Appendix: Rudiments of Supersymmetry

In this appendix we provide the basic ideas on the construction of supersymmetric field theories. The emphasis is in providing some basic results to be used in the general lectures. We mainly follow the notation and discussion in [1], to which we refer the reader interested in more details and proofs. For useful tables of supermultiplet components, for diverse extended supersymmetries in diverse dimensions, see [3, 2].

1 Preliminaries: Spinors in 4d

Before discussing supersymmetry, it is useful to briefly review two-component 4d spinors (Weyl spinors), their properties, some useful notation, and their relation to the more familiar four-component Dirac spinors. It is important to realize that the following discussion has nothing to do with supersymmetry, but just with spinor representations of the 4d Lorentz group, and that two-component spinors appear in many contexts, for instance in the Standard Model.

The 4d Lorentz group contains two inequivalent spinor representations, usually denoted left- and right-handed spinors. These representations are two-dimensional, so the spinors are denoted two-component, and sometime Weyl spinors. The two representations are exchanged under (Dirac) conjugation (transposition and complex conjugation), namely the conjugate of a left-handed object transforms as a right-handed spinor.

We use the following notation, we denote a left-handed spinor as ψ_α, a right-handed spinor as $\bar{\psi}^{\dot{\alpha}}$. We also denote the conjugate of a right-handed spinor by ψ^α and the conjugate of a left-handed spinor by $\bar{\psi}_{\dot{\alpha}}$.

A Lorentz transformation is represented on spinors in terms of a matrix M in $SL(2,\mathbf{C})$ (Notice that it contains six independent real parameters).

Spinors transform as

$$\begin{aligned} \psi'_\alpha = M_\alpha{}^\beta \psi_\beta \quad &; \qquad \bar{\psi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \\ \psi'^\alpha = \psi^\beta (M^{-1})_\beta{}^\alpha \quad &; \qquad \bar{\psi}'^{\dot{\alpha}} = \bar{\psi}^{\dot{\beta}} (M^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}} \end{aligned} \tag{1}$$

Namely, ψ_α and ψ^α are rotated by M as column and row vectors, while $\bar{\psi}_{\dot{\alpha}}$ and $\bar{\psi}^{\dot{\alpha}}$ are rotated by M^*.

Thus, contractions of the form $(\ldots)^\alpha(\ldots)_\alpha$ and $(\ldots)_{\dot{\alpha}}(\ldots)^{\dot{\alpha}}$ are invariant.

Vector representations can be constructed from the spinor representations. For that purpose, we introduce the matrices $\sigma^\mu{}_{\alpha\dot{\alpha}}$

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

Considering linear combinations of the form $P = P_\mu \sigma^\mu$, the inherited action of M is

$$P'_{\alpha\dot{\alpha}} = M_\alpha{}^\beta P_{\beta\dot{\beta}} (M^*)^{\dot{\alpha}\dot{\beta}} = (MPM^\dagger)_{\alpha\dot{\alpha}} \tag{3}$$

Indeed this is a Lorentz transformation on the 4-vector (P_μ), since the transformation preserves $\det P = -[-(P_0)^2 + (P_1)^2 + (P_2)^2 + (P_3)^2]$, which is precisely (minus) the norm of P_μ. Hence, any vector can be expressed in terms of bi-spinor components (and vice-versa).

It is useful to introduce the tensors

$$(\epsilon^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad ; \qquad (\epsilon_{\alpha\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

(and similarly for dotted indices). They are Lorentz invariant, namely they satisfy

$$\epsilon_{\alpha\beta} \,=\, M_\alpha{}^\gamma \, M_\beta{}^\delta \, \epsilon_{\gamma\delta} \qquad ; \qquad \epsilon^{\alpha\beta} \,=\, \epsilon^{\gamma\delta} \, (M^{-1})_\gamma{}^\alpha \, (M^{-1})_\delta{}^\beta \tag{5}$$

as may be checked by using their explicit expressions.

These properties imply that the tensors can be used to raise and lower indices

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \qquad ; \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \tag{6}$$

(and similarly for dotted indices). What this means is that e.g. the object $\epsilon^{\alpha\beta}\psi^\beta$ transforms as an object $()^\alpha$, (i.e. as a column vector on which M acts), which we denote ψ^α. We introduce the shorthand notation

$$\chi\psi = \chi^\alpha\psi_\alpha \qquad ; \qquad \bar{\chi}\bar{\psi} = \chi_{\dot{\alpha}}\psi^{\dot{\alpha}} \tag{7}$$

Using the ϵ tensors, we can also define

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu{}_{\beta\dot{\beta}} \tag{8}$$

They satisfy

$$(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta = -2\eta^{\mu\nu}\delta_\alpha{}^\beta \quad ; \quad (\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} = -2\eta^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{9}$$

In terms of them, the generators of the Lorentz group are given by

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{4}[\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}] \quad ; \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}[\dot{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\nu{}_{\alpha\dot{\beta}} - \dot{\sigma}^{\nu\dot{\alpha}\alpha}\sigma^\mu{}_{\alpha\dot{\beta}}] \tag{10}$$

Given two Weyl spinors of opposite chiralities χ_α, $\bar{\psi}^{\dot{\alpha}}$ (and equal global and gauge quantum numbers), one can construct a four-component Dirac spinor by superposing them as a column vector

$$\Psi_D = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{11}$$

on which the Dirac matrices are realized as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{12}$$

which satisfy the Clifford algebra relations, as follows from (9). Also, given a single Weyl spinor, say χ_α, in a real representation of all all global and gauge symmetries, one can construct a four-component fermion, by taking its conjugate to play the role of the right-handed piece, as follows

$$\Psi_M = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{13}$$

Such spinors Ψ_M are thus subject to a reality condition, and are denoted Majorana. Notice that Weyl spinors in complex representations of the global or gauge symmetries cannot be turned into Majorana spinors, since the spinor and its conjugate cannot belong to the same multiplet.

2 4d $N = 1$ Supersymmetry algebra and representations

In this section we discuss the basic structure of 4d $N = 1$ supersymmetry algebra, and its realization in terms of fields.

2.1 The supersymmetry algebra

The 4d $N = 1$ supersymmetry algebra contains two spinorial generators Q_α, $\bar{Q}_{\dot{\alpha}}$, which behave as Grassman variables, and hence obey anticommutation relations. The algebra is given by

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2\sigma^\mu{}_{\alpha\dot{\alpha}} P_\mu \\ \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \\ \{P_\mu, Q_\alpha\} &= \{P_\mu, \bar{Q}_{\dot{\alpha}}\} = 0 \end{aligned} \tag{14}$$

(in addition, we have the natural commutators that imply that the Q's are in the spinor representations).

OBS: The above algebra is invariant under $U(1)$ transformations rotating the supercharges Q_α, $\bar{Q}_{\dot{\alpha}}$ by opposite phases.

$$Q_\alpha \to e^{i\lambda} Q_\alpha \qquad ; \qquad \bar{Q}_{\dot{\alpha}} \to e^{-i\lambda} \bar{Q}_{\dot{\alpha}} \tag{15}$$

This symmetry is known as R-symmetry.

Since the supergenerators Q_α, $\bar{Q}_{\dot{\alpha}}$, are Grassman quantities, when realized on quantum fields they relate bosons and fermions. Each multiplet providing a representation of the supersymmetry algebra (supermultiplet) thus contains bosons and fermions. Since the operator P^2, which is the mass square operator, commutes with the Q's, bosons and fermions in the same multiplet are mass degenerate. Similarly, the supergenerators commute with any global and gauge symmetry of the theory [1], so all fields in a supermultiplet belong to the same representation of global and gauge symmetries.

An important property is that the total number of *physical* bosonic and fermionic degrees of freedom is equal within a supermultiplet. To show this, we define the operator $(-1)^F$, which is equal to $+1$ for bosons and -1 for fermions, and hence satisfies $(-1)^F Q_\alpha = -Q_\alpha (-1)^F$. We can then compute, in two different ways, $\mathrm{Tr}\,[(-1)^F \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}]$, where the trace is taken over states of fixed momentum in a supermultiplet,

$$\begin{aligned}
1) \qquad \mathrm{Tr}\,[(-1)^F \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}] \; &= \mathrm{Tr}\,[(-1)^F Q_\alpha \bar{Q}_{\dot{\alpha}} + (-1)^F \bar{Q}_{\dot{\alpha}} Q_\alpha] = \\
&= \mathrm{Tr}\,[-Q_\alpha (-1)^F \bar{Q}_{\dot{\alpha}} + Q_\alpha (-1)^F \bar{Q}_{\dot{\alpha}}] = 0 \\
2) \qquad \mathrm{Tr}\,[(-1)^F \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}] \; &= 2 \sigma^\mu{}_{\alpha\dot{\alpha}} P_\mu \mathrm{Tr}\,[(-1)^F]
\end{aligned} \tag{16}$$

Hence $\mathrm{Tr}\,[(-1)^F] = 0$ in a supermultiplet.

