Open strings

In this lecture we discuss open strings. The motivation is clear: they are strings of a kind very different from the ones we have studied up to now, so it is interesting to analyze their main features. Moreover, it is essential to have some familiarity with open strings to construct the type I superstring (see next lecture) since it contains sectors of open strings.

1 Generalities

Open strings are string with endpoints; they are described by worldsheets with boundaries, see figure 1

The basic interaction between open strings is that two endpoints glue together; the basic interaction vertex corresponds to two open strings joining into a single one, figure **??**a). Notice that the endpoints that glue together may belong to the same open string, so that this basic interaction also implies the existence of a vertex of two open strings joining into a closed one, figure **??**b). This has the remarkable consequence that **theories with open strings necessarily contain closed strings** (notice that we know that there exist theories of closed strings with no open strings; i.e. closed strings may be consistent by themselves, but open string theories necessarily must be coupled to closed string theories).

The worldsheet geometry forces us to include two sectors (open strings and closed strings) in the theory. The total spectrum of spacetime particles is given by the spectrum of oscillation modes of the closed string plus the spectrum of oscillation modes of the open string.

Any amplitude is obtained by summing over geometries of 2d surfaces interpolating between in and out states. This genus expansion contains contributions from surfaces with handles and boundaries, which is weighted by

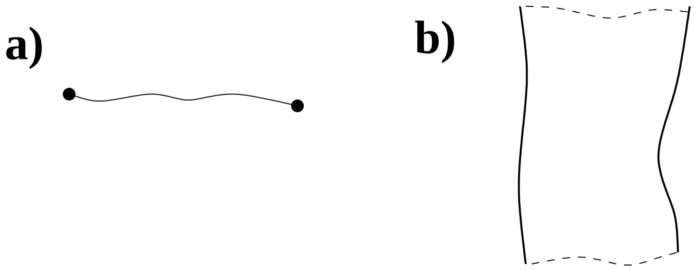

Figure 1: Open strings have endpoints. As open strings move in time they sweep out a worldsheet with boundaries.

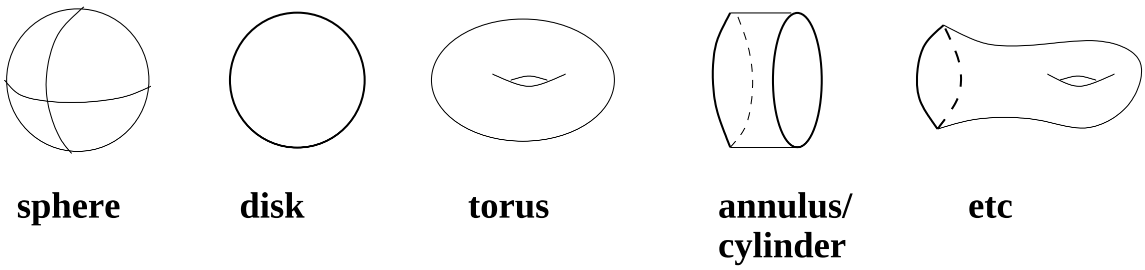

Figure 2: .

a factor of $g_s^{-\chi}$ where $\chi = 2 - 2g - n_b$, with g, n_b the number of handles and boundaries. Some examples are given in fig 2.

Finally, we would like to make the following important remark. The fact that open strings couple to closed strings implies that the local structure of the worldsheet of open strings is the same as that of closed strings. This implies that the local 2d dynamics for open and closed strings must be the same (with the only differences arising, as we will see, from boundary conditions on the 2d fields).

A related issue is that there exist diagrams which admit two different interpretations, regarded as open string diagrama or closed string diagramas. Namely, the annulus can be regarded as vacuum diagram of open string states running in a loop, or as a tree level diagram of closed string appearing from and disappearing into the vacuum. Both interpretations are possible because the local structure of the worldsheet is the same for open and closed strings. Both interpretations are related by a relabeling of the worldsheet coordinates σ, t. The requirement that a single geometry can receive both interpretations is a strong consisntency condition known as open/closed duality.

2 Open bosonic string

For this analysis we follow section 1.3 of [1]. This is an open string whose local worldsheet dynamics is described by 26 2d bosons $X^\mu(\sigma, t)$ and a 2d metric $g_{ab}(\sigma, t)$, with the Polyakov action

$$S_{\rm P} \,=\, -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi\,(-g)^{1/2}\,g^{ab}(\sigma,t)\,\partial_a X^\mu\,\partial_b X^\nu \eta_{\mu\nu} \tag{1}$$

The corresponding closed string sector is therefore the closed bosonic string. Here we center on the quantization of the open string sector, that is quantization of the above 2d field theory living on the interval (with boundary conditions to be specified below).

2.1 Light-cone gauge

The gauge freedom of the 2d theory is fixed in the same way as we did for the closed bosonic string. Again we have several steps

1. Reparametrization of t

Fix the t reparametrization freedom by setting the so-called light-cone condition

$$X^+(\sigma,t) \,=\, t \tag{2}$$

2. Reparametrization of σ

For slices of constant t, define a new spatial coordinate σ' for each point of the slice, as the (diffeomorphism and Weyl) invariant distance to one of the endpoints

$$\sigma' = c(t) \int_{\sigma_0}^{\sigma} f(\sigma, t)\, d\sigma \tag{3}$$

where

$$f(\sigma) = (-g)^{-1/2}\, g_{\sigma\sigma}(\sigma, t) \tag{4}$$

and $c(t)$ is a σ independent coefficient used to impose that the total length of the string is fixed, a constant in t which we call ℓ. Notice that, in contrast with closed string, there is a preferred reference line (so we do not impose level matching constraints to get physical states). In what follows σ' will be denoted simply σ.

