Non-perturbative effects in (weakly coupled) string theory

1 Motivation

We have seen that non-perturbative states are very important in the structure of string theory at finite string coupling. In this lecture we will discuss that non-perturbative states are also essential even in the weakly coupled regime in certain situations, in which the purely perturbative sector of the theory is incomplete and leads to divergent answers for physical quantities.

There are different situations of this kind in string theory. In this lecture we center on two particular examples: enhanced gauge symmetries in type IIA/M-theory on K3, and conifold singularities in Calabi-Yau compactifications.

2 Enhanced gauge symmetries in type IIA theory on K3

2.1 K3

K3 is the only compact topological space with four dimensions admitting a Calabi-Yau metric, i.e. of $SU(2)$ holonomy (besides the four-torus $\mathbf{T^4}$, which has trivial holonomy). We now state without proof some of its properties, see [1] for a more extensive discussion.

Its Hodge numbers are

$$
\begin{array}{ccccc}
 & & h_{0,0} & & \\
 & h_{1,0} & & h_{0,1} & \\
h_{2,0} & & h_{1,1} & & h_{0,2} \\
 & h_{2,1} & & h_{1,2} & \\
 & & h_{2,2} & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 0 & &
\end{array}
$$

The lattice of homology classes (with integer coefficients) turns out to be even and self-dual. We can split the corresponding harmonic forms in self-dual and anti-self dual forms, with respect to the 4d metric. This introduces a signature in the homology lattice, with 20 self-dual forms (given by 19 of the $(1,1)$ forms and a linear combination of the $(0,0)$ and the $(2,2)$ forms) and 4 anti-self-dual forms (one $(1,1)$ form and a combination of the $(0,0)$ and $(2,2)$). Hence, the (integer) homology of K3 has the very suggestive form of a even self-dual lattice with a lorentzian $(20,4)$ signature.

The moduli space of Calabi-Yau metrics on K3 is 58-dimensional. There are 38 parameters specifying the complex structure on K3 (i.e, telling us how to cook up comlex coordinates starting from real ones), and 20 parameters specifying the Kahler class.

At particular points (or more precisely, at some locus) in this metric moduli space, K3 develops singularities, which are always of orbifold type [1] $\mathbf{C^2}/\mathbf{\Gamma}$, with Γ a discrete subgroup of $SU(2)$. This limits correspond to points in moduli space where some 2-cycles within K3 have been tuned to zero size, see figure 1. The simplest such situation is $\mathbf{C^2}/\mathbf{Z_2}$, where just one 2-cycle collapses to zero size.

Notice that tuning more parameters, one can go to a limit where the whole K3 has the form of a toroidal orbifold, of the kind studied in the

[1] That is, the only singular local geometries that are consistent with $SU(2)$ holonomy are of orbifold type. In three complex dimensions there exist singularities consistent with $SU(3)$ holonomy, which are not of orbifold type.

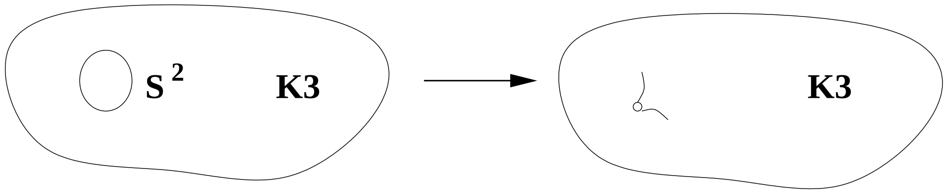

Figure 1: In K3, singularities arise when some 2-cycles are tuned to have zero size.

lecture on orbifold compactification. For instance, there exist points in the moduli space of metrics in K3 where it is of the forms $\mathbf{T^2/Z_2}$, At each of the 16 fixed points of the orbifolds the local geometry is $\mathbf{C^2/Z_2}$ and there is a zero size 2-cycle.

2.2 Type IIA on K3

We are interested in studying compactification of type IIA theory on K3. Since K3 has $SU(2)$ holonomy, each 10d gravitino leads to one 6d gravitino. The resulting 6d theory has therefore 16 unbroken supercharges and (being non-chiral) corresponds to 6d $\mathcal{N} = (1,1)$ supersymmetry. The main massless supermultiplets are

• the gravity multiplet, containing the graviton $G_{\mu\nu}$, a 2-form $B_{\mu\nu}$, a real scalar ϕ, four gauge bosons A_μ, two gravitinos $\psi_{\mu\alpha}$, $\psi_{\mu\dot{\alpha}}$, and two Weyl fermions ψ_α, $\psi_{\dot{\alpha}}$, all of opposite chiralities.