[1] Except for R-symmetries, see below.

2.2 Structure of supermultiplets

Let us consider the construction of the supermultiplet for massive fields of mass M. Going to the rest frame for such particles, the relevant piece of the algebra (14) becomes

$$
\begin{aligned}
\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2M\delta_{\alpha\dot{\alpha}} \\
\{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0
\end{aligned}
\tag{17}
$$

By defining $a_\alpha = Q_\alpha/\sqrt{2M}$, $a_\alpha^\dagger = \bar{Q}_{\dot{\alpha}}/\sqrt{2M}$, these are the anticommutators for two decoupled fermionic harmonic oscillators. The supermultiplet is built by starting with a lowest helicity state $|\Omega\rangle$, obeying $a_\alpha|\Omega\rangle = 0$, and appying operators $a_\alpha^\dagger$, namely

State	Helicity
$\|\Omega\rangle$	j
$a_\alpha^\dagger\|\Omega\rangle$	$j \pm \frac{1}{2}$
$a_1^\dagger a_2^\dagger\|\Omega\rangle$	j

In building a quantum field theory with the corresponding fields, it is important to notice that CPT flips the chirality (and conjugates the global and gauge representations), so a CPT-invariant supermultiplet may require using two of the above basic multiplets.

Two of the most useful supermultiplets are the following:

- The massive scalar supermultiplet is obtained by starting with a $j = 0$ state $|\Omega\rangle$. It contains states of helicities 0, $\pm 1/2$, 0. It thus contains a Weyl spinor and a complex scalar. This is CPT-invariant if the supermultiplet belongs to a real representation of the gauge and global symmetries. If not, two of these basic multiplets, in conjugate representations, must be combined to form a CPT-invariant set.

- The massive vector multiplet is obtained by starting with a $j = 1/2$

state $|\Omega\rangle$. It contains states of helicities 1/2, 1, 0, 1/2. Combining it with its CPT conjugate, the total multiplet contains one massive vector boson, one real scalar and two Weyl fermions.

Let us now consider the construction of supermultiplets for massless fields. Since they have light-like momentum $P^2 = 0$, they do not have rest frame, but we may use a reference system where $P = (-E, 0, 0, E)$. In this frame, the supersymmetry algebra is

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2 \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \tag{18}$$

Defining the rescaled operators

$$a = \frac{1}{2\sqrt{E}} Q_1 \qquad ; \qquad a^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}} \tag{19}$$

they correspond to a fermionic harmonic oscillator. The multiplet is constructed by starting with a lowest helicity state $|\Omega\rangle$, satisfying

$$a|\Omega\rangle = Q_2|\Omega\rangle = \bar{Q}_{\dot{2}}|0\rangle = 0 \tag{20}$$

Hence the multiplet contains the states $|\Omega\rangle$ and $a^\dagger|\Omega\rangle$, with helicities j and $j+1/2$, respectively. As before, one may need to combine this multiplet with its CPT conjugate to formulate a quantum field theory.

Some of the most useful massless supermultiplets are:

- The chiral supermultiplet, obtained by taking $|\Omega\rangle$ of helicity $j = 0$, so it contains states of helicity $j = 0, 1/2$. This should be combined with its CPT conjugate, with helicities $j = 0, -1/2$. This complete chiral supermultiplet contains a complex scalar and a 4d Weyl fermion. This multiplet can transform in an arbitrary representation of the gauge and global symmetries, hence contains a chiral fermion, which is necessarily massless. If the multiplet happens to transform in a real representation, it is possible to write a mass term for it (see later), so it is equivalent to a massive scalar supermultiplet.

- The massless vector supermultiplet, obtained by taking $|\Omega\rangle$ of helicity $j = 1/2$, so it contains states of helicities $j = 1/2, 1$. Combined with its CPT conjugate, with helicities $j = -1, -1/2$, the multiplet contains a 4d Weyl spinor and a massless vector boson. The multiplet transforms in the adjoint representation of the gauge group, which is real, so the 4d Weyl spinor can be recast as a 4d Majorana spinor.

- The supergravity multiplet, containing states of helicity $j = 3/2, 2$. Combined with its CPT conjugate, of helicities $j = -2, -3/2$, it contains a graviton and a gravitino (a spin 3/2 particle). We will not discuss it in detail, since interacting theories involving this multiplet have spacetime diffeomorphism invariance, and include gravity (and in fact local supersymmetry), they are known as supergravity theories, and lie beyond the scope of this lecture

3 Component fields, chiral multiplet

The supersymmetry transformation parameters are anticommuting spinors ξ^α, $\bar{\xi}_{\dot\alpha}$. Formally, the supersymmetry variation

is $\delta_\xi = \xi Q + \bar{\xi}\bar{Q}$. The supersymmetry algebra can be expressed as

$$\begin{aligned} [\xi Q, \bar{\eta}\bar{Q}] &= 2\,\xi\sigma^\mu\bar{\eta}\,P_\mu \\ [\xi Q, \eta Q] &= [\bar{\xi}\bar{Q}, \bar{\eta}\bar{Q}] = 0 \end{aligned} \tag{21}$$

We would like to construct a representation of the supersymmetry algebra, using the massive scalar multiplet, which contains as *physical* degrees of freedom a 4d Weyl spinor ψ_α and a complex scalar Φ. The supersymmetry transformations of these fields are

$$\begin{aligned} \delta_\xi\Phi &= \sqrt{2}\,\xi\psi \\ \delta_\xi\psi_\alpha &= i\sqrt{2}\,\sigma^\mu{}_{\alpha\dot\alpha}\bar{\xi}^{\dot\alpha}\,\partial_\mu\Phi + \sqrt{2}\,\xi_\alpha F \end{aligned} \tag{22}$$

Namely

$$\bar{Q}_{\dot{\alpha}}\Phi = 0 \qquad\qquad Q_\alpha \Phi = \sqrt{2}\psi_\alpha$$
$$\bar{Q}_{\dot{\alpha}}\psi_\alpha = -i\sqrt{2}\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu\Phi \qquad\qquad Q_\alpha\psi_\beta = \sqrt{2}\epsilon_{\alpha\beta}F \tag{23}$$

The field F appearing in the transformation of the fermions is discussed below.

The transformations acting on Φ satisfy the supersymmetry algebra. In order for the transformations acting on ψ to satisfy the supersymmetry algebra, we have two choices

i) Take $F = -m\Phi^*$, and use the equation of motion of a free massive

fermion for ψ, namely $-i\bar{\sigma}^\mu\partial_\mu\psi = m\psi$. Since we are using equations of motion, the algebra closes on-shell.

ii) Consider F to be and independent field, and require $\delta_\xi F = i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu\partial_\mu\psi$. Since the equations of motion are not involved, the algebra closes off-shell.

Notice that the viewpoint i) is disadvantageous, since the equations of motion are different for different theories, and this complicates the construction of interacting theories. On the other hand, from the viewpoint ii) the transformations obey the supersymmetry algebra relations, no matter what the dynamics of the theory is. It is important to notice that the field F does not really describe a new physical degree of freedom. Since the dimension of ξ, $\bar{\xi}$ is $1/2$, F has dimension 2, and it is not possible to write a kinetic term for it, and it is called an auxiliary field. Hence we still have equality of the number of bosonic and fermionic *physical* degrees of freedom in the supermultiplet.

In principle, one can construct supersymmetry transformations for fields in other supermultiplets. However it is non-trivial to do so for more complicated supermultiplets. The task is facilitated by a technique, known as superfield formalism.

4 Superfields

4.1 Superfields and supersymmetry transformations

Let us consider the set of component fields in a supermultiplet. Since they form an irreducible representation, the whole set can be generated from any one of them, say A, by acting with the supergenerators. It is useful to consider the following formal expression

$$F(x,\theta,\bar\theta) = e^{\theta Q+\bar\theta\bar Q} A \tag{24}$$

Different component fields in the supermultiplet arise as coefficients in the power-expansion of F in θ, $\bar\theta$. Since the latter are Grassman variables, the power-expansion is a finite expression, of the form

$$\begin{aligned} F(x,\theta,\bar\theta) &= f(x) + \theta\phi(x) + \bar\theta\bar\xi(x) + \theta\theta\, m(x) + \bar\theta\bar\theta\, n(x) + \\ &+ \theta\sigma^\mu\bar\theta\, v_\mu(x) + \theta\theta\,\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\,\theta\psi(x) + \theta\theta\,\bar\theta\bar\theta\, d(x) \end{aligned} \tag{25}$$

where all the fields are related to each other by the action of Q, $\bar Q$. Expressions of the form (25), providing a formal sum of the component fields in a supermultiplet, are refered to as superfields. Formally, they are functions over a superspace parametrize by the supercoodinates $z=(x,\theta,\bar\theta)$. A whole branch of mathematical physics is the study of the geometry of superspace (supergeometry), but we will not need much of its machinery.