3. Weyl invariance

Now we use Weyl invariance to impose that

$$g = -1 \qquad \forall \sigma, t \tag{5}$$

The gauge fixing conditions imply, just like for the closed bosonic string, that

$$\partial_\sigma g_{\sigma\sigma} = 0 \tag{6}$$

The quantization is very similar to quantization of the closed bosonic string, and the result is exactly the same local dynamics (e.g. hamiltonian). The reader satisfied with this explanation is welcome to jump to eq. (17).

2.2 Boundary conditions

It is now convenient to obtain what kind of boundary conditions we need to impose at $\sigma = 0, \ell$. To obtain them let us vary the action (1)

$$\delta S_P = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi\, g^{ab}\, \partial_a X^\mu\, \partial_b X_\mu =$$

$$= -\frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} dt \int_0^{\ell} d\sigma\, \partial_a (\, g^{ab}\, \delta X^\mu\, \partial_b X_\mu\,) \, + \, \frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi\, \delta X^\mu\, \partial_a (\, g^{ab} \partial_b X_\mu\,)$$

$$= -\frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} dt\, (\, g^{\sigma b}\, \delta X^\mu\, \partial_b X_\mu\,)\, |_{\sigma=0}^{\sigma=\ell} \, + \, \frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi\, \delta X^\mu\, \partial_a (\, g^{ab} \partial_b X_\mu\,) \quad (7)$$

The second term is the variation that leads to the equations of motion for the 2d fields just like in the closed string. To recover them, we need the first term to vanish. If δX^μ is unconstrained [1] , we then need

$$g^{\sigma b} \partial_b X^\mu(\sigma, t)\, |_{\sigma=0}^{\sigma=\ell} = 0 \quad (8)$$

Using this for $X^+ = t$, we get

$$g_{\sigma t} = 0 \;\text{ at }\; \sigma = 0, \ell. \quad (9)$$

For the transverse coordinates X^i we get

$$g^{\sigma\sigma} \partial_\sigma X^\mu(\sigma, t)\, |_{\sigma=0}^{\sigma=\ell} = 0 \quad (10)$$

We cannot satisfy this equation by requiring $g_{\sigma\sigma} = 0$ at $\sigma = 0, \ell$, since (6) would then imply $g_{\sigma\sigma} \equiv 0$ is non-dynamical, in contrast with the situation in closed bosonic strings. Therefore we have to impose

$$\partial_\sigma X^i\, |_{\sigma=0,\ell} \, = \, 0 \quad (11)$$

These are Neumann boundary conditions on both open string endpoints, so this kind of open strings are also called Neumann-Neumann or NN.

2.3 Hamiltonian

The lagrangian in light-cone gauge is

$L \;\; = -\frac{1}{4\pi\alpha'} \int_0^\ell d\sigma \quad [-2\, g^{tt} \partial_t X^+\, \partial_t X^- \, + g^{tt} \partial_t X^i\, \partial_t X^i \, - \, 2\, g^{\sigma t}\, \partial_t X^+\, \partial_\sigma X^- \, +$

[1] This is not the case for open string sectors describing lower-dimensional D-branes (to be studied in later lectures).

$$
\begin{aligned}
& +2\, g^{\sigma t}\, \partial_\sigma X^i\, \partial_t X^i\; + g^{\sigma\sigma}\, \partial_\sigma X^i\, \partial_\sigma X^i\,] \; = \\
= -\tfrac{1}{4\pi\alpha'} \textstyle\int_0^\ell d\sigma \quad & [\, g_{\sigma\sigma}\, (2\, \partial_t X^- \; - \partial_t X^i\, \partial_t X^i) \; - \; 2\, g_{\sigma t}\, (\partial_\sigma X^- - \partial_\sigma X^i\, \partial_t X^i) \; + \\
& g^{-1}_{\sigma\sigma}\, (1 - g^2_{\sigma t})\, \partial_\sigma X^i\, \partial_\sigma X^i\,] \qquad (12)
\end{aligned}
$$

Defining the center of mass and relative coordinates $x^-(t)$, $Y^-(\sigma, t)$

$$
\begin{aligned}
x^-(t) &= \frac{1}{\ell} \int_0^\ell d\sigma\, X^-(\sigma, t) \\
X^-(\sigma, t) &= x^-(t) \, + \, Y^-(\sigma, t) \qquad (13)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
L \; = \; & -\frac{\ell}{2\pi\alpha'}\, g_{\sigma\sigma}\, \partial_t x^-(t) \; - \; \frac{1}{4\pi\alpha'} \int_0^\ell d\sigma\, [\, -\, g_{\sigma\sigma}\, \partial_t X^i\, \partial_t X^i \; + \\
& -\, 2\, g^{\sigma t}\, (\partial_\sigma Y^- - \partial_\sigma X^i\, \partial_t X^i) \; + \; g^{-1}_{\sigma\sigma}\, (1 - g^2_{\sigma t})\, \partial_\sigma X^i\, \partial_\sigma X^i\,] \qquad (14)
\end{aligned}
$$

The $Y^-(\sigma, t)$ acts as a Lagrange multiplier imposing

$$
\partial_\sigma g_{\sigma,t}(\sigma, t) \; = \; 0 \quad \forall \sigma, t \qquad (15)
$$

