Rudiments of differential geometry/topology

Useful references for this lecture are [1] and sections 12, 14 and 15 of [2].

1 Differential manifolds; Homology and co-homology

1.1 Differential manifolds

An n-dimensional differential manifold M is a topological space, together with an atlas, that is a collection of charts $(U_\alpha, x_{(\alpha)})$ where U_α are open sets of M and $x_{(\alpha)}$ is a one to one map between U_α and an open set in $\mathbf{R^n}$, such that

i) M is covered by the U_α, that is $\bigcup_\alpha U_\alpha = M$.

ii) If $U_\alpha \bigcap U_\beta$ is non-empty, the map

$$x_{(\beta)} \circ x_{(\alpha)}^{-1} \, : \, x_{(\alpha)}(U_\alpha \bigcap U_\beta) \, \in \mathbf{R^n} \, \rightarrow \, \mathbf{x}_{(\beta)}(\mathbf{U}_\alpha \bigcap \mathbf{U}_\beta) \, \in \mathbf{R^n} \tag{1}$$

is differentiable.

Namely, the charts attach coordinates to the points in the U_α, such that on intersections $U\alpha \bigcap U_\beta$ the $x_{(\beta)}$ are smooth functions of the $x_{(\alpha)}$. This is illustrated in figure 1. Namely, a differential manifold is a space that at each point looks locally like $\mathbf{R^n}$ (with respect to differential structures).

By abuse of notation, we will often refer to a point $P \in M$ by its coordinates x (in some chart). Also, we will denote the map $x_{(\beta)} \circ x_{(\alpha)}^{-1}$ as $x_{(\beta)}(x_{(\alpha)})$.

We refer to any introductory book on differential geometry for examples of the description of familiar manifolds (like the n-sphere $\mathbf{S^n}$ or the n-torus $\mathbf{T^n}$ in the above language).

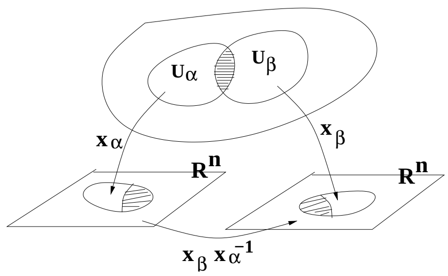

Figure 1: Charts in a differential manifold.

In this lecture we will center on orientable manifolds. An orientable manifold is such that the sign of the determinant of the jacobian matrix $J_i^j = \partial x^j_{(\beta)}/\partial x^i_{(\alpha)}$ is the same in all intersections $U_\alpha \bigcap U_\beta$.

In a differential manifold we can introduce the concept of a differentiable (or smooth) function. A function $f : M \to \mathbf{R}$ is differentiable if the functions

$$f \circ x^{-1}_{(\alpha)} : \, x_{(\alpha)}(U_\alpha) \, \in \mathbf{R^n} \, \to \, \mathbf{R} \tag{2}$$

are differentiable. And similarly for functions taking values in $\mathbf{R^n}$, $\mathbf{C}$, $\mathbf{C^n}$, etc.

We denote by $\mathcal{F}$ the set of smooth (real) functions over M. By abuse of language we often write $f(x)$ to denote $f \circ x^{-1}_{(\alpha)}$.

1.2 Tangent and cotangent space

A tangent vector to M at a point $P \in U_\alpha$ is a linear mapping from the set of smooth functions $\mathcal{F}$ to $\mathbf{R}$. A basis of tangent vectors is the set $\{\partial_i\}$, $i = 1, \ldots, n$, which act as

$$\partial_i \ : \ \mathcal{F} \to \mathbf{R}$$

$$f \longmapsto \left.\frac{\partial f}{\partial_{x^i_{(\alpha)}}}\right|_P \tag{3}$$

The tangent space to M at P, denoted $T_P(M)$, is the vector space generated by linear combinations of the ∂_i, acting as

$$\begin{aligned} V = V^i \partial_i \ &: \ \mathcal{F} \rightarrow \mathbf{R} \\ & \quad f \longmapsto V^i \left.\frac{\partial f}{\partial_{x^i_{(\alpha)}}}\right|_P \end{aligned} \tag{4}$$