The use of superfields facilitates the computation of supersymmetry transformations of the component fields. Let us introduce a formal sum of such variations

$$\begin{aligned} \delta_\xi F(x,\theta,\bar\theta) &= \delta_\xi f(x) + \theta\delta_\xi\phi(x) + \bar\theta\delta_\xi\bar\xi(x) + \theta\theta\,\delta_\xi m(x) + \bar\theta\bar\theta\,\delta_\xi n(x) + \\ &+ \theta\sigma^\mu\bar\theta\,\delta_\xi v_\mu(x) + \theta\theta\,\bar\theta\delta_\xi\bar\lambda(x) + \bar\theta\bar\theta\,\theta\delta_\xi\psi(x) + \theta\theta\,\bar\theta\bar\theta\,\delta_\xi d(x) \end{aligned} \tag{26}$$

We formally write $\delta_\xi F \equiv (\xi Q + \bar{\xi}\bar{Q}) \times F$. The operation $(\xi Q + \bar{\xi}\bar{Q})\times$ thus maps a superfield to the superfield constructed using the susy variations of the component fields. Notice that it does not interfere with the θ, $\bar{\theta}$.

We would like to represent the action of $(\xi Q+\bar{\xi}\bar{Q})\times$ in terms of differential operators in superspace. The simplest operators in superspace are derivatives $\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}$ and $\bar{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$ (in addition to the familiar $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Using Hausdorff formula, $e^{A+B} = e^A e^B e^{-[A,B]/2}$ (for A, B, commuting with $[A,B]$), we have

$$\begin{aligned}
\xi^\alpha \partial_\alpha \left(e^{\theta Q + \bar{\theta}\bar{Q}} \times \right) &= \xi^\alpha \partial_\alpha \, e^{\theta Q} e^{\bar{\theta}\bar{Q}} e^{-\theta\sigma^\mu\bar{\theta}P_\mu} \times = \\
&= (\xi Q + i\sigma^\mu\bar{\theta}\partial_\mu) \times e^{\theta Q+\bar{\theta}\bar{Q}} \times \\
\bar{\xi}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} \, e^{\theta Q+\bar{\theta}\bar{Q}} \times &= (\bar{\xi}\bar{Q} - i\theta\sigma^\mu\bar{\xi}\partial_\mu) \times e^{\theta Q+\bar{\theta}\bar{Q}} \times
\end{aligned} \tag{27}$$

From this we learn that the action of ξQ, $\bar{\xi}\bar{Q}$ on component fields can be represented in terms of differential operators acting on superfields. By abuse of notation, these differential operators are also denoted Q_α and $\bar{Q}_{\dot{\alpha}}$

$$\begin{aligned}
Q_\alpha &= \partial_\alpha - i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \\
\bar{Q}_{\dot{\alpha}} &= \bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu
\end{aligned} \tag{28}$$

Namely, given a superfield $F(x,\theta,\bar{\theta})$, we can compute the supersymmetry variation of its components, which are encoded in the superfield of variations (26) $\delta_\xi F = (\xi Q + \bar{\xi}\bar{Q}) \times F$, by computing $\delta_\xi F$ using the action of the differential operators (28), namely $\delta_\xi F = (\xi Q + \bar{\xi}\bar{Q})F$. Comparing terms in both θ-expansions leads to the supersymmetry variations.

An important observation is that the component field corresponding to highest power in θ, $\bar{\theta}$ in the expansion, always transforms as a total divergence. This is because θ, $\bar{\theta}$ have dimension $-1/2$, so that this component field is the one of highest dimension in the supermultiplet. On the other hand, the supergenerators Q_α, $\bar{Q}_{\dot{\alpha}}$, have dimension $1/2$. Thus the supersymmetry variation of the highest-dimension component field is necessarily the derivative of a lower-dimension component field. This observation will be the key idea in the construction of supersymmetric field theory actions.

Superfields are useful since they provide linear representations of the supersymmetry algebra. Actually, a completely general superfield corresponds to a reducible representation. Different irreducible representations correspond to superfields satisfying different constraints, consistent with the action of the operators (28). This will be discussed below. For that purpose, it is useful to define the differential operators

$$D_\alpha \;=\; \partial_\alpha + i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \qquad ; \qquad \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu \tag{29}$$

They anticommute with the operators (28)

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \tag{30}$$

4.2 The chiral superfield

A chiral superfield $\Phi(x,\theta,\bar{\theta}$ is characterized by $\bar{D}_{\dot{\alpha}}\Phi = 0$. It is useful to describe it in terms of a new position variable $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, in terms of which the differential operators (29) read

$$D_\alpha \;=\; \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^\mu} \qquad ; \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \tag{31}$$

Hence a chiral superfield has the expansion

$$\Phi(y,\theta,\bar{\theta}) = \Phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{32}$$

We can readily identify that this describes a chiral (or scalar) supermultiplet (by abuse of language, one often uses the same notation for the superfield and for its complex scalar component field, hoping the context will disentangle any possible ambiguity). Indeed we can reproduce the supersymmetry transformations of the component fields, by using the differential operators (28), which in these coordinates read

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} \qquad ; \qquad \bar{Q}_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - 2i\theta^\alpha\sigma^\mu{}_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu} \tag{33}$$

and comparing

$$\begin{aligned}
(\xi Q + \bar\xi\bar Q) \times \Phi(y,\theta,\bar\theta) &= \delta_\xi\Phi(y) + \sqrt{2}\theta^\alpha\delta_\xi\psi_\alpha(y) + \theta\theta\delta_\xi F(y) \qquad (34)\\
(\xi Q + \bar\xi\bar Q)\Phi(y,\theta,\bar\theta) &= \xi^\alpha\frac{\partial}{\partial\theta^\alpha}\Phi(y,\theta,\bar\theta) + (\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - 2i\theta^\alpha\sigma^\mu{}_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu})\xi^{\dot\alpha}\Phi(y,\theta,\bar\theta) = \\
&= \sqrt{2}\xi\psi + \sqrt{2}\theta^\alpha(-i\sqrt{2}\sigma^\mu{}_{\alpha\dot\alpha}\bar\xi^{\dot\alpha}\partial_\mu\Phi + \xi_\alpha F) + \theta\theta\, i\sqrt{2}\bar\xi\bar\sigma^\mu\partial_\mu\psi
\end{aligned}$$

In terms of the original coordinates, we have

$$\begin{aligned}
\Phi(x,\theta,\bar\theta) &= \Phi(x) + i\theta\sigma^\mu\bar\theta\,\partial_\mu\Phi(x) + \frac{1}{4}\theta\theta\,\bar\theta\bar\theta\,\Box\Phi(x) + \\
&\quad + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar\theta + \theta\theta F(x)
\end{aligned} \tag{35}$$

Notice that the highest-dimension component is the same, expressed in terms of x or y.

An antichiral field satisfies the condition that D_α annihilates it. Clearly the the adjoint superfield $\Phi^\dagger$ of a chiral superfield is antichiral. In terms of x, θ, $\bar\theta$, it reads

$$\begin{aligned}
\Phi^\dagger(x,\theta,\bar\theta) &= \Phi^*(x) - i\theta\sigma^\mu\bar\theta\,\partial_\mu\Phi^*(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\Box\Phi^*(x) + \\
&\quad + \sqrt{2}\bar\theta\bar\psi(x) + \frac{i}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi(x) + \theta\theta F^*(x)
\end{aligned} \tag{36}$$

The supermultiplet has a simpler expression in terms of the variable $y^{\dagger\,\mu} = x^\mu - i\theta\sigma^\mu\bar\theta$, it reads

$$\Phi^\dagger(y^\dagger,\theta,\bar\theta) = \Phi^*(y^\dagger) + \sqrt{2}\bar\theta\bar\psi(y^\dagger) + \theta\theta F^*(y^\dagger) \tag{37}$$

An important property of chiral multiplets is that their product is also a chiral superfield. This is straightforward using the expression in terms of

y coordinates. By using power-series, one can show that any holomorphic function of chiral multiplets $W(\Phi_k(x,\theta,\bar{\theta})$ is also a chiral multiplet. For future convenience, one can show that its highest-dimension component is given by

$$W(\Phi)|_{\theta\theta} \;=\; \frac{\partial^2 W}{\partial\Phi_k\partial\Phi_l}\psi_k\psi_l + F_k\left(\frac{\partial W}{\partial\Phi_k}\right)^* + F_k^*\frac{\partial W}{\partial\Phi_k} \tag{38}$$

where in the right-hand side Φ denotes the scalar component field, not the superfield.