From (9) we get

$$
g_{\sigma,t}(\sigma, t) = 0 \quad \forall \sigma, t \qquad (16)
$$

The lagrangian becomes

$$
L \; = \; -\frac{\ell}{2\pi\alpha'}\, g_{\sigma\sigma}\, \partial_t x^-(t) \; + \; \frac{1}{4\pi\alpha'} \int_0^\ell d\sigma\, [\, g_{\sigma\sigma}\, \partial_t X^i\, \partial_t X^i \; - \; g^{-1}_{\sigma\sigma}\, \partial_\sigma X^i\, \partial_\sigma X^i\,]
$$

exactly as for closed strings. Following the computations there, the hamiltonian then reads

$$
H \; = \; \frac{\ell}{4\pi\alpha' p^+} \int_0^\ell d\sigma\, [\, 2\pi\alpha'\, \Pi_i\, \Pi_i \; + \; \frac{1}{2\pi\alpha'}\, \partial_\sigma X^i\, \partial_\sigma X^i\,] \qquad (17)
$$

2.4 Oscillator expansions

From the above hamiltonian, the equations of motion for the 2d fields $X^i(\sigma,t)$ read

$$\partial_t^2 X^i \;=\; \partial_\sigma^2 X^i \tag{18}$$

where we have again set $\ell = 2\pi\alpha' p^+$. Again, the general solution will be a superposition of left- and right-moving waves $X_L^i(\sigma+t)$, $X_R^i(\sigma-t)$. These have the general oscillator expansion

$$\begin{aligned}
X_L^i(\sigma+t) \;&=\; \frac{x^i}{2} \,+\, \frac{p_i}{2p^+}\,(t+\sigma) \,+\, i\sqrt{\frac{\alpha'}{2}}\,\sum_\nu \frac{\alpha_n^i}{n}\, e^{-\pi i\,\nu\,(\sigma+t)/\ell} \\
X_R^i(\sigma-t) \;&=\; \frac{x^i}{2} \,+\, \frac{p_i}{2p^+}\,(t-\sigma) \,+\, i\sqrt{\frac{\alpha'}{2}}\,\sum_\nu \frac{\tilde{\alpha}_n^i}{n}\, e^{\pi i\,\nu\,(\sigma-t)/\ell}
\end{aligned} \tag{19}$$

with ν a modding to be fixed by the boundary conditions. Notice that for convenience the exponents we use differ from those in closed strings in a factor of two.

Now we have to impose the boundary conditions

$$\partial_\sigma X_L^i \,+\, \partial_\sigma X_R^i \;=\; 0 \;\text{ at }\; \sigma = 0, \ell \tag{20}$$

We compute

$$\partial_\sigma X_L^i \,+\, \partial_\sigma X_R^i \;=\; i\sqrt{\frac{\alpha'}{2}}\,\frac{i\pi}{\ell}\,\sum_\nu \left[-\alpha_\nu^i e^{-\pi i\nu(\sigma+t)/\ell} \,+\, \tilde{\alpha}_\nu^i e^{\pi i\nu(\sigma-t)/\ell}\right] \tag{21}$$

Imposing the boundary condition at $\sigma = 0$ we obtain

$$\alpha_\nu^i = \tilde{\alpha}_\nu^i \tag{22}$$

The boundary conditions for open strings relate the left and right movers, which are no longer independent. This alos means that the Hilbert space of an open string will be exactly like one of the sides (say the left-moving sector) of a closed string (the right-moving one not being an independent one). Notice that this also means that open strings can couple only left-right symmetric closed string sectors; for instance, there are no heterotic open strings.

Imposing the boundary condition at $\sigma = \ell$ we obtain

$$\alpha^i_\nu \sin \pi\nu \;=\; 0 \tag{23}$$

Which implies $\nu \in \mathbf{Z}$

The hamiltonian in terms of the oscillator modes reads

$$H \;=\; \frac{p_i p_i}{2p^+} + \frac{1}{2\alpha' p^+}\left[\sum_{n>0} [\,\alpha^i_{-n}\alpha^i_n\right] + E_0 \tag{24}$$

with $E_0 = 24 \times (-1/24) = -1$. This is exactly the hamiltonian for the left-moving sector of the closed bosonic string, except for a factor of two arising from that in the oscillator expansion.

2.5 Spectrum

The spectrum is obtained just like the left-moving sector of the closed string theory. The spacetime mass formula is

$$\alpha' m^2 \;=\; N_B \;-\; 1 \;\text{ with }\; N_B \;=\; \sum_{n>0} \alpha^i_{-n}\alpha^i_n \tag{25}$$

We define the vacuum by $\alpha^i_n|0\rangle_o = 0$ for $n > 0$, and construct the Hilbert space by applying creation oscillators to it. The lightest modes are

State	$\alpha' m^2$	$SO(24)$
$\|0\rangle_o$	-1	1
$\alpha^i_{-1}\|0\rangle_o$	0	24

(Notice that we get the right Lorentz little group for the massless particles). We obtain a 26d $U(1)$ massless gauge boson and a neutral tachyonic 26d scalar.

To the open string states we have to add the closed string states. Recall thay are given by

State	$\alpha' m^2$	$SO(24)$
$\|0\rangle_c$	-4	1
$\alpha^i_{-1}\tilde{\alpha}^j_{-1}\|0\rangle_c$	0	24×24

where $|0\rangle_c$ is the closed string vacuum. This leads to the 26d closed string tachyon and the massless 26d graviton, 2-form and dilaton.

