of the path integral over the (interacting) 2d field theory is done as a loop expansion in the 2d world, the $\alpha'$ expansion.

Both expansion are typically very involved, and most results are known at one loop in either expansion. The issue of the $\alpha'$ expansion makes it very difficult to use string theory in regimes where very large curvatures of spacetime are present, like black hole or big-bang singularities.

This is a bit unfortunate, because $\alpha'$ mainly encode effects which encode the fact that the fundamental object in string theory is an extended object, rather than a point particle. For instance, the geometry seen by string theory, at scales around $L_s$, is different from the geometry a point particle would see. This new notion of geometry (which is still vague in many formal respects) is called stringy geometry (or quantum geometry, by B. Greene, because it corresponds to taking into account loops in $\alpha'$, in the 2d quantum field theory).

Happily, there still exist some simple enough situations where $\alpha'$ effects are tractable, and can be seen to be spectacular. For instance, the fact the complete equivalence of string theory on two different spacetime geometries, once stringy effects are taken into account (T-duality).

We conclude with an important issue. We have emphasized the importance of conformal invariance of the 2d worldsheet field theory in order to have a consistent string theory (with finite amplitudes, etc). Therefore, the interacting 2d field theory given by (14) should correspond to a conformal field theory. In general, this can be checked only order by order in the $\alpha'$ expansion, and in practice the results are known at leading order (one loop in $\alpha'$). In perturbation theory in the 2d field theory, conformal invariance means that the (one-loop in $\alpha'$ beta functions for the coupling constants in the 2d field theory lagrangian) vanish.

Notice that in a sense, the background fields play the role of these coupling constants. The condition that their beta function equals zero amounts to the constraint that the background fields obey some differential equation. The amazing thing is that these differential equations are exactly the equations of motion that one obtains from the spacetime effective action for the spacetime fields (7). That is, string propagation is consistent (2d action is conformal field theory) exactly in background which obey the equations of motion from the spacetime effective action (derived from scattering amplitudes, etc, i.e. from a different method). I regard this as an amazing self-consistency property of string theory.

It should be pointed out that these statements remain valid for string theories beyond the closed bosonic theory we are studying for the moment.

A final comment concerns an alternative approach to study string theory beyond flat space. A whole lot is known about two-dimensional field theory which are exactly conformal [3]. Some of them can be solved exactly, namely one can give expression for any 2d correlator, exactly i.e. to all orders in the 2d loop expansion. One can then imagine using conformal field theories of this kind (so-called exactly solvable conformal field theories) to describe the string worldsheet. The question is then to identify what is the background where the string is propagating. In several cases this can be done and corresponds to very exotic possibilities, for instance Witten's black hole, compact curves spaces with very small size (or order the string length, etc. The importance of these models is that by construction all $\alpha'$ effects are included. Another motivation is that in this language it is clear that spacetime is in a sense a derived concept in string theory, and that the worldsheet theory is more fundamental (this view was dominant before '95, and is perhaps slightly changed nowadays; still it has a point).

## 1.7 Compactification

In this section we study a special and very important class of backgrounds, which lead, in the low-energy limit, to effective theories with smaller number of dimensions than the original one. We center on constructing models which produce four-dimensional physics, of course (although people often study e.g. six-dimensional models, etc).

The idea is to consider string propagation in a spacetime of the form

$$X_{26} = M_4 \times X_{\text{comp.}} \tag{17}$$

where $M_4$ is 4d Minkowski and $X_{\text{comp}}$ is a compact 22-dimensional manifold (with Euclidean signature), called the compactification manifold, or internal space.