On the other hand, non-holomorphic functions like $\Phi^\dagger\Phi$ are not chiral superfields. For future convenience, we list the highest-dimension component of the latter

$$\begin{aligned}\Phi_1^\dagger(x,\theta,\bar{\theta})\Phi_2(x,\theta,\bar{\theta})|_{\theta\theta\bar{\theta}\bar{\theta}} \;=\;& F_1^*F_2 + \frac{1}{4}\Phi_1^*\Box\Phi_2 + \frac{1}{4}\Box\Phi_1^*\,\Phi_2 - \frac{1}{2}\partial_\mu\Phi_1^*\partial^\mu\Phi_2 + \\ &+\frac{i}{2}\partial_\mu\bar{\psi}_1\bar{\sigma}^\mu\psi_2 - \frac{i}{2}\bar{\psi}_1\bar{\sigma}^\mu\partial_\mu\psi_2\end{aligned} \tag{39}$$

We are now ready to construct supersymmetric lagragians for fields in chiral supermultiplets. The key idea is that, since the highest-dimensional component of a supermultiplet (usually a product of basic supermultiplets) transforms as a total derivative, its spacetime integral is invariant under supersymmetry transformations. The strategy then is to construct product superfields whose highest-dimensional component corresponds to kinetic and interactions terms. Finally, recalling the rules of integration over Grassman variables,

$$\int d\theta = 0 \qquad ; \qquad \int d\theta\,\theta = 1 \tag{40}$$

an efficient way to extract the highest component of a supermultiplet is to integrate it over the supercoordinates θ and/or $\bar{\theta}$. For instance

$$\int d^2\theta\Phi(x,\theta,\bar{\theta}) = F(x) \tag{41}$$

A typical supersymmetric action for a set of chiral supermultiplets has the structure

$$S \;=\; \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \Phi_i^\dagger \Phi_i \;+\; \int d^4x\, d^2\theta\, W(\Phi_i) \;+\; \int d^4x\, d^2\bar{\theta}\, W(\Phi_i)^* \quad (42)$$

The first term can be generalized to $\int d^4x\, d^2\theta\, d^2\bar{\theta}\, K(\Phi_i, \Phi_i^\dagger)$, with K a real function, known as Kahler potential. Expanding in components, this implies that the space parametrized by scalars in chiral multiplets is Kahler (in the geometric sense). We will however stick to the canonical kinetic term above, but occasionally refer to these more general possible actions.

Using (38), (39), the action in component fields reads (integrating by parts in certain terms)

$$S \;=\; -\left[\partial_\mu \Phi_i^* \partial^\mu \Phi_i + i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i - F_i^* F_i - \frac{\partial^2 W}{\partial \Phi_i \partial \Phi_j} \psi_i \psi_j - F_i \left(\frac{\partial W}{\partial \Phi_i}\right)^* - F_i^* \frac{\partial W}{\partial \Phi_i}\right] \quad (43)$$

We see that the auxiliary fields F_i are indeed non-dynamical. We can use their equations of motion, to obtain $F_i = -\partial W / \partial \Phi_i$. Replacing in the above expression, we have

$$S \;=\; -\left[\partial_\mu \Phi_i^* \partial^\mu \Phi_i + i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i - \sum_i \left|\frac{\partial W}{\partial \Phi_i}\right|^2 + \frac{\partial^2 W}{\partial \Phi_i \partial \Phi_j} \psi_i \psi_j\right] \quad (44)$$

The first two pieces are standard kinetic terms. The fourth describes scalar-fermion interactions, and the third is a scalar potential

$$V(\Phi_i) = \sum_i \left|\frac{\partial W}{\partial \Phi_i}\right|^2 \quad (45)$$

It is positive-definite, and vanishes for scalar vevs such that

$$F_i = -\frac{\partial W}{\partial \Phi_i} = 0 \quad (46)$$

These are know as F-term constraints, which are a necessary condition for a supersymmetric vacuum of the theory.

An important property of supersymmetric field theories is that the superpotential is not renormalized in perturbation theory. That is, because of the relations imposed by supersymmetry, all radiative corrections to the terms arising from the superpotential vanish to all orders in perturbation theory. The proof of this statements involves the structure of Feynman diagrams in superspace, and we will not discuss it. In particular examples (for instance for the Wess-Zumino model, i.e. a theory with one chiral multiplet and a cubic superpotential), one can show it very explicitly exploiting the holomorphy of the superpotential, see [**?**] for detailed discussion. Both arguments show that there are important non-renormalization theorems involving terms in the action which involve intergration over half the superspace coordinates. Another important observation is that the non-renomalization theorem in general does not hold beyond perturbation theory, hence non-perturbative corrections to the superpotential may appear. In some situations, they may be exactly computable using the constraints from supersymmetry and reasonable assumptions about the field theory dynamics. These non-perturbative corrections usually have a nice physical interpretation (like instanton effects or gaugino condensation). See [**?**] for more complete discussion.

4.3 The vector superfield

A vector superfield V is characterized by the condition $V = V^\dagger$. The expansion in component fields can be expressed as

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x) + i\theta\chi(x) i\bar\theta\bar\chi(x) + \frac{i}{2}\theta\theta M(x) - \frac{i}{2}\bar\theta\bar\theta M^*(x) - \theta\sigma^\mu\bar\theta\, V_\mu(x) + \\ &+ i\theta\theta\,\bar\theta\,[\,\bar\lambda(x) + \frac{i}{2}\bar\sigma^\mu\partial_\mu\xi(x)\,] - i\bar\theta\bar\theta\,\theta\,[\,\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar\xi(x)\,] + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\, D(x) \end{aligned} \quad (47)$$

The peculiar choice of components in the $\theta^2\bar\theta$, $\bar\theta^2\theta$ and $\theta^2\bar\theta^2$ terms, is for future convenience.

As we will see, the content of component fields of the vector superfield is that of a massless vector superfield. Thus, it should describe the supersymmetric version of a gauge boson. Hence there is a supersymmetric version of a gauge transformation. For vector multiplets associated to $U(1)$, it is given by

$$V \longrightarrow V + (\Lambda + \Lambda^\dagger) \tag{48}$$

where $\Lambda(y, \theta, \bar\theta) = A + \sqrt{2}\theta\psi + \theta\theta F$ is a chiral superfield. Since

$$\begin{aligned} \Lambda + \Lambda^\dagger &= \Lambda + \Lambda^* + \sqrt{2}(\theta\psi + \bar\theta\bar\psi) + \theta\theta F + \bar\theta\bar\theta F^* + i\theta\sigma^\mu\bar\theta\partial_\mu(\Lambda - \Lambda^*) + \\ &+ \frac{i}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi + + \frac{i}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi + \frac{1}{4}\theta\theta\bar\theta\bar\theta\Box(\Lambda + \Lambda^*) \end{aligned} \tag{49}$$

the transformation of component fields is

$$\begin{aligned} V_\mu \to V_\mu - i\partial_\mu(\Lambda - \Lambda^*) \quad &; \quad C \to C + \Lambda + \Lambda^+ \\ \lambda \to \lambda \quad & \quad \xi \to \xi - i\sqrt{2}\psi \\ D \to D \quad & \quad M \to M - 2iF \end{aligned} \tag{50}$$

So one can use the gauge transformation parameters $\Lambda + \Lambda^*$, ψ, F to gauge away C, ξ and M. The vector supermultiplet then reduces to [2]

$$V(x, \theta, \bar\theta) = -\theta\sigma^\mu\bar\theta V_\mu + i\theta\theta\,\bar\theta\bar\lambda - i\bar\theta\bar\theta\,\theta\lambda + \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\,D \tag{51}$$

This partial gauge fixing, known as Wess-Zumino gauge, still allows for standard gauge transformations $V_\mu \to V_\mu - i\partial_\mu(\Lambda - \Lambda^*)$. Hence the vector supermultiplet provides the supersymmetric generalization of the Yang-Mills gauge

[2]Notice that supersymmetry transformations do not preserve the WZ gauge. Hence any supersymmetry transformation should be followed by a compensating gauge transformation to bring the supermultiplet to the WZ gauge.

potential V_μ. In order to build gauge-invariant kinetic terms, we introduce the field-strength superfields

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\, D_\alpha V \qquad ; \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\, \bar{D}_{\dot{\alpha}}V \tag{52}$$

They are chiral superfields, which are invariant under the gauge transformations (48). In terms of components fields (in coordinates y, θ, $\bar{\theta}$), we have

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}\nu)_\alpha{}^\beta\theta_\beta\, F_{\mu\nu}(y) + \theta\theta\, \sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(y) \tag{53}$$

where $F_{\mu\nu} = \partial_{[\mu}V_{\nu]}$. There is a similar expression for $\bar{W}_{\dot{\alpha}}$ in terms of $y^\dagger$. Hence the above superfields provide the supersymmetric completion of the gauge-invariant field strength.