• the vector multiplet, with one gauge boson A_μ, four real scalars, and two Weyl fermions of opposite chiralities.

As usual, it will be thus enough to identify the bosonic fields in the 6d theory, since the fermions simply complete the supermultiplets.

Since K3 is curved (unless we are sitting at the point of moduli space corresponding to some global orbifold geometry) the 2d worldsheet theory is not free, and we can discuss compactification only in the supergravity approximation. This will provide the spectrum in the limit where all length scales in K3 are large (in particular all 2-cycles are large), usually refered to as large volume regime. Denoting Σ_a the 22 $(2,2)$ 2-cycles, Π, $\overline{\Pi}$ the $(2,0)$ and $(0,2)$ 2-cycles, and Π_a the 20 $(1,1)$ 2-cycles, the Kaluza-Klein reduction of the massless 10d bosonic fields gives

IIA		Gravity	Vector
G	$\to$	$G_{\mu\nu}$	38+20 scalars
B	$\to$	B_2	$\int_{\Sigma_a} B$
ϕ	$\to$	ϕ	
A_1	$\to$	A_1	
C_3	$\to$	C_3, $\int_\Pi C_3$, $\int_{\overline{\Pi}} C_3$	$\int_{\Pi_a} C_3$

We thus obtain the 6d $\mathcal{N}=(1,1)$ supergravity multiplet and 20 vector multiplets (with gauge group $U(1)^{20}$).

The structure of the moduli space is (locally) of the form

$$\frac{SO(20,4)}{SO(20)\times SO(4)} \tag{1}$$

In principle this can be determined from supergravity, in the large K3 volume regime. However it turns out to be completely determined by supersymmetry, so it is exactly of this form (locally), with no α' or g_s corrections. The above structure is related as we know to the moduli space of 24-dimensional $(20,4)$ lorentzian even self-dual lattices up to rotations within the 20d and 4d signature eigenspaces. In K3, it can be regarded as the moduli space of ways of splitting the 24d lattice of homology classes into sublattices of self-dual and anti-self-dual forms. More technical considerations involving mirror symmetry moreover allows to determine the global structure of moduli space of IIA on K3 [2], which turns out to be

$$\frac{SO(20,4)}{SO(20)\times SO(4)\times SO(20,4;\mathbf{Z})}. \tag{2}$$

2.3 Heterotic on T^4 / Type IIA on K3 duality

This is a prototypical example of string duality below ten dimensions. Let us provide a list of supporting evidence for it; for details, see [3, 4].

• The spectrum of heterotic string theory on $\mathbf{T^4}$ (either for the $E_8 \times E_8$ or the $SO(32)$ theories, since they are equivalent upon toroidal compactification) , at a generic point of its moduli space (see lecture on toroidal compactification of superstrings) is given by the 6d $\mathcal{N} = (1,1)$ supergravity multiplet and 20 vector multiplets (with gauge group $U(1)^{20}$). The bosonic fields arise from $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ, the 24 abelian gauge bosons $G_{m\mu}$, $B_{m\mu}$, A^I_μ and the 80 scalars G_{mn}, B_{mn}, A^I_m, with $m = 1,\ldots,4$, $I = 1,\ldots,16$.

• The structure of the moduli space of both theories agrees, even globally. As we know, $\mathbf{T^4}$ compactifications of heterotic string theory have (2) as their moduli space (with the lattice corresponding to the Narain lattice of left- and right-moving momenta).

• The low-energy effective actions of bothe theories is the same, up to a redefinition of the fields. Defining the 6d dilaton by $e^{-2\phi_6} = V_{X_4} e^{-2\phi}$, with V_{X_4} the volume of the internal space, the actions agree up to the field redefinition

$$\begin{aligned} \phi_6' &= \phi_6 \quad ; \quad H_3 &&= e^{-2\phi_6} *_6 H_3 \\ G' &= e^{-2\phi_6}\, G \quad ; \quad A^{I}_a{}' &&= A^I_a \end{aligned} \tag{3}$$

The relations work as above in any direction of the duality. The above mapping implies that when the IIA theory has large 6d coupling, it admits a dual perturbative description in terms of weakly coupled heterotic strings, and vice versa.

- The spectrum of BPS states agrees in both theories. For instance,

Heterotic on $\mathbf{T^4}$		Type IIA on K3
F1	$\longleftrightarrow$	NS5 wrapped on K3
NS5 wrapped on $\mathbf{T^4}$	$\longleftrightarrow$	F1
momentum k_i		D2 wrapped on any
winding w_i	$\longleftrightarrow$	of the 22 2-cycles or
momentum P_I		D0, or D4 wrapped on K3

The tensions of these objects agree, and objects related as above have equivalent world-volume field theories.