We would like to briefly mention that, in contrast with the closed string tachyon, there is a general consensus on the meaning of the open string tachyon. It signals an instability because we are expanding the theory around a maximum of the potential for this field. In order to correct this, we should look for a minimum of the tachyon potential and expand the theory around it. The potential indeed has a minimum, and very surprisingly the proposal is that the theory sitting at this minimum is just the closed bosonic string theory, with no open string sector.

The intuition underlying this proposal by A.Sen (and which is a bit advanced for this lecture) is that the open string sector is associated to an underlying objetc which is filling spacetime (a D25-brane). The open string tachyon signals an instability of this object, which decays and disappears. The theory left over is just closed string theory with no open string sector.

Although open string sectors of the bosonic theory are 'unstable' in this sense, it is useful to study them to learn more about string theory, and as background material for other open string sectors without this kind of tachyons.

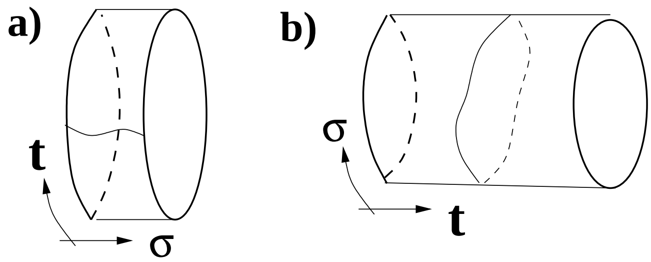

Figure 3: The annulus diagram regarded in the open and in the closed string channel.

2.6 Open-closed duality

In this section we would like to study how theories with open strings deal with ultraviolet regimes. Consider the simplest 1-loop open string diagram, namely the vacuum to vacuum amplitude given by the annulus. This corresponds to an open string evolving for some time $2T\ell$ and glueing back to itself, see figure 3a.

This can be computed easily as a trace over the open string Hilbert space. An important difference with respect to the torus in the closed bosonic string is that now we have a fixed reference line, we cannot glue back the open string with a shift in σ; hence we do not have the analog of τ_1. One could imagine to glue back the strin up to an exchange of the roles of the two string endpoints, but this would lead to a worldsheet with the topology of the Moebius string, rather than an annulus. Such worldsheets exist for unoriented open strings, which couple to unoriented closed string. Since the closed string theories we have studied are oriented, so are our open strings, and we will not consider Moebius strips. In next lecture, type I superstring is an unoriented string theory and will contain such diagrams.

Let us evaluate the annulus amplitude. It is given by a sum over all possible annulus geometries, namely integrating over the parameter T we have

$$Z = \int_0^\infty \frac{dT}{2T} Z(T) \tag{26}$$

with

$$Z(T) = \mathrm{tr}_{\mathcal{H}_{\mathrm{op.}}} e^{-2T\ell H_{\mathrm{op.}}} \tag{27}$$

Recalling

$$H_{\mathrm{op.}} = \frac{\sum_i p_i^2}{2p^+} + \frac{1}{2\alpha' p^+} (N_B - 1) \tag{28}$$

we have

$$Z(T) = \mathrm{tr}_{\,\mathrm{mom.}} e^{-2\pi\alpha' T \sum_i p_i^2} \mathrm{tr}_{\,\mathrm{osc.}} e^{-2\pi T(N-1)} \tag{29}$$

Defining $q = e^{2\pi i(iT)}$, the traces will organize in modular functions with parameter $\tau = iT$. Computing the traces in a by now familiar way we have

$$Z = \int_0^\infty \frac{dT}{2T} (8\pi^2\alpha' T)^{-12} \eta(iT)^{-24} \tag{30}$$

Open-closed duality is the fact that the annulus diagram can be regarded, in a dual channel, as a diagram where closed strings appear from and disappear into the vacuum, at tree level, see figure 3b). Notice that the ultraviolet regime in the open string channel corresponds to the infrared in the closed string channel, see figure 4. Hence the ultraviolet regime is mapped to an infrared regime due to the appearance of a dual channel once stringy energies are reached.

In order to see more manifestly how the amplitude (30) can be regarded as a closed string one, notice that in exchanging the roles of σ and t in the annulus there is a redefinition of the new σ to bring it back to the light-cone convention (total length equal to ℓ for closed strings) and hence the

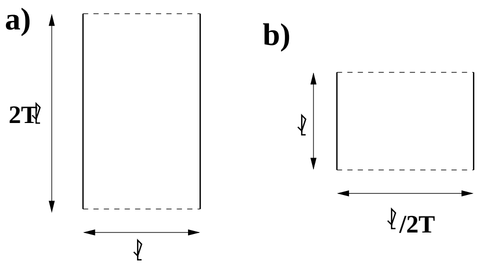

Figure 4: Open-closed duality. An open string propagating a time $2T\ell$ is geometrically the same as a closed string propagating a time $T'\ell$ with $T' = 1/(2T)$.

closed string propagates for a time $T'\ell$ with $T' = 1/(2T)$. Using the modular transformation properties

$$\eta(i/(2T')) \;=\; (2T)^{1/2}\,\eta(2iT') \tag{31}$$

we can write

$$Z \;=\; \int_0^\infty \frac{dT'}{2T'}\,(8\pi^2\alpha')^{-12}\,\eta(2iT')^{-24} \tag{32}$$

The same amplitude now has the structure of a sum over closed string states with some peculiarities: there is not power-like dependence on T', meaning that the closed states are created out of the vacuum with zero momentum (due to momentum conservation); also, there is no analog of τ_1 since the closed string does not come back to itself; finally, due to the absence of integration over τ_1 (because there is no τ_1) the level matching on closed states has to be imposed explicitly, this leads to the argument of the oscillator η functions to be doubled.