The gauge and Lorentz invariant expression $W^\alpha W_\alpha$ has a highest-dimension component

$$W^\alpha W_\alpha = \ldots + \theta\theta\,(\,-2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + D^2 + \frac{i}{2}\varepsilon_{\mu\nu\sigma\rho}F^{\mu\nu}F^{\sigma\rho}\,) \tag{54}$$

precisely of the form of the kinetic term (and theta-term) for the $U(1)$ gauge boson, and the gauginos. Hence the action for the gauge boson can be constructed as

$$S = \int d^4x\, d^2\theta\, W^\alpha W_\alpha \,+\, \int d^4x\, d^2\bar{\theta}\, \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} \tag{55}$$

OBS: For $U(1)$ gauge group, it is also possible to introduce an additional term in the action, known as Fayet-Illiopoulos term, of the form

$$S_{FI} = \chi_{FI}\int d^4x \int d^2\theta\, d^2\bar{\theta}\, V \;=\; \int d^4x\, D \tag{56}$$

where χ_{FI} is a constant.

The discussion of non-abelian gauge bosons is similar, with slightly more general definitions. Vectors superfields have the same structure, but transform in the adjoint representation. The gauge parameters are given by a set of chiral multiplets in the adjoint representation of the gauge group G,

$$\Lambda_{ij} = T^a{}_{ij}\Lambda_a \tag{57}$$

The gauge transformation is given by

$$e^V \to e^{-i\Lambda^\dagger} e^V e^{i\Lambda} \tag{58}$$

This also allows for a WZ gauge, leaving V_μ^a, λ^a, D^a as degrees of freedom, with the standard gauge transformations for V_μ^a. The non-abelian field-strength superfields are given by

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_a e^V \tag{59}$$

which transforms under (58) as

$$W_\alpha \to e^{-i\Lambda^\dagger} W_\alpha e^{i\Lambda} \tag{60}$$

The supersymmetric Yang-Mills action is given by (55), with an implicit trace over gauge indices.

4.4 Coupling of vector and chiral multiplets

We would like to discuss the construction of actions describing the interaction of gauge and chiral supermultiplets. As expected, the coupling of chiral multiplets to gauge vector multiplets is obtained by a suitable modification of the chiral multiplet kinetic term so as to make it gauge invariant.

Let us start with the case of a $U(1)$ vector multiplet, and several chiral multiplets ϕ_i, transforming under $U(1)$ with charges q_i. Namely, under a gauge transformation $V \to V + i(\Lambda - \Lambda^\dagger)$,

$$\Phi_i \to e^{-iq_i\Lambda}\Phi_i \quad ; \quad \Phi_i^\dagger \to e^{iq_i\Lambda^\dagger}\Phi_i^\dagger \tag{61}$$

Hence the expression $\Phi_i^\dagger e^{q_i V}\Phi_i$ is gauge invariant, and is the gauge-invariant generalization of $\Phi^\dagger\Phi_i$.

The full lagrangian for the vector and chiral multiplet interactions is

$$\begin{aligned} S &= \int d^4x\, d^2\theta\, W^\alpha W_\alpha + \int d^4x\, d^2\bar\theta\, \bar W_{\dot\alpha}\bar W^{\dot\alpha} + \\ &+ \int d^4x\, d^2\theta\, d^2\bar\theta\, \Phi_i^\dagger\, e^{q_i V}\, \Phi_i + \int d^4x\, d^2\theta\, W(\Phi_i) + \int d^4x \int d^2\bar\theta\, W(\Phi)^* \end{aligned} \quad (62)$$

In fact, one can generalize the gauge kinetic term to an expression $\int d^4x d^\theta f(\Phi) W^\alpha W_\alpha$, where f is a holomorphic function (known as gauge kinetic function) and Φ are chiral multiplets. Notice that this can be regarded as promoting the gauge coupling to a chiral superfield. In the following we however stick to the simplest situation of constant f.

The term containing the chiral-vector coupling is

$$\begin{aligned} &\int d^4x\, d^2\theta\, d^2\bar\theta\, \Phi^\dagger e^{qV}\Phi = FF^* + \Phi\Box\Phi^* + i\partial_\mu\bar\psi\bar\sigma^\mu\psi + \frac{1}{2}(\,qD\Phi^*\Phi\,) + \qquad (63) \\ &+qV_\mu\,(\,\frac{1}{2}\bar\psi\bar\sigma^\mu\psi + \frac{1}{2}\Phi^*\partial_\mu\Phi - \frac{i}{2}\partial_\mu\Phi^*\Phi\,) - \frac{i}{\sqrt{2}}q\,(\,\Phi\bar\lambda\bar\psi - \Phi^*\lambda\psi\,) - \frac{1}{4}q^2\, V_\mu V^\mu \Phi^*\Phi \end{aligned}$$

One can integrate out the auxiliary field D, by using its equations of motion. The field D appears in

$$\mathcal{L}_D = \frac{1}{2}D^2 + \frac{1}{2}\sum_i q_i D\Phi_i^*\Phi_i + \chi_{FI} D \quad (64)$$

so the equations of motion give $D = -1/2\,\sum_i q_i\Phi_i^*\Phi_i + \chi_{FI}$. The D-term lagrangian becomes a potential term

$$V_D = \frac{1}{2}(\frac{1}{2}\sum_i q_i\Phi_i^*\Phi_i - \chi_{FI})^2 \quad (65)$$

The condition $D=0$ that it vanishes is a necessary condition for a supersymmetric vacuum, known as D-term condition.

For non-abelian gauge symmetries, chiral multiplets transform in a representation R of the gauge group,

$$\Phi \to e^{-i\Lambda}\Phi \quad (66)$$

where Φ is regarded as a column vector and $\Lambda_{ij} = (t_a^R)_{ij}\Lambda^a$ is a matrix acting on it. The action for the complete system is given by

$$\begin{aligned} S \;&=\; \frac{1}{4g^2}\int d^4x\, d^2\theta\, W^\alpha W_\alpha \;+\; \frac{1}{4g^2}\int d^4x\, d^2\bar\theta\, \bar W_{\dot\alpha}\bar W^{\dot\alpha} \;+ \\ &+\int d^4x \int d^2\theta\, d^2\bar\theta\, \Phi_i^\dagger e^{t_a^R V_a}\Phi_i \;+\; \int d^4x\, d^2\theta\, W(\Phi_i) \;+\; \int d^4x\, d^2\bar\theta\, W(\Phi)^* \end{aligned} \tag{67}$$

After integrating out the D field, the D-term potential has the explicit expression

$$V_D \;=\; \frac{1}{2}\sum_a\Big(\frac{1}{2}\sum_k \Phi_k^\dagger (t_a^{R_k})\Phi_k\Big)^2 \tag{68}$$

where the sum in k runs over all chiral multiplets in non-trivial representations (denoted R_k) of the gauge group G.

In conclusion, the most general $N = 1$ supersymmetric action (up to two derivatives) for a system of chiral and vector multiplets is specified by three functions: the Kahler potential $K(\Phi, \Phi^\dagger)$, which is a real function and defines the chiral multiplet kinetic term, the superpotential $W(\Phi)$, which is holomorphic and defines chiral multiplet interactions, and the gauge kinetic functions $f(\Phi)$, which are holomorphic and define the gauge boson kinetic term.

4.5 Moduli space

Supersymmetric gauge field theories often contain flat directions in the scalar potential, namely there is a continuous set of (inequivalent) supersymmetric vacuum states of the theory, parametrized by the vacuum expectation values (vevs) for scalar fields. The scalars parametrizing flat directions in the scalar potential are known as moduli (moduli fields in string theory, like the dilaton etc, are indeed examples of such fields), and are massless. The set of vevs corresponding to supersymmetric minima of the theory is known as moduli space.

The conditions that scalar vevs should satisfy to belong to the moduli space are that the F-terms and D-terms vanish, namely

$$
\begin{aligned}
\frac{\partial W}{\partial \Phi_i} &= 0 \\
\sum_i \Phi_i^\dagger (t_a^{R_i}) \Phi_i &= 0
\end{aligned}
\tag{69}
$$

where i runs through the chiral multiplets in the theory (in a representation R_i of the gauge group) and a runs through the generators of the gauge group.

Notice that supersymmetry is essential in maintaining the direction flat after quantum corrections. Indeed the F-term conditions are obtained from the superpotential, which is protected against quantum corrections by supersymmetry. On the other hand, the D-term conditions follow from gauge invariance, and are uncorrected as well. In non-supersymmetric theories, fields which look like moduli at tree level typically acquire mass terms from radiative corrections, and moduli space is lifted (a non-trivial scalar potential develops).