The fundamental string of one theory corresponds to the wrapped five-brane of the other. Namely starting with the IIA theory and going to the limit of large 6d coupling the wrapped fivebrane becomes weakly coupled and sets the lightest scale, hence dominating the dynamics. In fact, it is possible to see that the world-volume theory on this wrapped fivebrane is that of a heterotic string (and viceversa of the IIA F1 vs the heterotic NS5).

2.4 Enhanced non-abelian gauge symmetry

The above duality suggests that there must exist an interesting phenomenon at particular points (loci) in the moduli space of type IIA on K3. Indeed, at particular points (or rather, subspaces) of the moduli space of heterotic theory on $\mathbf{T^4}$, some abelian gauge symmetries get enhanced to non-abelian ones. Recalling the left-moving spacetime mass formula

$$\alpha' M_L^2/2 = N_B + \frac{P_L^2}{2} - 1 \tag{4}$$

we see that when the parameters are tuned such that some state has $P_L^2 = 2$, we get two new massless state, corresponding to $\pm P_L$. They corresponds to a 6d vector multiplet, and carry charges ± 1 under some linear combination of the $U(1)$ gauge factors in the generic gauge group. Thus, they enhance the corresponding $U(1)$ gauge group to $SU(2)$.

This process has clear generalizations. If the parameters are tuned in such a way that additional states reach $P_L^2 = 2$, then we obtaine enhancements to larger gauge factors. In general, any non-abelian gauge symmetry with Lie algebra of type A, D or E (or products thereof) and rank ≤ 24 is possible (Note that only these algebras are possible since they are the only ones with all roots of length square equal to 2).

The states becoming massless are BPS states, so we know that there are new massless states in heterotic theory, even at strong coupling. By duality, this implies that type IIA must have enhanced non-abelian gauge symmetries at particular points in K3 moduli space, even at weak coupling. This is a very surprising conclusion: we have seem that compactification of type IIA theory on large and smooth K3 spaces leads to abelian gauge symmetries. Moreover one can use 2d conformal field theory tecniques to show (exactly in α') that any regular conformal field theory describing propagation of IIA string theory on K3 necessarily leads only to abelian gauge symmetries.

Interestingly enough, it is possible to show that there are points in moduli space of K3 where the 2d conformal field theory breaks down, i.e. the perturbative prescription to compute things in string theory gives infinite answers. Hence we suspect that it is at these points in moduli space where non-abelian gauge symmetries may arise, due to non-perturbative effects (present even at weak coupling!). These points in moduli space correspond to K3 geometries where some 2-cycle is collapsed to zero size and where the integral of B along the 2-cycle vanishes. The simplest situation corresponds to geometries with one collapsed 2-cycle C on which $\int_C B = 0$. As discussed above, this corresponds to the geometry of a local $\mathbf{C^2}/\mathbf{Z_2}$ orbifold singularity.

Now it is easy to identify how gauge symmetry enhancement occurs. The 6d theory contains a $U(1)$ gauge boson arising from $\int_C C_3$. The theory contains 6d particle states charged under it with charges ± 1, arising from D2-branes wrapped on C (with the two possible orientations). It is possible to see that these states are BPS [2], and their mass is (exactly) given by

$$M = \frac{|V_C + ib|}{g_s} \tag{5}$$

where V_C denotes the volume of C and $b = \int_C B$. Hence, at the point or zero size and zero B-field the D2-brane particle is exactly massless, no matter how small the string coupling is. This effect is very surprising, since we see that the non-perturbative sector of the theory leads to significant effects (new massless particles!) even in the weak coupling regime. It is reasonable (and correct) to guess that these new particles in 6d belong to vector multiplets $\mathcal{N} = (1,1)$ supersymmetry, and therefore enhance the gauge symmetry from $U(1)$ to $SU(2)$. Clearly, these D2-brane particles are the duals to the $P_L^2 = 2$ states in heterotic theory. Note that this is in agreement with the mapping of BPS states proposed above.

Several comments are in order

• Notice that in the IIA picture we have perturbative states (the $U(1)$ gauge boson) and non-perturbative ones (the D2-brane particles) on an equal footing. Indeed, they are related by an exact gauge symmetry of the theory.

• In the above discussion we used heterotic/IIA duality to motivate the appearance of enhanced gauge symmetries in IIA compactifications on K3. However, the whole argument about the appearance of new massless charged states could have been done based simply on our knowledge of D-branes and

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[2] Understanding this requires some discussion of the supersymmetry unbroken by D-branes wrapped on cycles in Calabi-Yau space. We chose to skip this discussion for our introductory overview.

the BPS formulae, without any use of string duality. Clearly, we have enough understanding of non-perturbative states in string theory to look for them without help from duality, as we will do in next section.