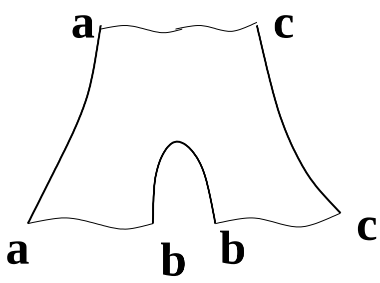

Figure 5: Open string interaction vertex with Chan-Paton factors.

3 Chan-Paton factors

We now turn to the discussion of an essentially new feature of open strings. It is consistent to have more than one kind of open string sector in a string theory. The most general possibility is to introduce a discrete degree of freedom, in one out of N possible states, at each string endpoint. Hence, each open string is characterized by two indices, a, b, with $a, b = 1, \ldots, N$, denoted Chan-Paton indices, specifying in which states the endpoints are. Notice that the labels are ordered for oriented open strings.

These degrees of freedom are non-dynamical, so the label of an endpoint simply propagates unchanged along the endpoint worldline. The rules for interactions are clear, there is one label per boundary, and one should sum over all possible labels in internal boundaries. The basic interaction vertex is shown in figure 5.

The quantization of open strings with Chan-Paton factors is straightforward. Since Chan-Paton indices are non-dynamical, they do not enter in the hamiltonian, and the quantization of each ab sector proceeds as for a single open string without Chan-Paton factors. The existence of the indices only implies that there are N^2 states of each kind. The lightest states are as

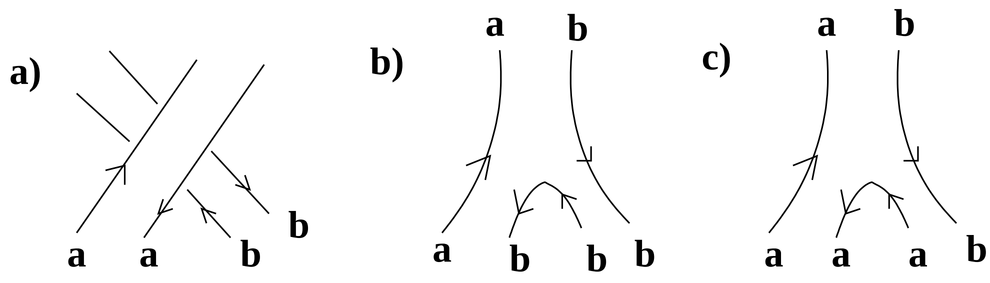

Figure 6: Interactions between open string with Chan-Paton factors.

follows

State	$\alpha' m^2$	$SO(24)$
$\|0\rangle_{ab}$	-1	1
$\alpha_{-1}^i \|0\rangle_{ab}$	0	24

where $|0\rangle_{ab}$ denotes the groundstate of the ab open string. Hence we obtain N^2 gauge bosons and N^2 scalar tachyons. The N^2 gauge bosons A_{ab} can be seen to correspond to a gauge group $U(N)$. This can be seen by analyzing their interactions as follows, see fig 6.

• The gauge bosons A_{aa}, A_{bb} for $a \neq b$ do not interact among themselves, since they do not have common indices, fig 6a. This means that the corresponding generators of the gauge group commute. In fact, they generate a $U(1)^N$ Cartan subalgebra.

• The gauge boson A_{ab} interacts with, i.e. is charged under A_{aa}, A_{bb}, as shown in figures 6b,c. The orientations of the boundary are inherited from the orientation on whe worldsheet. The orientations imply that 6b, c differ by a sign. Fixing a convention, we say that A_{ab} carries charge $+1$ and -1 under A_{aa}, A_{bb}.

Since charge under Cartan generators correspond to weights, and since weights in the adjoint representation (in which gauge bosons must transform) are roots, we obtain that the gauge group has $N^2 - N$ non-zero roots of the form

$$\underline{(+, -, 0, \ldots, 0)} \tag{33}$$

Going back to the lecture on group theory, we see that these are the non-zero roots of $U(N)$.

Performing a similar discussion it is easy to see that all states in the open string tower transform in the adjoint representation of $U(N)$.

An alternative way to understand the appearance of $U(N)$ is to consider general states, linear combinations of the basic states $|\ \rangle_{ab}$

$$|\ \rangle = \sum_{ab} \lambda_{ab} |\ \rangle_{ab} \tag{34}$$

where the matrix of coefficients λ is hermitian. These hermitian matrices are providing an N-dimensional representation of the $U(N)$ generators. Notice that a single Chan-Paton index a can be thought of as transforming in the fundamental or antifundamental representation of the $U(N)$ group, depending on whether it sits at the endpoint where the string starts from or ends at.

It is very remarkable that the simple non-dynamical Chan-Paton degrees of freedom lead to the rich dynamics of non-abelian gauge symmetry from the viewpoint of spacetime. Also very remarkably, we have uncovered a brand new way to obtain non-abelian gauge symmetries in string theory.