A vector field is a set of tangent vectors, one per point of M, smoothly varying with P. Namely, a set of linear combinations with coefficients given by functions, defined on the U_α

$$V_{(\alpha)} = V^i_{(\alpha)}(x_{(\alpha)})\partial_i \tag{5}$$

with the conditions that they agree on intersections $U_\alpha \bigcap U_\beta$, namely

$$V^i_{(\alpha)}(x_{(\alpha)}) = \frac{\partial x_{(\alpha)^i}}{\partial x^j_{(\beta)}} V^j_{(\beta)}(x_{(\beta)}) \tag{6}$$

We will define analogously the concept of field for other vector spaces below. In section 2.1 we will see that they are simply sections of the corresponding fiber bundle.

The cotangent space $T_P(M)^*$ of M at P is the vector space dual to $T_P(M)$. Namely it is the vector space of linear mappings from $T_P(M)$ to $\mathbf{R}$. We can understand this better by introducing a basis for $T_P(M)^*$, which is given by the set $\{dx^i\}$, which act as

$$\begin{aligned} dx^i \ &: \ T_P(M) \rightarrow \mathbf{R} \\ & \quad \partial_j \longmapsto \delta^i_j \end{aligned} \tag{7}$$

A general linear combination $u = u_i dx^i$ is hence defined by

$$\begin{aligned} u \ &: \ T_P(M) \rightarrow \mathbf{R} \\ & \quad \partial_j \longmapsto u_j \end{aligned} \tag{8}$$

The element of $T_P(M)^*$ are also called 1-forms, see below.

A tensor of type (k,l) is a linear mapping from $(T_P(M)^*)^k \times T_P(M)^l$ to $\mathbf{R}$. It is the vector space of linear combinations

$$T \; = \; T^{i_1 \ldots i_k}_{j_1 \ldots j_l} \, dx^{j_1} \otimes \ldots \otimes dx^{j_l} \otimes \partial_{i_1} \otimes \ldots \otimes \partial_{i_k} \tag{9}$$

with the obvious definition of the elements of the basis.

A simple examples is given by the metric, which is a tensor field of type $(0,2)$, $g = g_{ij} dx^i \otimes dx^j$, or $g_{ij} = g(\partial_i, \partial_j)$.

1.3 Differential forms

A differential p-form is a tensor of type $(0,p)$, which has completely antisymmetric component (this statement is true in any coordinates). So they are of the form

$$A_{(p)} = A_{i_1 \ldots i_p} \, dx^{i_1} \otimes \ldots \otimes dx^{i_p} \tag{10}$$

with completely antisymmetric $A_{i_1 \ldots i_p}$.

Equivalently, it is the vector space of linear combinations of the basis elements

$$dx^{i_1} \wedge \ldots \wedge dx^{i_p} = \frac{1}{p!} \epsilon_{i_1 \ldots i_p} dx^{i_1} \otimes \ldots dx^{i_p} \tag{11}$$

(with $i_1 < \ldots < i_p$), namely

$$A_{(p)} = A_{i_1 \ldots i_p} \, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \tag{12}$$

The vector space of p-forms is denoted $\Lambda^p(M)$. We define p-form fields as usual, which will be denoted p-forms by abuse of language.

We define the wedge product of a p-form $A_{(p)}$ and a q-form $B_{(q)}$ to be the $(p+q)$-form

$$A_{(p)} \wedge B_{(q)} \; = \; \frac{1}{p!q!} A_{i_1 \ldots i_p} \, B_{j_1 \ldots j_q} \, dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q} \tag{13}$$

Notice the property $A_{(p)} \wedge B_q = (-1)^{pq} B_{(q)} \wedge A_{(p)}$. Often, wedge products are assumed and not explicitly displayed.