• Once the additional multiplet of non-perturbative origin is included, the physics of the configuration is completely non-singular. Equivalently, the divergent behaviour of the perturbative sector can be understood as due to incorrectly not including all the massless fields in the dynamics (as often stated, due to integrating out (= to not incluing) the non-perturbative state, incorrectly since it is a massless state that clearly must be included in discussing the low energy dynamics of the system).

• Let us emphasize again that this non-perturbative effect takes place no matter how small the string coupling is.

• The point $V_C = 0$, $b = 0$ is singular from the viewpoint of the 2d world-sheet theory, which only sees perturbative physics. This may seem puzzling at first sight: In the lesson on orbifold compactification we studied orbifold singularities with cycles collapsed to zero size, and they were perfectly well described by simple (in fact, free) 2d worldsheet theories. The key difference, realized in [5], is that the orbifold describe by a free 2d worldsheet theory corresponds to a point in moduli space where $V_C = 0$ but $b \neq 0$ (in fact $b = 1/2$ for $\mathbf{C^2/Z_2}$). In this situation, the D2 particle is very massive at weak coupling, and the perturbative description is accurate and non-singular (gives finite answers for all observables in the theory).

• There is a generalized version of this that explains other non-abelian gauge symmetry enhancements. There is a classification of $\mathbf{C^2\Gamma}$ singularities with Γ a discrete subgroup of $SU(2)$. In this classification there is an infinite A series (corresponding to cyclic $\mathbf{Z_k}$ groups), and infinite D series (dihedral groups) and an E series with three groups (denoted E_6, E_7, E_8). When parameters of K3 are tuned so that it develops a $\mathbf{C^2/\Gamma}$ singularity of A, D, E type (with zero B-fields over the collapsed 2-cycles), non-perturbative states become massless and enhance the gauge symmetry to the corresponding A, D, E gauge group. This provides the IIA dual to the configurations of enhanced gauge symmetries in heterotic compactifications. Moreover, it also establishes a 'physics proof' of the so-called McKay correspondence in mathematics, which establishes a relation between the geometry of orbifold singularities $\mathbf{C^2/\Gamma}$ and Lie algebras.

2.5 Further comments

It is interesting to consider dual realizations of this gauge symmetry enhancement. Indeed, we will find out that it is related to a very familiar phenomenon we have already encountered.

The local geometry of $\mathbf{C^2/Z_2}$ is identical to that of a 2-center Taub-NUT geometry in the limit where the two centers coincide. In fact, it is possible to display the 2-cycle collapsing to zero size in quite an explicit way, see figure 2a. Both spaces differ only in their asymptotic behaviour at infinity, but this is not important for the phenomenon of gauge symmetry enhancement. Therefore, we conclude that multi - Taub-NUT spaces develop enhanced gauge symmetry when two centers coincide, and the B-field is tuned to zero.

Performing now a T-duality along the isometric direction in the Taub-NUT space, the two centers of the Taub-NUT geometry turn into two parallel NS5-branes of IIB theory, sitting at points in the transverse $\mathbf{R^4}$. Their separation in $\mathbf{R^3}$ is determined by the volume and B-field of the T-dual 2-cycle. The non-perturbative D2-brane state now corresponds to a D1-brane stretched between the NS5-branes, which clearly becomes massless when the NS5-branes coincide. Performing now an S-duality on this configuration we obtain two D5-branes; the state related to the original D2-brane is now a fundametnal string stretched between the D5-branes. In this language, the

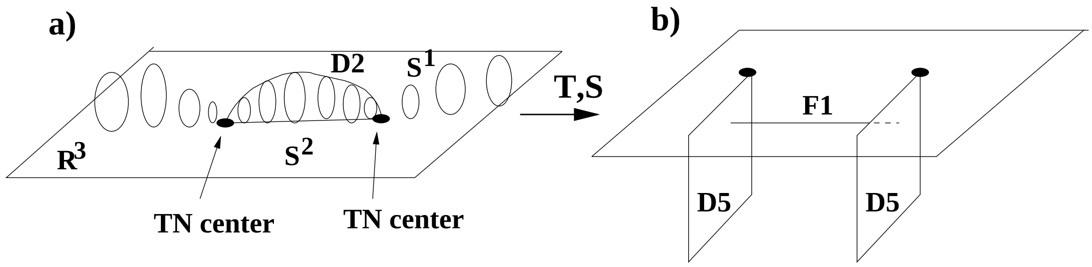

Figure 2: The $\mathbf{S^1}$ fibration over a segment joining two centers in a multi Taub-NUT geometry defines a homologically non-trivial 2-cycle with the topology of a 2-sphere. Its volume vanishes as the two centers of the Taub-NUT are tuned to coincide.

exotic phenomenon of enhanced symmetry due to the D2-brane state is the familiar phenomenon of enhancement of 6d gauge symmetry on the volume of D5-branes when they are coincident, due to the appearance of new massless open (fundamental) strings, see fig 2b.