As a final comment, it is easy to see that open-closed duality is satisfied for any choice of the Chan-Paton rank N. The annulus amplitude is exactly as the above up to a multiplicity factor of N^2. Upon going to the closed channel, this implies there is an additional factor of N on the disk diagrams creating or annihilating the closed string from or into the vacuum.

Notice finally that the number of open string tachyons increases with N. Hence the more open string sectors the theory has, the more unstable it is in this sense. As with the single open string case, condensation of these tachyons leads to the disappearance of the open string sectors, leaving behind just the closed bosonic string theory.

4 Open superstrings

Let us try to consider describing open superstrings. We know that they will couple to some closed superstring, which must be of the kind studied in previous lectures. Since the local 2d worldsheet must be left-right symmetric, the natural possibility to be considered is open string theories coupling to type IIB closed string sectors.

At the end of this section we will see that in superstrings there is an additional consistency condition, called RR tadpole cancellation condition, which is not satisfied by the models we are about to construct. Nevertheless, the material we cover will turn out to be useful for the construction of type I theory, which is consistent, in next lecture.

4.1 Hamiltonian quantization

In the light-cone gauge the dynamical 2d fields are $X_L^i(\sigma + t)$, $\psi_L^i(\sigma + t)$, $X_R^i(\sigma - t)$, $\psi_R^i(\sigma - t)$, with $i = 2, \ldots, 9$. The quantization of the bosonic piece works exactly like in the open bosonic string, and will not be reviewed here.

Centering on the 2d fermions, let us simply state, without entering into details, that there are two possible boundary conditions which lead to the correct equations of motion locally on the worldsheet. The possibilities are

$$\begin{aligned}\psi_L^i &= e^{2\pi i\rho}\,\psi_R^i \quad \text{at } \sigma=0\\ \psi_L^i &= e^{2\pi i\rho'}\,\psi_R^i \quad \text{at } \sigma=\ell\end{aligned} \tag{35}$$

with $\rho,\rho'=0,1/2$. Redefining $\psi_R^i(\sigma-t)\to e^{-2\pi i\rho'}\psi_R^i(\sigma-t)$ we can trivialize the condition at $\sigma=\ell$, hence we are left with two possible sectors, which we call NS and R

$$\begin{array}{llll}\text{NS} & \psi_L^i=-\psi_R^i \quad \text{at } \sigma=0 & \text{R} & \psi_L^i=\psi_R^i \quad \text{at } \sigma=0\\ & \psi_L^i=\psi_R^i \quad \text{at } \sigma=\ell & & \psi_L^i=\psi_R^i \quad \text{at } \sigma=\ell\end{array}$$

The mode expansion in both cases reads

$$\begin{aligned}\psi_L^i(\sigma+t) &= i\sqrt{\frac{\alpha'}{2}}\sum_\nu \psi_\nu^i\, e^{-\pi i\nu(\sigma+t)/\ell}\\ \psi_R^i(\sigma-t) &= i\sqrt{\frac{\alpha'}{2}}\sum_\nu \tilde{\psi}_\nu^i\, e^{\pi i\nu(\sigma-t)/\ell}\end{aligned} \tag{36}$$

For NS boundary conditions, we have

$$\begin{array}{lll}\sigma=0 & \displaystyle\sum_\nu (\psi_\nu^i+\tilde{\psi}_\nu^i)\, e^{-\pi i\nu t/\ell} = 0 & \rightarrow \quad \psi_\nu^i=-\tilde{\psi}_\nu^i\\ \sigma=\ell & \displaystyle\sum_\nu \psi_\nu^i\,\cos\pi\nu\, e^{-\pi i\nu t/\ell} = 0 & \rightarrow \quad \nu\in\mathbf{Z}+\mathbf{\frac{1}{2}}\end{array} \tag{37}$$

For R boundary conditions, we have

$$\begin{array}{lll}\sigma=0 & \displaystyle\sum_\nu (\psi_\nu^i-\tilde{\psi}_\nu^i)\, e^{-\pi i\nu t/\ell} = 0 & \rightarrow \quad \psi_\nu^i=\tilde{\psi}_\nu^i\\ \sigma=\ell & \displaystyle\sum_\nu \psi_\nu^i\,\sin\pi\nu\, e^{-\pi i\nu t/\ell} = 0 & \rightarrow \quad \nu\in\mathbf{Z}\end{array} \tag{38}$$

So left and right movers are linked together. NS fermions are half-integer modded and R fermions have integer moddings. Everything behaves as with the left moving sector of a superstring.

4.2 Spectrum for NS and R sectors

Being careful with the factor of 2 from the different exponent in the oscillator expansions, the hamiltonian and mass formula are similar to the left moving ones in a superstring. They are given by

$$
\begin{aligned}
H &= \frac{\sum_i p_i^2}{2p^+} + \frac{1}{2\alpha' p^+}\,(\,N_B\,+\,N_F\,+\,E_0\,) \\
\alpha' m^2 &= N_B\,+\,N_F\,+\,E_0
\end{aligned}
\tag{39}
$$

with $E_0 = -1/2, 0$ for NS, R sectors.