The recipe to write the worldsheet action is as above. In general, it corresponds to a nonlinear sigma model, an interacting theory, and we can

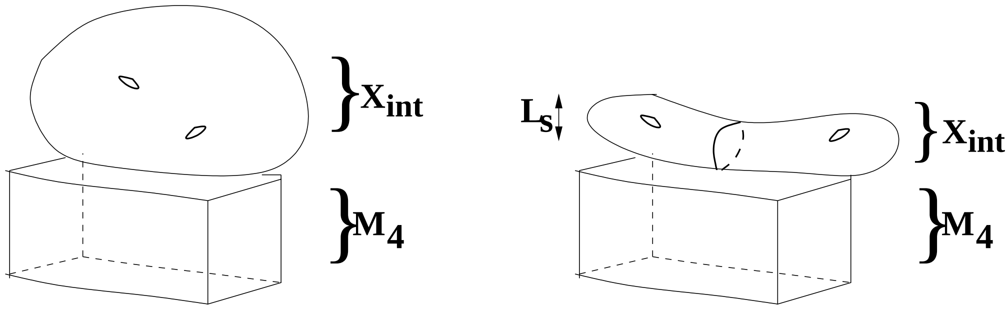


Figure 11: Picture of compactification spacetimes; thick small lines represent string states which are light in the corresponding configuration. When the internal manifold has size of the order of $L_s$, stringy effects (which do not exist in theories of point particles) become relevant; for instance, string winding modes (where a closed string winds around some internal dimension) may be light and appear in the low energy spectrum (even if they do not appear, they may modify importantly the low-energy effective action).

study it only in the $\alpha'$ expansion (and often at leading order). From the spacetime viewpoint, this means that we study the point particle limit, we use the effective field theory (7), which is basically 26d Einstein theory (plus other fields in this background). This approximation is good as long as the typical size of the compactification manifold is larger than the string scale. In this regime, our theory looks a standard Kaluza-Klein theory.

In very special cases (mainly when the compactification manifold is a torus) the sigma model reduces to a free field theory, which is solved exactly (in the sense of the $\alpha'$ expansion). In such cases, the theory can be studied reliably even for small sizes of the compactification manifold. When these sizes are of the order of the string length, stringy effects become spectacular, and there happen things which are unconceivable in a theory of point particle. For instance, a typical stringy effect is having closed strings wrapping around the non-trivial curves in the internal space. For large volumes, these states are hugely massive, and do not affect much the low-energy physics. For stringy volumes, such states can be very light (as light as other 'point-particle' like modes, or even massless!) and do change the low-energy physics.

Let us first consider large volume compactifications for the moment (so we work in the effective field theory approach) and explain why the low-energy physics is four-dimensional. Consider first a toy model of a 5d spacetime of the form $X_5 = M_4 \times S^1$, on which a 5d massless scalar field $\varphi(x^0, \ldots, x^4)$ propagates with 5d action

$$S_{5d\varphi} = \int_{M_4\times S^1} d^5x\, \partial_M\varphi\partial^M\varphi \tag{18}$$

Since $x^4$ parametrizes a circle, it is periodic, and we can expand the $x^4$ dependence in Fourier modes

$$\varphi(x^0, \ldots, x^4) = \sum_{k\in\mathbf{Z}} e^{2\pi i k x^4/L}\, \varphi_k(x^0, \ldots, x^3) \tag{19}$$

where $L$ is the lenght of $\S^1$.

From the 4d viewpoint, we see a bunch of 4d scalar fields $\phi_k(x^0, \ldots, x^4)$, labeled by the integer index $k$, the 5d momentum. The 4d spacetime mass of those fields increases with $k^2$. To see that, take the 5d mass-shell condition

$$P^2 = 0 \quad \text{that is } P_{4d}^2 + p_5^2 = 0 \tag{20}$$

For the field $\phi_k$, we have

$$P_{4d}^2 + (k/L)^2 = 0 \tag{21}$$

which means that the 4d mass of the field $\phi_k$ is $m_k^2 = (k/L)^2$

At energies much lower than the compactification scale $M_c = 1/L$, $E \ll 1/L$, the only mode which is observable is the zero mode $\phi_0(x^0, \ldots, x^3)$. So we see just a single 4d field, with a 4d action, which is obtained by replacing $\phi(x^0, \ldots, x^4)$ in (18) by the only component we are able to excite $\phi_0(x^0, \ldots, x^3)$. The $x^4$ dependence drops and we get

$$S_{eff} \; = \int_{M_4} d^4x \, L \, \partial_\mu \varphi_0 \partial^\mu \varphi_0 \tag{22}$$

So we recover 4d physics at energies below $M_c$. This is the Kaluza-Klein mechanism, or Kaluza-Klein reduction. The massive 4d fields $\phi_k$ are known as Kaluza-Klein (KK) excitations or KK replicas of $\phi_0$.