We would like to conclude by briefly mentioning that compactification of type IIB theory on K3 leads to even more exotic physics [6]. Type IIB theory does not contain abelian $U(1)$ gauge symmetries associated to 2-cycles. Rather it contains abelian 2-forms, arising from the KK reduction of C_4, belonging to tensor multiplets of 6d $\mathcal{N} = (2,0)$ supersymmetry. Similarly, IIB theory does not have D2-brane states and hence does not lead to new massless particles in K3 with collapsed 2-cycles and zero B-field. Instead it leads to BPS tensionless string states, charged under the 2-form fields, arising from D3-branes wrapped on the collapsed 2-cycles. This surprising answer is completely consistent with T-duality with the type IIA answer, once we compactify both IIB and IIA on a further circle. Winding states of these IIB tensionless strings are mapped by T-duality to momentum states of the IIA massless particles.

These configurations can be used to define exotic theories in 6d if we take the limit of decoupling gravitational interactions. In particular, they can be used to define the so-called $(0,2)$ superconformal field theoty, or the so-called little string theory. Their discussion is however beyond our scope in these lectures.

3 Type IIB on $\mathbf{CY}_3$ and conifold singularities

We now have enough understanding of BPS states in string theory to analyze non-perturbative effects in other situations, even without the help from string duality. For this section see [7].

3.1 Breakdown of the perturbative theory at points in moduli space

Recall that type IIB on Calabi-Yau threefolds, with Hodge numbers $(h_{1,1}, h_{2,1})$, gives rise to the $\mathcal{N}=2$ 4d supergravity multiplet, $(h_{1,1}+1)$ vector multiplets and $h_{2,1}$ hypermultiplets. The latter two kinds of multiplets contain scalars spanning a moduli space. We are interested in looking for regions in this moduli space where non-perturbative effects may be relevant, even at weak coupling.

There is a non-renormalization theorem for 4d $\mathcal{N}=2$ supersymmetry that ensures that (to all orders in perturbation theory) the geometry of the moduli space of vector multiplets (the moduli space metric, which controls the kinetic terms of moduli in the effective action) does not depend on scalars in hypermultiplets, and vice versa. In type IIB, both the dilaton and the overall volume of the Calabi-Yau belong to hypermultiplets. This implies that the geometry of the vector multiplet space does not depend on the dilaton (i.e. does not suffer any quantum corrections in g_s) or on the volume scalar (i.e. does not suffer any α'/R^2 corrections). The moduli space metric determined in the classical supergravity approximation is exacty in g_s and α'.

On the other hand it is known that there are points in the moduli space of complex structures (i.e. vector multiplet moduli space) of Calabi-Yau manifolds where the effective action obtained from supergravity is singular. Since we have argued that the supergravity result is exact, there is no α' or g_s correction (to any order in perturbation theory) which removes this singularity. This means that even the α'-exact worldsheet theory (describing compactification on the Calabi-Yau space at this point in complex structure moduli space) is singular, and gives divergent answers for certain physical quantities.

This breakdown of the perturbative prescription suggests that at this points in moduli space there is some non-perturbative effect playing an essential role, even if the string coupling is weak. Our aim in this section is to discuss this effect.

3.2 The conifold singularity

Let us discuss, the generic, simplest, case where compactification on a CY_3 leads to a breakdown of the perturbative theory. It corresponds to sitting at a point in complex structure moduli space, such that the CY_3 has a region which locally develops a so-called conifold singularity. Namely, a piece of the CY_3 can be locally described as the complex hypersurface in $\mathbf{C^4}$ given by the equation

$$(z_1)^2 + (z_2)^2 + (z_3)^2 + (z_4)^2 = \epsilon \qquad (6)$$

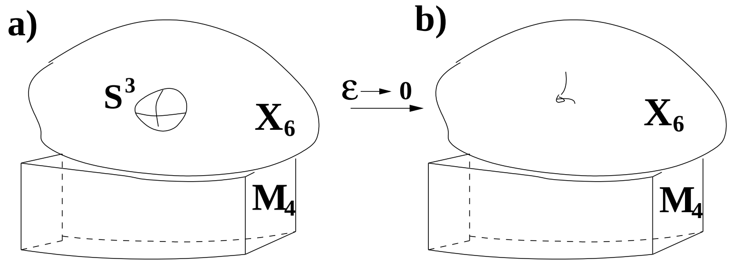

Figure 3: Tuning a modulus in the Calabi-Yau geometry, a 3-cycle shrinks and the geometry develops a conifold singularity.