In the NS sector, we take the groundstate annihilated by positive modding operators

$$
\alpha_n|0\rangle = 0 \quad , \quad \psi_{n-1/2}|0\rangle = 0 \quad , \quad \text{for } n > 0 \tag{40}
$$

and build the Hilbert space by applying negative modding oscillators to it. The lightest states are

State	$\alpha' m_L^2/2$	$SO(8)$
$\|0\rangle$	$-1/2$	$\mathbf{1}$
$\psi^i_{-1/2}\|0\rangle$	0	$\mathbf{8_V}$

In the R sector, we define the groundstates as annihilated by positive modding operators

$$
\alpha_n|0\rangle = 0 \quad , \quad \psi_n|0\rangle = 0 \quad , \quad \text{for } n > 0 \tag{41}
$$

The groundstate is degenerate due to fermion zero modes, and hence forms a representation of the Cliffor algebra generated by them. Introducing the operators $A_a^{\pm} = \psi_0^{2a} \pm i\psi_0^{2a+1}$, and the state $|0\rangle$ annihilated by the raising operator, the groundstates are

$$
\begin{array}{cc}
|0\rangle & A^+_{a_1}|0\rangle \\
A^+_{a_1}A^+_{a_2}|0\rangle & A^+_{a_1}A^+_{a_2}A^+_{a_3}|0\rangle \\
A^+_1A^+_2A^+_3A^+_4|0\rangle &
\end{array}
\tag{42}
$$

The two columns correspond to the two chiral irreps of $SO(8)$, 8_S and 8_C respectively. Finally the spectrum is obtained by applying negative modding oscillators to these groundatates. The lightest modes are the groundstates themselves

	State		$\alpha' m_L^2/2$	$SO(8)$
R	$\frac{1}{2}(\pm,\pm,\pm,\pm)$	$\#- =$ even	0	$\mathbf{8_S}$
	$\frac{1}{2}(\pm,\pm,\pm,\pm)$	$\#- =$ odd	0	$\mathbf{8_C}$

4.3 GSO projection

A natural question is now how (or whether) to combine NS and R sectors in constructing the open string spectrum (as was required by modular invariance in closed superstrings). Clearly, the fact that the open strings we want to construct couple to type IIB closed string imposes a constraint on the physical spectrum of the open string. Indeed, the physical spectrum of the closed sector had a GSO projection; if no constraint is imposed on the open string spectrum, it would be possible to create unphysical closed string states (with the wrong GSO behaviour) by scattering open string states.

In other words, open/closed duality (the fact that the open 1-loop annulus diagram can be regarded as a closed string amplitude (with only GSO projected states propagating) requires the open string sector to have a specific mixture of NS and R boundary condition, i.e. a GSO projection.

Indeed, it turns out that the GSO projection in the open string sector is exactly that on one of the sides in a type II superstring. Namely, it eliminates the NS groundstate, and the 8_S R groundstate. Hence the open string tachyon disappears, and the only massless states are a 10d $U(1)$ gauge boson and a 10d chiral fermion. They fill out a vector multiplet of 10d $\mathcal{N}=1$ supersymmetry.

The complete spectrum is given by this open string spectrum, plus the closed type IIB string spectrum, which at the massless level is 10d $\mathcal{N}=2$ supersymmetry. This supersymmetry in the closed sector is not a symmetry of the full theory, and it would be broken to $\mathcal{N}=1$ by interactions with open strings.

Let us finish by mentioning that addition of Chan-Paton indices is straightforward and leads to the same result as for bosonic open strings, namely the gauge group becomes non-abelian $U(N)$ and all states transform in the adjoint representation. This leads to a new situation, very different from heterotic, with non-abelian gauge symmetries and charged fermions. So it in principle provides an interesting starting point for model building of theories similar to the Standard Model (see future lectures on D-branes worlds).

4.4 Open-closed duality

Let us verify that the annulus constructed in the open string channel indeed reproduces a GSO projected closed string amplitude in the dual channel.

The annulus amplitude is

$$Z \,=\, \int_0^\infty \frac{dT}{2T}\, Z(T) \tag{43}$$

with

$$\begin{aligned} Z(T) \;&=\; \mathrm{tr}_{\,\mathcal{H}_{\rm op.}}\, e^{-2T\ell H_{\rm op.}} \,=\, \mathrm{tr}_{\,\rm mom.} e^{-2\pi\alpha' T\sum_i p_i^2}\, \mathrm{tr}_{\,\rm bos.}\, e^{-2\pi T(N_B-E_0^B)} \,\times \\ &\times\; \left(\mathrm{tr}_{\,NS,GSO}\, e^{-2\pi T(N_F-E_0^F)} \,-\, \mathrm{tr}_{\,R,GSO}\, e^{-2\pi T(N_F-E_0^F)}\right) \end{aligned} \tag{44}$$

We have

$$\begin{aligned} \mathrm{tr}_{\,\rm mom.} e^{-2\pi\alpha' T\sum_i p_i^2} \;&=\; (8\pi^2\alpha' T)^{-4} \\ \mathrm{tr}_{\,\rm bos.}\, e^{-2\pi T(N_B-E_0^B)} \;&=\; \eta(iT)^{-8} \\ \mathrm{tr}_{\,NS,GSO}\, e^{-2\pi T(N_F-E_0^F)} \;&=\; \frac{1}{2}\left(\mathrm{tr}_{\,NS} q^{N_F+E_0^F} \,+\, \mathrm{tr}_{\,NS}(q^{N_F+E_0^F}\,(-)^F\right) \,= \end{aligned}$$

$$
\begin{aligned}
& \frac{1}{2}\,\eta^{-4}\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4-\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4\right)\\
\mathrm{tr}\,_{R,GSO}\,e^{-2\pi T(N_F-E_0^F)} \quad & = \frac{1}{2}\left(\mathrm{tr}\,_R q^{N_F+E_0^F}+\mathrm{tr}\,_R(q^{N_F+E_0^F}\,(-)^F\right)=\\
& \frac{1}{2}\,\eta^{-4}\left(\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4-\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\right) \qquad (45)
\end{aligned}
$$

In total

$$
Z(T)\;=\;\frac{1}{2}\,(8\pi^2\alpha'\tau_2)^{-4}\,\eta^{-8}\,\eta^{-4}\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4-\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4-\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4\pm\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\right) \qquad (46)
$$

It is clear that replacing $T=1/(2T')$ and using the modular properties of the eta and theta functions we recover a correctly GSO projected closed string amplitude.