As explained in the first lecture, the Kaluza-Klein reduction works for any higher dimensional field. An important new feature arises when the original higher dimensionl field has non-trivial Lorentz quantum numbers. The procedure is then to first decompose the representation of the $SO(d)$ higher-dimensional Lorentz group with respect to the 4d one $SO(4)$ (i.e. separate different components according to their behaviour under 4d Lorentz), and finally perform KK reduction for each piece independently. For instance, for a 5d graviton we have the KK reduction

$$\begin{aligned} G_{MN}(x^0,\ldots,x^4) \to \ & G_{\mu\nu}(x^0,\ldots,x^4) \to & G^{(0)}_{\mu\nu}(x^0,\ldots,x^3) \\ & G_{\mu 4}(x^0,\ldots,x^4) \to & G^{(0)}_{\mu 4}(x^0,\ldots,x^3) \\ & G_{44}(x^0,\ldots,x^4) \to & G^{(0)}_{44}(x^0,\ldots,x^3) \end{aligned} \qquad (23)$$

where the first step is just decomposition in components, and the second is KK reduction. We therefore obtain, at the massless level, a 4d graviton, a 4d $U(1)$ gauge boson, and a 4d scalar. Recall that diffeormophism invariance in 5d implies gauge invariance of the 4d vector gauge boson. Also notice that the vev for the scalar field is $G_{44}$, which is related to the length of the internal circle. Therefore, it is not an external parameter, but the vev of a 4d dynamical scalar field. On the other hand, the compactification is consistent (solves the 5d equations of motion) no matter what circle radius we choose; this implies that in the 4d effective action there is no potential for the 4d scalar, it parametrizes what is called a flat direction of the potential, the field is called a modulus (and it is similar to the string theory dilaton in many respects).

Obs: If the higher-dimensional field theory contains massive fields with mass $M$, the 4d KK tower has masses $m_k^2 = M^2 + (k/L)^2$, so they will not be observable at energies below $M$.

The lesson learned here is very general, and can be applied to compactification of any theory on any internal manifold, and an arbitrary set of fields. In particular, it can be applied to string theory. Massless 26-dimensional string states will lead to massless 4d fields corresponding to the zero modes in the KK reduction. KK replicas are not visible at energies below $M_c$. Massive 26-dimensional string states give massive 4d states, with masses at least or order $M_s$, which is huge, and are not observable at low energies.

Let us skip the details of KK reduction in manifolds more general than tori, and simply say that in general the role played by the momentum $k$ in toroidal directions is played by the eigenvalues of the laplace operator in the internal manifold (which are also quantized in units of $1/L$, where $L$ is the typical length of the internal space).

# References


[1] J. Polchinski, 'String theory', Vol 1.

[2] M.Green, J.Schwarz, E.Witten, 'Superstring theory', vol1.

[3] P. Ginsparg, 'Applied conformal field theory', Lectures given at Les Houches Summer School in Theoretical Physics, Les Houches, France, Jun 28 - Aug 5, 1988. Published in Les Houches Summer School 1988:1-168;
A.N. Schellekens, 'Introduction to conformal field theory', Based on lectures given at Grundlagen und neue Methoden der Theoretischen Physik, Saalburg, Germany, 3-16 Sep 1996 and at the Universidad Autonoma, Madrid, Oct-Dec 1995, Published in Fortsch.Phys.44:605-705,1996