We define the exterior derivative d as a mapping from p-form fields to $(p+1)$-form fields. For a p-form (field) $A_{(p)}$ its exterior derivative $(dA)_{(p+1)}$ is defined by

$$dA \;=\; \partial_{i_0} A_{i_1 \ldots i_p} \, dx^{i_0} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \tag{14}$$

Notice the property

$$d(A_p \wedge B_{(q)}) = dA_{(p)} \wedge B_{(q)} + (-1)^p A_{(p)} \wedge dB_{(q)} \tag{15}$$

However, the main property of exterior differentiation for this lecture is

$$d^2 = 0 \tag{16}$$

in the sense that for any p-form $A_{(p)}$, $d(dA) = 0$. This follows easily from the symmetry of double partial derivation $\partial_i \partial_j = \partial_j \partial_i$.

We refer to introductory books on differential forms to check that d reproduces the familiar formulae for the gradient, divergence and curl of 3d vector calculus.

1.4 Cohomology

A p-form field $A_{(p)}$ is said to be closed if $dA = 0$. A p-form $A_{(p)}$ is said to be exact if there exists a $(p-1)$-form B_{p-1} (globally defined on M, see below) such that $A_p = dB_{(p-1)}$. Clearly, because $d^2 = 0$ every exact form is also a closed form.

$$A_{(p)} = dB_{(p-1)} \;\rightarrow\; dA = ddB = 0 \tag{17}$$

It is natural to ask to what extent the reverse is true. In general, it is not. There exist manifolds where there are closed forms which are not exact. We will see one example below.

However, there is one important case where the reverse is true, and every closed form is also exact:

Poincare lemma: In $\mathbf{R^n}$, any closed p-form, $p > 0$, is also exact.

(since there are no (-1)-forms, clearly 0-forms can never be exact). A simple example is provided by 1-forms in $\mathbf{R}$. Any 1-form $A = f(x)dx$ in $\mathbf{R}$ can be written as $A = dF$, where F is the 0-form (i.e. function)

$$F(x) = \int_0^x f(y)\,dy \tag{18}$$

This is very important, and can be exploited to define a topological invariant for any differentiable manifold M, the cohomology of M. The argument is as follows.

Recall that M is a bunch of open sets U_α isomorphic to $\mathbf{R^n}$, glued in some way (specified by the transition functions $x_{(\beta)}(x_{(\alpha)})$). A p-form (field) $A_{(p)}$ is a bunch of p-forms $A^\alpha_{(p)}$ defined on the U_α's, which agree on the intersections $U_\alpha \bigcap U_\beta$

$$A^\alpha_{i_1 \ldots i_p} = \frac{\partial x^{i_1}_{(\alpha)}}{\partial x_{(\beta)^{j_1}}} \cdots \frac{\partial x^{i_p}_{(\alpha)}}{\partial x_{(\beta)^{j_p}}} A^\beta_{j_1 \ldots j_p} \tag{19}$$

A closed p-form satisfies $dA = 0$ globally, hence $dA^\alpha = 0$ on every U_α. Since each U_α is essentially $\mathbf{R^n}$, Poincare ensures that there always exists some $(p-1)$-form B^α in U_α such that $A^\alpha = dB^\alpha$. However, there is no guarantee that the B^α glue in the right way at intersections to define a global $(p-1)$-form B satisfying $A = dB$ globally. If this is not the case then A is closed but not exact.

In this argument, the local structure of M is not relevant, only the global structure, defined by how the U_α patch together, is relevant. Therefore, the existence of closed forms which are not exact is an statement which depends only on the global topology of M, and not on its local properties.

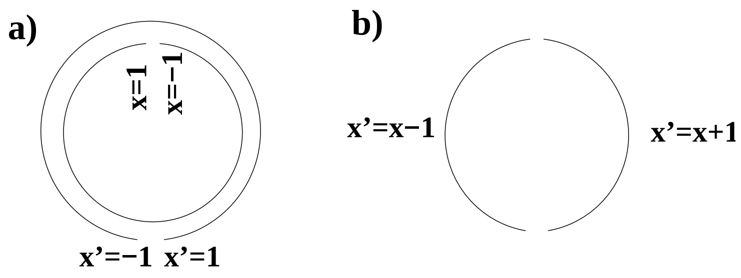

Figure 2: Covering the circle with two charts.