4.5 RR tadpole cancellation condition

Although everything looks fine, clearly there must be something wrong in the above construction. In previous lectures we mentioned that the field content of type IIB theory is free of gravitational anomalies in a very intricate and miraculous manner. Here we are seemingly constructing a bunch of theories which include the anomaly free type IIB field content, plus a bunch of additional chiral fields arising from the open string sectors.

The additional sets of fields in these theories are anomalous, so it is *not* possible that these theories with open string sectors are consistent.

Indeed we are going to learn that in theories with open superstrings there is a consistency condition which we had not satisfied, and which renders inconsistent all the above theories unless $N=0$, namely unless open string sectors are absent.

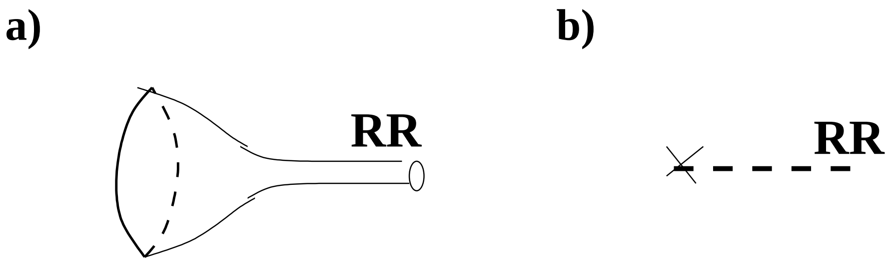

Figure 7: Disk diagram leading to a tadpole term for some closed string mode.

Let us discuss the physical idea, since the computations will be done in some more detail in the lecture on type I superstrings. The key idea is that the theory contains tadpole interactions due to disk diagrams of the kind shown in figure 7. From the spacetime viewpoint, these are terms in the effective action, which are linear in the closed sector field, schematically

$$Q \int d^{10}x \, \varphi(x) \tag{47}$$

with Q the coefficient of the disk tadpole, and φ the corresponding closed string field.

It is possible to compute explicitly in string theory which closed string fields get this kind of tadpoles, but much can learnt from simple considerations. First, the terms should be Poincare invariant in order to appear in the effective action. In the RR sector, massless fields are p-forms in spacetime, for all possible even p degrees. The only p-form for which the tadpole term is Poincare invariant is the 10-form C_{10}. This field is very peculiar, since its field strength would be an 11-form which is identically zero in a 10d spacetime. Hence, and although it has a vertex operator in string theory, it has not kinetic term. The only place where it appears in the spacetime action is in fact the tadpole term. Hence we have

$$S_{C_{10}} = Q \int_{M_{10}} C_{10} \tag{48}$$

The equation of motion for this field is therefore

$$Q = 0 \tag{49}$$

Namely, rather than a condition on the field, it is a consistency condition on the theory. It requires that the RR tadpole is absent from the theory. This is the RR tadpole cancellation condition.

It is possible to check that the coefficient of the tadpole diagram is non-zero if there are open string sectors. Indeed, the standard way to compute the disk (see lecture on type I) is to compute the annulus and take the infinite T' limit in the closed string channel, where the amplitude factorizes as the square of the disk. Recalling that with N Chan-Paton factors, the annulus goes like N^2, the disk and hence the tadpole is proportional to N. Consequently (**??**) requires $N = 0$, namely no open string sectors.

This is our result. The derivation was a bit crude, in particular since it involved spacetime considerations. Nevertheless the result is robust and has been derived (in a very technical way) purely from worldsheet consideration [**?**].

We would like to conclude with two comments. In addition to the RR tadpole, there is also a tadpole for NSNS fields. This tadpole is not a dangerous one, since all fields in the NSNS sector have kinetic terms, hence their equations of motion impose conditions on the fields and not consistency conditions on the theory [2]. This is analogous to open bosonic strings,

[2] In any event, supersymmetry relates NSNS and RR tadpoles, so that often in imposing RR tadpole cancellation conditions one obtains NSNS tadpole cancellation, although the latter is not required for consistency.

where disk tadpoles exist for fields with kinetic terms, hence do not signal inconsistencies.

Finally, let us mention what theories are affected by the inconsistency. The precise statement is that it is not possible to couple open strings to type IIB closed string in a 10d Poincare invariant way. In further lectures we will encounter consistnte situations with open superstrings, which avoid the above problem: either because the open strings are unoriented and couple to an unoriented version of type IIB theory (but not to just type IIB theory); or beacuse the open string sectors do not preserve 10d Poincare invariance (see lecture on D-branes).

References

[1] J. Polchinski, 'String theory', Vol 1.

[2] J. Polchinski, Y. Cai, 'Consistency of open superstring theories', Nucl. Phys. B296 (1988) 91.