The complex structure modulus is described by the parameter ϵ, and the problematic configuration corresponds to tuning $\epsilon \to 0$.

The above geometry corresponds, as $\epsilon \to 0$ to a local singularity, which is not an orbifold, but still is quite simple and well-knonw to mathematicians (algebraic geometers). It is possible to see that the geometry (6) contains a 3-cycle with the topology of a 3-sphere of size controlled by $|\epsilon|$. Namely, let $\epsilon = |\epsilon|e^{i\theta}$, and define $z_i' = z_i e^{-i\theta/2}$. If we let $x_i = \mathrm{Re}\, z_i'$, $y_i = \mathrm{Im}\, z_i'$, the 3-sphere is given by

$$y_i = 0 \quad , \quad (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = |\epsilon| \tag{7}$$

As $\epsilon \to 0$ the 3-cycle C collapses to zero size (see figure 3). In the configuration with a zero size 3-cycle, the perturbative theory breaks down.

The cure of the problem is now clear. Type IIB string theory on this CY_3 contains non-pertubative particle states arising from D3-branes wrapped on the 3-cycle C. It is possible to see that this state is BPS [3] and that its mass

[3] The 3-cycle has the property of being Special lagrangian, which implies that D-branes wrapped on it preserve some of the supersymmetry unbroken by the CY_3.

is given by

$$M = \frac{|\epsilon|}{g_s} \tag{8}$$

Thus it becomes massless precisely when $\epsilon \to 0$, suggesting that this states solves the problem of the perturbative sector, as is indeed the case.

An important difference with respect to the case of IIA theory on K3, is that the massless states belong to hypermultiplets of $\mathcal{N}=2$ 4d supersymmetry. They are charged under the (perturbative) $U(1)$ gauge symmetry arising from $\int_C C_4$. Therefore the effective action for the light modes in this region in moduli space, is simply a $U(1)$ vector multiplet coupled to a charged hypermultiplet of mass equal to ϵ. In $\mathcal{N}=1$ susy language, we have a $U(1)$ vector multiplet V, a neutral chiral multiplet Φ (whose vev corresponds to ϵ) and two chiral multiplets of H, H' of charges ± 1. The action is of the form1

$$\mathcal{L} \;=\; \int d^2\theta W_\alpha W_\alpha + \int d^4\theta \left(H^\dagger e^V H - H'^\dagger e^V H'\right) + \int d^2\theta \Phi H H' \tag{9}$$

This is a perfectly nice an smooth effective action. However, integrating out the massless fields H, H' leads to the singular behaviour of the perturbative sector. The pathological behaviour of the perturbative theory can be regarded as a consequence of missing important dynamical degrees of freedom in the low energy dynamics.

Again, let us emphasize that the appearance of these non-perturbative states takes place no matter how small the string coupling is.

3.3 Topology change

For this section see [8].

We have seen that the conifold geometry can be regarded as a limit of a smooth geometry (6), containing a 3-cycle, in the limit where the 3-cycle collapses to zero size. Mathematically, the conifold geometry can also be

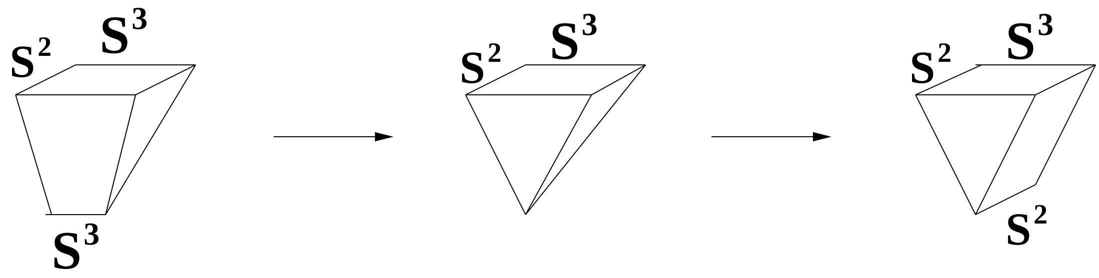

Figure 4: Topology change in the neighbourhood of a conifold singularity. Starting with a finite size $\mathbf{S^3}$ we tune a modulus to shrink it; at this stage massless state appear; a vev for them parametrizes growing an $\mathbf{S^2}$ out of the conifold singualrity.

regarded as a limit of a (different) smooth geometry, containing a 2-cycle, in the limit where the 2-cycle collapses to zero size (and the B-fieldthrough it is tuned to zero).