Let us provide some typical examples of theories with flat directions.

Consider a $U(1)$ gauge theory with one neutral chiral multiplet Φ, and two chiral multiplets Φ_1, Φ_2 with charge $+1$, and two Φ_1, Φ_2 with charge -1. We introduce a superpotential

$$
W = \Phi \Phi_1 \Phi_1' - \Phi \Phi_2 \Phi_2' \tag{70}
$$

The F-term conditions on scalars give

$$
\Phi_1 \Phi_1' = \Phi_2 \Phi_2' \quad ; \quad \Phi \Phi_i = 0 \quad ; \quad \Phi \Phi_i' = 0 \tag{71}
$$

while the D-term conditions read

$$
|\Phi_1|^2 + |\Phi_2|^2 - |\Phi_1'|^2 - |\Phi_2'|^2 = 0 \tag{72}
$$

These equations are satisfied for the choice of vevs

$$\langle \Phi \rangle = 0 \quad ; \quad \langle \Phi_1 \rangle = v \quad ; \quad \langle \Phi_1' \rangle = w \quad ; \quad \langle \Phi_2 \rangle = w \quad ; \quad \langle \Phi_2' \rangle = v \quad ; \tag{73}$$

So the moduli space is parametrized by two complex parameters. There is a complex two-dimensions manifold of vacuum configurations for this theory [3].

Let us provide a second example, with non-abelian gauge symmetry. Consider a $U(N)$ supersymmetric gauge theory with three chiral multiplets Φ_i in the adjoint representation (thus regarded as $N \times N$ matrices, and superpotential

$$W = \mathrm{tr}\ (\ \Phi_1\Phi_2\Phi_3\ -\ \Phi_1\Phi_3\Phi_2\ ) \tag{74}$$

This theory has a very non-trivial moduli space [4]. The F-term conditions read

$$[\Phi_i, \Phi_j] = 0 \tag{75}$$

This implies that the matrices of vevs for these fields should be commuting. Then one can use gauge transformations to simultaneously diagonalize them, so that the vevs are

$$(\Phi_i)_{mn} = (v_i)_n \delta_{mn} \qquad \text{(no sum)} \tag{76}$$

For adjoint multiplets expressed as $n \times n$ matrices, the D-term condition is

$$\sum_i [\,(\Phi_i^\dagger)_{mn}(t_a^{\mathrm{fund}})_{np}(\Phi_i)_{pm} - (\Phi_i^\dagger)_{mn}(t_a^{\mathrm{fund}})_{mq}(\Phi_i)_{nq}\,]\ =\ 0 \tag{77}$$

[3]As we will see later, this theory is in fact $N = 2$ supersymmetric, with V and Φ forming an $N = 2$ vector multiplet, and Φ_i, Φ_i' forming two hypermultiplets. The moduli space is parametrized by the vevs of a hypermultimplet, given by a combination of the latter.

[4]As we will see later, this theory is in fact $\mathcal{N} = 4$ supersymmetric.

These are automatically satisfied, upont substitution of the above vevs.

Hence the moduli space is parametrized by the n triples of complex eigenvalues $(v_i)_n$. Some realizations of this gauge theory in string theory (in terms of configurations of D-branes) allow for a simple geometric interpretation of this moduli space.

5 Extended 4d supersymmetry

5.1 Extended superalgebras

N-extended supersymmetry is generated by N Weyl spinor supercharges Q_α^I, $\bar{Q}_{\dot{\alpha}I}$, with $I = 1, \ldots, N$. Since each supercharge contains two-components, the number of supercharges is $4N$. The algebra that they satisfy is

$$\begin{aligned} \{Q_\alpha^I, \bar{Q}_{\dot{\alpha}J}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I{}_J \\ \{Q_\alpha^I, Q_\beta^J\} &= \varepsilon_{\alpha\beta} Z^{IJ} \\ \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} &= \varepsilon_{\dot{\alpha}\dot{\beta}} (Z^*)_{IJ} \end{aligned} \tag{78}$$

with Z^{IJ} antisymmetric in its indices.

This is the most general superalgebra consistent with 4d Lorentz invariance. The Z^{IJ} (and their conjugates Z^*) commute with all supercharges Q, $\bar{Q}$, and are known as central charges. Each state (each supermultiplet) has a particular value for the corresponding operators. For the most familiar supermultiplets, the value of the central charges is zero, so we ignore them for most of our discussion (however, the supermultiplets describing soliton states of certain supersymmetric theories have non-trivial central charges. Thus, we will make some useful comments on this case, towards the end).

Some remarks are in order: Notice that the R-symmetry of the superalgebra is (for zero central charges) $U(N)$, where the $SU(N)$ acts on the indices I (in the fundamental or antifundamental representation), while the $U(1)$ acts on supercharges as an overall phase rotation (just like in $N=1$ supersymmetry). Notice also the fact that the N-extended supersymmery algebra contains the supersymmetry algebras of M-extended supersymmetry, for $M<N$. This implies that the supermultiplets of extended supersymmetries naturally decompose as sums of supermultiplets of their subalgebras.

5.2 Supermultiplet structure

Let us start by considering the construction of supermultiplets, in a sector of zero central charges, so that the superalgebra reads

$$\begin{aligned} \{Q_\alpha^I, \bar{Q}_{\dot{\alpha}J}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^I{}_J \\ \{Q_\alpha^I, Q_\beta^J\} &= 0 \quad ; \quad \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} = 0 \end{aligned} \tag{79}$$

Let us start discussing massless supermultiplets. In the reference frame where the momentum is $(P_\mu)=(-E,0,0,E)$, the non-trivial piece of the superalgebra reads

$$\{Q_\alpha^I, \bar{Q}_{\dot{\beta}J}\} = 2\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \delta^I{}_J \tag{80}$$

As in the $N=1$ case, the supercharges Q_2^I, $\bar{Q}_{\dot{2}J}$ are realized as zero, and we introduce

$$a^I = \frac{1}{2\sqrt{2}} Q_1^I \quad ; \quad a_I^\dagger = \frac{1}{2\sqrt{2}} \bar{Q}_{\dot{1}I} \tag{81}$$

which satisfy

$$\{a^I, a_J^\dagger\} = \delta^I{}_J \quad ; \quad \{a^I, a^J\} = \{a_I^\dagger, a_J^\dagger\} = 0 \tag{82}$$

We construct the supermultiplet by starting with a state $\Omega\rangle$ of lowest helicity j, annihilated by the a^I (and the Q_2, $\bar{Q}_{\dot{2}}$), and applying the operators $a_I^\dagger$ to it. The number of states in such multiplet is 2^N. As in the $N = 1$ case, CPT invariance may require to combine these basic multiplets with their conjugates to be realized in a local field theory.

We will discuss some explicit examples of massless supermultiplets below.

The construction of massive supermultiplets is also a simple generalization of the $N = 1$ case. In the rest frame, we have

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}_{\dot{\alpha}J}\} &= 2M\delta_{\alpha\dot{\alpha}}\delta^I{}_J \\ \{Q^I_\alpha, Q^J_\beta\} &= 0 \quad ; \quad \{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} = 0 \end{aligned} \tag{83}$$

Rescaling the operators as

$$a^I_\alpha = \frac{1}{\sqrt{2M}} Q^I_\alpha \quad ; \quad a^{I\,\dagger}_\alpha = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha}I} \tag{84}$$

we have a set of $2N$ decoupled fermionic harmonic oscillators, which lead to a supermultiplet of 2^{2N} degrees of freedom.