To give a simple example, consider the circle $\mathbf{S^1}$, described using two charts with local coordinates x, x', as shown in figure 2, running in $(-1, 1)$, each covering $\mathbf{S^1}$ except the norht and south poles repectively. The intersection is disjoint, and on its two disconneted pieces the transition functions are $x' = x + 1$ and x'=x-1. Let us construct a global 1-form A, by glueing together the 1-form dx on U and dx' on U'; note they glue nicely with the above transition functions. The global 1-form is closed, and on U and U' is is locally exact, it reduces to dx or dx'. However, it is not possible to patch together x and x' to form a 0-form f such that $A = df$ globally (this would be as much as finding a coordinate valid globally on $\mathbf{S^1}$, which is not possible). By a strong and misleading abuse of language, the global 1-form is often referred to as dx, although we know that x is not a global 0-form.

The natural object which can be defined from these observations, and which depends only on the global structre of M is the de Rahm cohomology groups. Let $\mathcal{Z}^p$ be the set of closed p-form on M

$$\mathcal{Z}^p = \{ A_{(p)} \, | dA_{(p)} = 0 \} \tag{20}$$

and $\mathcal{B}_p$ the set of exact p-forms on M

$$\mathcal{B}^p = \{ A_{(p)} \, | \, A_{(p)} = dB_{(p-1)} \text{ for some } B_{(p-1)} \} \tag{21}$$

Since $\mathcal{B}^p \subset \mathcal{Z}^p$, we can define the quotient

$$H^p(M, \mathbf{R}) = \frac{\mathcal{Z}^\mathbf{p}}{\mathcal{B}^\mathbf{p}} \tag{22}$$

known at p^{th} de Rahm cohomology group of M. It is the set of closed forms of M modulo the equivalence relation

$$A_{(p)} \simeq A_{(p)} + dB_{(p-1)} \tag{23}$$

Namely, two closed p-forms define the same equivalence class in cohomology if they differ by an exact form. Notice that exact forms are also closed, they correspond to the zero (or trivial) class in cohomology (the class corresponding to an identically vanishing form). We denote by $[A]$ the cohomology class of a closed form A.

The sets $H^p(M, \mathbf{R})$ have the structure of finite-dimensional vector spaces (so in particular they are groups with respect to addition). Their structure depens only on the topology of M. Their dimensions, denoted b_p and known as Bettin numbers of M, are the simplest topological invariants of manifolds.

1.5 Homology

We now aim at defining a related class of topological quantities. To define them we need some additional concepts.

An m-dimensional submanifold N of M $(m < n)$ is a subset of M which has the structure of an m-dimensional differential manifold. We will be interested in allowing for submanifolds with boundary, so we define the concept of boundary of a manifold.

A manifold M with boundary is a topological set together with an atlas with two kinds of charts: the familiar $(U_\alpha, x_{(\alpha)}$ and charts $(V_\beta, x_{(\beta)})$, where V_β is isomorphic to an open set in 'half' $\mathbf{R}^\mathbf{n}$. As before, the charts cover M,

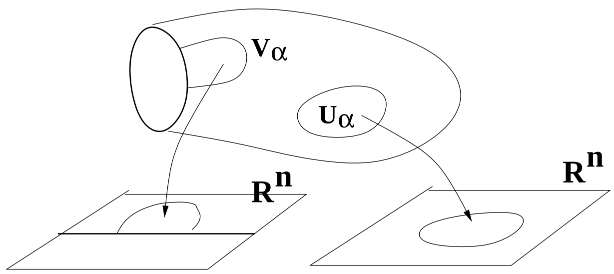

Figure 3: Manifolds with boundary are described by two kinds of charts.

and the $x_{(\alpha)}$, $x_{(\beta)}$ define differentiable transition functions. By 'half' $\mathbf{R^n}$ we mean the set of point

$$\mathbf{R^n_+} = \{(\mathbf{x^1}, \ldots, \mathbf{x^n}) | \mathbf{x^1} \geq \mathbf{0}\} \tag{24}$$

The boundary ∂M of M is the set of points which are anti-images of the points $x^1 = 0$ in the maps $x_{(\beta)}$. See figure 3. It is important, although we do not discuss it in detail, to notice that the orientation in a manifold induces a natural orientation on its boundary.