To understand this better, consider the equation (6) for $\epsilon = 0$ in terms of $x_i = \mathrm{Re}\, z_i$, $y_i = \mathrm{Im}\, z_i$. We get

$$x^2 - y^2 = 0 \quad , \quad x \cdot y = 0 \tag{10}$$

where x, y are 4-vectors with components x_i, y_i. Equivalently, introducing a new variable r taking positive values, we have

$$x^2 = r^2 \quad ; \quad y^2 = r^2 \; , \; x \cdot y = 0 \tag{11}$$

The first equation implies that x describes a 3-sphere of radius r, while the last equations implies that y describes a 2-sphere of radius r. The geometry of the conifold is a cone, with base $\mathbf{S^3} \times \mathbf{S^2}$ and radial coordinate r. At $r = 0$ both the 3-sphere and the 2-sphere have zero size.

The manifold (6) for non-zero ϵ corresponds to a smoothing of the conifold singularity by replacing the singular tip of the cone by a finite size 3-sphere, as illustrated in 4. This process is called deformation of the singularity. As mentioned above, there is also the possibility of smoothing the geometry by replacing the singular tip of the cone by a finite size 2-sphere, as illustrated in figure 4. This process is called small resolution of the singularity, and mathematically the smooth space is described by the equations

$$\begin{aligned} z_+ x + w_+ y &= 0 \\ w_- x + z_+ y &= 0 \end{aligned} \qquad (12)$$

in $\mathbf{C^4} \times \mathbf{P_1}$, where $\mathbf{C^4}$ is parametrized by $z_\pm = z_1 \pm i z_2$, $w_\pm = i(z_3 \pm i z_4)$, and $\mathbf{P_1}$ is parametrized by (x, y) (with the equivalence relation $(x, y) \simeq \lambda(x, y)$ with $\lambda \in \mathbf{C}^*$. The above equations define a smooth space, which is the same as the conifold singularity except at the tip of the cone. Namely, for each non-zero value of $z_\pm$, $w_\pm$, the above equations define a unique point, so the resolved space has a 1-1 mapping to the conifold singularity (away from the thip). When $z_\pm = w_\pm = 0$, then (x, y) are unconstrained and instead of just a singular point we obtain a whole $\mathbf{P_1}$. The resolved conifold thus corresponds to a smooth space, containing a 2-sphere, given by the $\mathbf{P_1}$. When its size goes to zero, the space becomes the conifold singularity.

Starting with a deformed conifold, we can imagine the process of shrinking the 3-cycle to zero size to reach the singular conifold geometry, and then growing a 2-cycle to obtain a resolved conifold. This process changes the topology of the space, since we have $\Delta(h_{1,1}, h_{2,1}) = (1, -1)$. This process is possible mathematically, but only passing through singular geometries. However, we have just seen that physically string theory is smooth even at the singular geometry. Therefore it is reasonable to wonder whether string theory can smoothly interpolate between the two topologically different geometries.

It can be shown that this is not really possible in the above situation, where the CY_3 has only one conifold point. The new geometry does not contain any 3-cycle, hence the low energy theory should not have any $U(1)$ gauge symmetry. This suggests that the transition to the new geometry must be triggered by a vev for the massless charged hypermultiplet. However, the field theory (9) does not have a flat direction where the multiplets H, H' acquire non-zero vevs. This cannot be done due to the conditions to minimize the scalar potential: these include the D-flatness constraint for the $U(1)$ gauge symmetry

$$|H|^2 - |H'|^2 = 0 \tag{13}$$

and the F-flatness constraint

$$\frac{\partial W}{\partial \Phi} = H\,H' = 0 \tag{14}$$

In other words, since in the Higgsing of $U(1)$ the vector multiplet must eat one hypermultiplet, we are left with not scalars whose vev parametrize the new branch.

On the other hand, this kind of topology changing transitions are possible at points in complex structure moduli space where the CY_3 develops several conifold singularities, such that the 3-cycles at the conifold points are not homologically independent. For instance, we can imagine a CY_3 with N conifold singularities, with the property that the homology classes of the corresponding 3-cycles add to zero in homology. In such situation the gauge symmetry is $U(1)^{N-1}$; equivalently there are N gauge bosons $U(1)^N$, but there is a relation between them, namely their sum is identically zero. On the other hand, we still get N independent charged hypermultiplets arising from D3-branes wrapped on the N collapsing 3-spheres. So in $\mathcal{N}=1$ multiplet language we have N pairs of chiral multiplets H_i, H_i' with charges $\pm q_a^i$ under the a^{th} $U(1)$ factor, with $a = 1, \ldots, N$ and $\sum_a q_a^i = 0$.