Finally, let us briefly sketch the construction of massive multiplets in a sector of non-zero central charges. Using the R-symmetry of the theory, we may bring the antisymmetric matrix Z^{IJ} to a block form e.g. for N even (on which we center in what follows)

$$Z = \varepsilon \otimes D = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \tag{85}$$

with $D = \mathrm{diag}\,(Z_1, \ldots, Z_{N/2})$. Splitting the indices I as (a, m), with $a = 1, 2$ and $m = 1, \ldots, N/2$, the central charges read $Z^{am,bn} = \epsilon^{ab}\delta^{mn}\, Z_n$ (no sum). The superalgebra reads

$$\begin{aligned} \{Q^{am}_\alpha, \bar{Q}_{\dot{\alpha}bn}\} &= 2M\delta_{\alpha\dot{\alpha}}\delta^a{}_b\delta^m{}_n \\ \{Q^{am}_\alpha, Q^{bn}_\beta\} &= \varepsilon_{\alpha\beta}\varepsilon^{ab}\delta^{mn}Z_n \\ \{\bar{Q}_{\dot{\alpha}am}, \bar{Q}_{\dot{\beta}bn}\} &= \varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{ab}\delta_{mn}Z_n \end{aligned} \tag{86}$$

We can define the linear combinations

$$\begin{aligned} a_\alpha^m &= \frac{1}{\sqrt{2}}[Q_\alpha^{1m} + \varepsilon_{\alpha\beta}\bar{Q}_{\dot\beta 2m}] \\ b_\alpha^m &= \frac{1}{\sqrt{2}}[Q_\alpha^{1m} - \varepsilon_{\alpha\beta}\bar{Q}_{\dot\beta 2m}] \end{aligned} \tag{87}$$

and their adjoints. They satisfy

$$\begin{aligned} \{a_\alpha^m, a_\beta^n\} &= \{b_\alpha^m, b_\beta^n\} = \{a_\alpha^m, b_\beta^n\} = 0 \\ \{a_\alpha^m, (a_\beta^n)^\dagger\} &= \delta_{\alpha\beta}\delta^{mn}(2M + Z_n) \\ \{b_\alpha^m, (b_\beta^n)^\dagger\} &= \delta_{\alpha\beta}\delta^{mn}(2M - Z_n) \end{aligned} \tag{88}$$

From this it follows that in a sector of given charges Z_n, the masses of the states satisfy $2M \geq |Z_n|$, for all n. This conditions is known as the BPS bound.

For generic mass M, we have $2\times 2\times N/2$ fermionic harmonic oscillators, so that supermultiplets contain 2^{2N} states. On the other hand if $2M = \pm Z_n$ for some n, then some of the operators anticommute, and are realized as zero, so there are $2N-1$ harmonic oscillators, and the representation contains 2^{2N-1} states, less than the generic supermultiplet. Supermultiplets saturating the BPS bound are known as BPS states, and contain less states than generic supermultiplets. This guarantees that BPS states cannot cease to be BPS, and their mass is given by the central charge, which is part of the algebra. Hence, for BPS states the mass is controlled by the symmetry of the theory and is protected agains quantum corrections by supersymmetry.

5.3 Some useful information on extended supersymmetric field theories

There is no simple superfield formalism for theories with extended supersymmetry, hence supersymmetry transformations must be checked on-shell.

The simplest way to describe the supermultiplets and the supersymmetric actions is thus to phrase them in terms of the supermultiplets and superfield formalism of an $N = 1$ subalgebra.

In the following we discuss some basic features of $N = 2, 4$ supersymmetric theories. $N = 8$ supersymmetry also appears in some applications, but the smallest supermultiplet already contains spin-2 particles, namely gravitons. They can be realized in theories describing gravitational interactions, namely supergravity theories. Their discussion is beyond the scope of this lecture. Finally, for even higher degree of supersymmetry, even the smallest massless supermultiplet already contains fields with spin higher than 2. It is not known how to write interacting theories for such fields, hence they are not usually considered.

5.3.1 $N = 2$ supersymmetric theories

The basic supermultiplets of $N = 2$ supersymmetric field theories are most simply described by specifying their decomposition under a $N = 1$ subalgebra of the theory. We describe some useful massless supermultiplets.

- The hypermultiplet: It decomposes as two chiral multiplets (in conjugate representations of the gauge and global symmetries) of $N = 1$ supersymmetry. Hence, one hypermultiplet contains two complex scalars and two Weyl fermions. Notice that the latter have same chirality and conjugate quantum numbers, hence the supermultiplet is non-chiral. It is possible to write supersymmetric mass terms for hypermultiplets, hence the massive hypermultiplet has the same supermultiplet structure.

- The $N = 2$ vector multiplet: It decomposes as one $N = 1$ vector multiplet, and a chiral multiplet (in the adjoint representation). Hence, it contains a gauge boson, two Majorana fermions, and one complex scalar.

Let us describe the general action (up to two derivatives) for an $N = 2$

supersymmetric theory with hyper- and vector multiplets. For $N=2$, the action is fully determined by the gauge quantum numbers of the hypermultiplets. Let us denote V, Σ the $N=1$ vector and chiral multiplets in the $N=2$ vector multiplets of the gauge group G, and Φ_i, Φ_i' the two chiral multiplets in the i^{th} hypermultiplet, in the representation R_i. The $N=2$ action has the standard $N=1$ form, with a superpotential fully determined by gauge symmetry and supersymmetry

$$W(\Phi_i, \Phi_i', \Sigma) = \sum_{i,a} \Phi_i \Sigma_a (t_a^{R_i}) \Phi_i' \tag{89}$$

The $N=2$ supersymmetry implies additional non-renormalization theorems beyond those in the $N=1$ theory. For instance, in $N=1$ language the Kahler potential for the chiral multiplets splits in two pieces, $K(\Sigma, \Sigma^\dagger)$ and $K(\Phi, \Phi', \Phi^\dagger, \Phi'^\dagger)$. This implies that the kinetic terms for scalars in vector multiplets do not depend on scalars in hypermultiplets, and viceversa. This implies that the scalar field space (and hence the moduli space) factorizes as the vector multiplet scalar field space times the hypermultiplet scalar field space. Moreover, the former is a Kahler space, while the latter is even more constrained, and is hyperKahler [5].

5.3.2 $N=4$ supersymmetric theories

Let us now describe some facts on $N=4$ supersymmetric theories [6].

The smallest supermultiplet is the $N=4$ vector multiplet. Under an $N=2$ subalgebra, it contains one $N=2$ vector multiplet and one hypermultiplet in the adjoint representation. In terms of $N=1$, it contains a

[5] Namely, admits three Kahler forms, with their product obeying the rules of quaternionic product.

[6] The supermultiplet structure and low-energy effective action of $N=3$ is exactly as in $N=4$, so $N=3$ supersymmetry is not so interesting.

vector multiplet and three chiral multiplets in the adjoint representation. Finally, in component fields, it contains one gauge boson, four Majorana fermions, and six real scalars.

Other supermultiplets contain spin-2 particles, namely gravitons, and so appear only in supergravity theories. Their discussion is beyond the scope of this lecture.

The general action for an $\mathcal{N} = 4$ theory is extremely constrained. It has the structure of an $N = 2$ theory, but with the gauge representation of hypermultiplets fixed by the $N = 4$ supermultiplet structure. Using $N = 1$ language, we denote V, Φ_1, Φ_2, Φ_3 the vector and chiral multiplets of the $N = 4$ vector multiplet. The superpotential is given by

$$W(\Phi_i) = \mathrm{Tr}\, \Phi_1\Phi_2\Phi_3 - \mathrm{Tr}\, \Phi_1\Phi_3\Phi_2 \tag{90}$$

Again, the action is protected by even more powerful non-renomalization theorem. In particular, the Kahler potential for scalar fields are forced to be canonical, and the gauge kinetic functions are non-renormalized. This implies that $N = 4$ supersymmetric theories are finite (this in fact holds even non-perturbatively).

6 Supersymmetry in several dimensions

6.1 Some generalities

In this section we sketch the basic structure of supermultiplets in theories in more than four dimensions. The basic ideas are completely analogous to those discussed for four-dimensional supersymmetry. The main difference arises because of the larger number of components of higher-dimensional spinors, as compared with four-dimensional ones.

A detailed discussion of the construction of irreducible spinor representation of the Lorentz group in an arbitrary number of dimensions can be found in appendix B of [4]. For our present purposes, it will be enough to just mention that in an even number of dimensions, $D = 2n$, the representation of the Clifford algebra has dimension 2^n.

This spinor representation of $SO(2n-1,1)$ is reducible into two Weyl spinor representations, of opposite chiralities, and with 2^{n-1} components each. Also, for odd n, namely $D = 2k + 4$ it is possible to define Majorana spinors, which satisfy a reality condition, and thus have 2^{n-1} components. In general, Majorana and Weyl conditions are incompatible (namely, the conjugation operation flips the chirality, so Majorana spinors contain components with opposite chiralities). However, for $D = 2k + 8$, the conjugation operation does not flip the chirality, and one can define spinors satisfying both the Majorana and Weyl conditions, and thus have 2^{n-2} components.

The basic features of supersymmetric theories in different dimensions mainly depend only on the total number of supercharges. Indeed, any superalgebra in a given dimension can be regarded as a superalgebra of lower dimensional supersymmetry, simply obtained by decomposing the Lorentz representations of supergenerators with respect to the lower-dimensional Lorentz group. This is usually knows as dimensional reduction. Notice that since spinor representations in higher dimensions have larger number of components than in lower dimensions, the original superalgebra in general descends to an extended superalgebra in the lower dimension. Clearly, the same kind of relation follows for representations of the superalgebras. Namely, supermultiplets of the higher-dimensional supersymmetry can be recast as supermultiplets of the lower-dimensional one. An important point is that, since higher-dimensional superalgebras are related to extended superalgebras in 4d, there is no superfield formalism to describe the structure of higher-dimensional supermultiplets.