The effective theory for these field does have a flat direction where the fields H_i, H_i' acquire vevs, as can be checked from the D- and F-term constraints in this case

$$\begin{aligned} \textstyle\sum_i q_a^i(|H_i|^2 - |H_i'|^2) = 0 \quad , \quad a = 1, \ldots, N \\ \textstyle\sum_i q_a^i H_i H_i' = 0 \end{aligned} \tag{15}$$

And there is a flat direction, corresponding to $\langle H_i' \rangle = v$, $\langle H_i \rangle = w$, i.e. i-independent vevs. More intuitively, we have N charged hypermultiplets to Higgs $U(1)^{N-1}$ vector multiplets. Clearly $N-1$ hypermultiplets are eaten in the Higgs mechanism, making the vector multiplets massive, while a last hypermultiplet is left. The two complex parameters v, w correspond to vevs for scalars in this overall hypermultiplet.

The geometric interpretation of this new branch is clear. Since there are no massless $U(1)$'s, all 3-spheres have disappeared from the geometry. Since there is a new massless hypermultiplet, there is a new 2-sphere. Indeed, there are N new 2-spheres at the N conifold points, which have been resolved, but the geometry forces the sizes of all these spheres to be equal [4]. String theory has managed to smoothly interpolate [5] between the two topologically different geometries, thanks to the crucial presence of massless non-perturbative states! (figure 5).

Some comments are in order

• Let us emphasize again that, at least in this particular setup, string theory is able to interpolate smoothly between spacetimes of different topologies. In a sense, this is a more drastic version of the statement that geometry is

[4] It would be a bit tricky to explain this, see [9].

[5] Notice that the topology change as we have discussed it is not really dynamical, but simpy and adiabatic change as some parameters of the model are varied. However, it is easy to imagine configurations where moduli change slowly with time, so that their vevs evolve in time, and we are really moving in moduli space as time goes by. In this setup the above topology change could occur dynamically during time evolution.

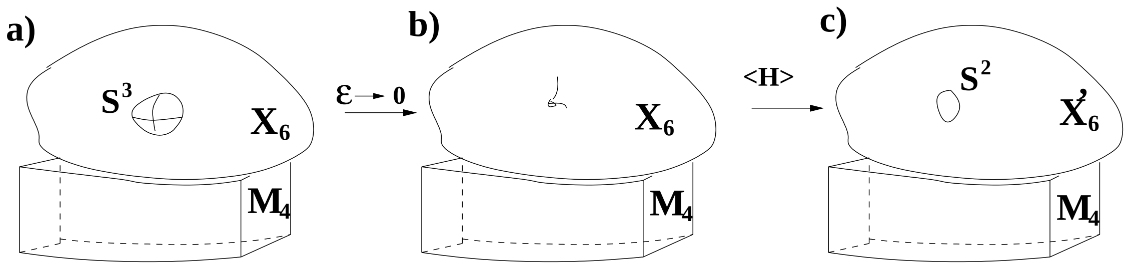

Figure 5: Topology change in CY spaces with conifold singularities.

dynamical in theories with gravity. In string theory, even the topology of spacetime is, to some extent, dynamical and can change.

• After the transition to the small resolution branch, the original hypermultiplet which was of non-perturbative origin, becomes just a perturbative hypermultiplet arising from the KK reduction of 10d type IIB theory on a CY_2 with a 2-cycle. This is a very striking phenomenon, but certainly it is implied by our discussion of topology change.

• The topology changing transitions allow to connect the moduli spaces of different CY compactifications. Indeed it has been checked that all known Calabi-Yau manifolds are connected by this kind of transition (or generalizations of it). This is conceptually very satisfying, and suggests that the election of particular compactification is as dynamical as the choice of vevs for some fields in a(n extended) moduli space.

• Finally, we would like to point out that there exist dual versions of this phenomenon, where it looks much more familiar. For instance, there exists a dual version in terms of heterotic theory compactified on $\mathrm{K3}\times\mathbf{T^2}$, where the above process corresponds to simply deforming the internal gauge bundle of the compactification.

4 Final comments

There are two finals comments we would like to make

• Non-perturbative effects can be important in string theory even in the weakly coupled regime. These effects are particularly crucial in situations where the perturbative sector of the theory is singular.

• The ideas in this lecture suggest a powerful tool to determine new interesting phenomena in string theory (and check its self-consistency). Namely, cook up situations where some singular behaviour arises, and try to identify what effects solve the problem Many new phenomena of string theory have been uncovered using this idea, and many more lessons still wait to be learnt.

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