In each dimension, it is conventional to define $N = 1$ supersymmetry as that generated by supercharges in the smallest spinor representation. Hence, N-extended supersymmetry corresponds to that generated by N supercharges in the smallest spinor representation. Since the number of components of spinors jumps with dimension in a non-trivial way, it is sometimes more useful to refer to the theories by its total number of supercharges, although we will use the conventional N-extended susy notation as well.

6.2 Some useful superalgebras and supermultiplets in higher dimensions

In this section we provide some useful supermultiplets of certain superalgebras in six and ten dimensions. It is by no means a complete classification, but rather a list of some structures which will appear in the main text. A detailed classification of superalgebras and supermultiplets may be found in [2] and [3].

6.2.1 Minimal Supersymmetry in six dimensions

In six dimensions $D = 6$, the Weyl spinor has $2^3/2 = 4$ complex components, hence the minimal supersymmetry, denoted $N = 1$, is generated by 8 supercharges. Thus $D = 6$ N-extended supersymmetry is generated by $8N$ supercharges.

Let us center on the minimal supersymmetry, with 8 supercharges, denoted $N = 1$ (sometimes also $N = (1, 0)$ or $(0, 1)$ to indicate the left or right chirality of the chosen supergenerators; clearly, both such superalgebras are isomorphic). The R-symmetry of the theory is $SU(2)_R$. Let us describe some useful massless supermultiplets of this theory, providing their quantum numbers under the Lorentz (massless) little group $SO(4)_L = SU(2) \times SU(2)$ and the R-symmetry $SU(2)_L$.

Vector multiplet: It contains fields transforming under $SU(2) \times SU(2) \times SU(2)_R$ as

$$(2,2;1) \, + \, (1,2;2) \tag{91}$$

namely a massless vector boson and a chiral right-handed Weyl spinor.

Hypermultiplet: It contains fields transforming as

$$(2,1;1) \, + \, (1,1;2) \tag{92}$$

Unless it transforms in a pseudoreal representation of the gauge and global symmetries, it must be combined with its CPT conjugate to form a physical field. Then it contains two complex scalar fields, and a chiral left-handed Weyl spinor.

Tensor multiplet: It contains fields transforming as

$$(3,1;1) \, + \, (1,1;1) \, + \, (2,1;2) \tag{93}$$

namely a self-dual two-form, a real scalar fields and a chiral left-handed Weyl spinor.

Graviton multiplet: It contains fields transforming as

$$(3,3;1) \, + \, (1,3;1) \, + \, (2,3;2) \tag{94}$$

namely a massless graviton, an anti-selfdual 2-form, and two left-handed gravitinos.

This superalgebra can be dimensionally reduced to 4d $N = 2$ supersymmetry. It is a simple exercise to match the above 6d supermultiplets with supermultiplets of 4d $N = 2$ supersymmetry.

6.2.2 Extended supersymmetry in six dimensions

Let us discuss some features of $N = 2$ supersymmetry in six dimensions. The superalgebra is generated by 16 supercharges, organized in two Weyl spinors. There are two possible inequivalent superalgebras, depending on the relative chirality of these two spinors. Namely, there is a 6d $N = (2, 0)$ superalgebra, where both supergenerators have the same chirality, and a 6d $N = (1, 1)$ superalgebra, where they have opposite chiralities. Let us describe some of their massless multiplets in turn.

The $N = (2, 0)$ supersymmetry has a $USp(4) = SO(5)$ R-symmetry. Some interesting massless supermultiplets are

Tensor multiplet: It contains fields transforming under $SU(2) \times SU(2) \times SO(5)_R$ as

$$(3, 1; 1) \, + \, (1, 1; 5) \, + \, (2, 1; 4) \tag{95}$$

namely a self-dual two-form, five real scalar fields and two chiral left-handed Weyl spinors. Notice that it decomposes as a hyper- and a tensor multiplet with respect to the 6d $N = 1$ subalgebra.

Graviton multiplet: It contains fields transforming as

$$(3, 3; 1) \, + \, (1, 3; 5) \, + \, (2, 3; 4) \tag{96}$$

namely, a graviton, five anti-selfdual 2-forms and four left-handed gravitinos.

The $N = (1, 1)$ supersymmetry has a $SO(4) = SU(2) \times SU(2)$ R-symmetry. Some interesting massless supermultiplets are

Vector multiplet: It contains fields transforming under $SU(2) \times SU(2) \times [SU(2) \times SU(2)]_R$ as

$$(2, 2; 1, 1) \, + \, (1, 1; 2, 2) \, + \, (2, 1; 1, 2) \, + \, (1, 2; 2, 1) \tag{97}$$

namely a massless vector boson, two complex scalars, one chiral left- and one chiral right-handed Weyl spinors. Notice that it decomposes as a hyper- and a vector multiplet with respect to the 6d $N = 1$ subalgebra.

Graviton multiplet: It contains fields transforming as

$$\begin{aligned}(3,3;1,1) \,+\, (3,1;1,1) \,+\, (1,3;1,1) \,+\, (1,1;1,1) \,+\, (2,2;2,2)+ \\ +\,(3,2;1,2) \,+\, (2,3;2,1) \,+\, (1,2;1,2) \,+\, (2,1;2,1)\end{aligned} \tag{98}$$

namely, a graviton, a two-form, a real scalar, four vector bosons, two left- and two right-handed gravitinos, and one left- and one right-handed spinor.

6.2.3 Supersymmetry in ten dimensions

In ten dimensions $D = 10$, the minimal spinor satisfies the Majorana and Weyl constraints and has $2^5/4 = 8$ complex components, hence the minimal supersymmetry, denoted $N = 1$, is generated by 16 supercharges. Thus $D = 6$ N-extended supersymmetry is generated by $8N$ supercharges. Indeed, for $N > 2$ the smallest massless supermultiplet contains fields with spin higher than two; it is not known how to write interacting theories for such fields, hence they are not usually considered.

Let us center on the minimal $N = 1$ supersymmetry, with 16 supercharges. The R-symmetry of the theory is trivial. Some useful massless supermultiplets of this theory are

Vector multiplet, containing fields in the $8_V + 8_C$ of the $SO(8)$ Lorentz little group. Namely, a massless vector boson and a chiral 10d spinor.

Graviton multiplet, containing fields transforming under $SO(8)$ as

$$35_V \,+\, 28_V \,+\, 1 \,+\, 8_S \,+\, 56_S \tag{99}$$

namely, a graviton, a 2-form, a real scalar, a right-handed gravitino and a right-handed spinor.

Concerning extended supersymmetry, with 32 supercharges organized in two Majorana-Weyl spinors, there are two possibilities, according to their relative chirality. The 10d $N = (2,0)$ supersymmetry is generated by spinors of same chirality. The R-symmetry is $SO(2)_R$. The only relevant massless supermultiplet is the graviton multiplet, with fields transforming as

$$\begin{aligned} &\mathbf{35}_V + \mathbf{28}_V + \mathbf{1} + \mathbf{35}_C + \mathbf{28}_C + \mathbf{1} + \\ &\quad + 2 \times (\mathbf{8}_C + \mathbf{56}_C) \end{aligned} \tag{100}$$

Namely, one graviton, two 2-forms, two real scalars, one self-dual 4-form and two right-handed gravitinos and two right-handed spinors.

The 10d $N = (1,1)$ supersymmetry is generated by spinors of opposite chirality. The R-symmetry is trivial. The only relevant massless supermultiplet is the graviton multiplet, with fields transforming as

$$\begin{aligned} &\mathbf{35}_V + \mathbf{28}_V + \mathbf{1} + \mathbf{8}_V + \mathbf{56}_V + \\ &\quad + + \mathbf{8}_C + \mathbf{56}_C + \mathbf{8}_S + \mathbf{56}_S \end{aligned} \tag{101}$$

Namely, one graviton, one 2-form, one real scalar, one 1-form, one 3-form, one left- and one right-handed gravitino and one left- and one right-handed spinor.

Finally, for completeness we provide the basic massless supermultiplet of 11d $N = 1$ supersymmetry, the gravity multiplet. It contains states transforming as $\mathbf{44} + \mathbf{84} + \mathbf{128}$ under the $SO(9)$ Lorentz little group. Notice that it maps to the gravity multiplet of 10d $N = (1,1)$ supersymmetry upon dimensional reduction.

Notice that going to higher dimensions requires introducing more supercharges, which implies that even the smallest massless supermultiplet already contains fields with spin higher than 2, so these theories are usually not considered. This underlies the statement that eleven is the maximal number of dimensions allowed by supersymmetry (with the extra assumption of not having massless fields with spins higher than 2). The maximal amount of supersymmetry is thus 32 supercharges.

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