A p-chain $a_{(p)}$ is a formal linear combination (with real coefficients) of p-dimensional submanifolds N_k (possibly with boundary) of M, namely $a = c_k N_k$.

The operation of taking the boundary, which we call ∂, can be regarded as a linear operator mapping a p-chain to a $(p-1)$-chain, by

$$\partial a_{(p)} = c_k \, \partial N_k \tag{25}$$

An essential property of ∂, which is geometrically obvious is that

$$\partial^2 = 0 \tag{26}$$

In the sense that for any p-chain, $\partial(\partial a) = \emptyset$ is empty.

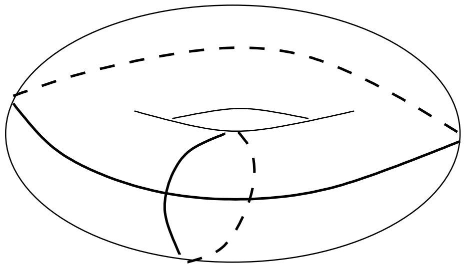

Figure 4: Non-trivial 1-cycles in a two-torus.

A p-chain $a_{(p)}$ without boundary is called a p-cycle, $\partial a_{(p)} = 0$. A p-chain is called trivial if it is the boundary of a $(p+1)$-chain, namely $a_{(p)} = \partial b_{(p+1)}$. Clearly, because $\partial^2 = 0$ any trivial p-chain is a p-cycle.

$$a_{(p)} = \partial b_{(b+1)} \rightarrow \partial a = \partial^2 b = 0 \tag{27}$$

It is natural to wonder to what extent the reverse is true. In general it is not: there exist manifolds M where there are p-cycles which are not trivial. An example of non-trivial 1-cycles is shown in figure 4.

However, there is an important n-dimensional manifold where any p-cycle $(p < n)$ is trivial [1]. This is the case for $\mathbf{R^n}$, see figure 5. Again, this implies that the existence of non-trivial p-cycles on a manifold M is determined by the global structure of M, how it is patched together. It is ia features insensitive to the local structure of M, since locally it looks like $\mathbf{R^n}$, where all p-cycles are trivial.

We are now ready to define the p^{th} homology group $H_p(M, \mathbf{R})$. Let $\mathcal{Z}_p$ be the set of p-cycles

$$\mathcal{Z}_p = \{a_{(p)} | \partial a_{(p)} = 0\} \tag{28}$$

[1]Since there are no $(n+1)$-cycles in an n-dimensional space, n-chains cannot be trivial.

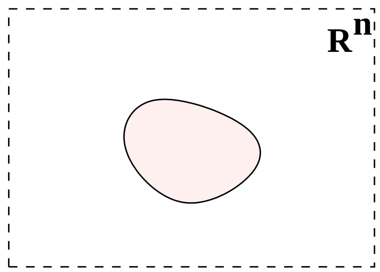

Figure 5: All cycles in $\mathbf{R^n}$ are boundaries of some higher dimensional chain.

and let $\mathcal{B}_p$ be the set of trivial p-chains

$$\mathcal{B}_p = \{a_{(p)} | a_{(p)} = \partial b_{(p+1)}\} \tag{29}$$

Since $\mathcal{B}_p \subset \mathcal{Z}_p$, we can define the quotient

$$H_p(M, \mathbf{R}) = \frac{\mathcal{Z}_\mathbf{p}}{\mathcal{B}_\mathbf{p}} \tag{30}$$

known as the p^{th} homology group of M. It is formed by the set of p-cycles modulo the equivalence

$$a_{(p)} = a_{(p)} + \partial b_{(b+1)} \tag{31}$$

namely two p-cycles define the same homology class if they differ by a boundary. Trivial p-cycles correspond to the zero class in homology. We denote by $[a]$ the homology class of a cycle a. The spaces $H_p(M, \mathbf{R})$ have the structure of vector spaces, and their structure depends only on the topology of M. The dimension of $H_p(M, \mathbf{R})$ will be seen to be equal to b_p, i.e. the dimension of $H^p(M, \mathbf{R})$.

Examples of non-trivial 1-homology classes on $\mathbf{T^2}$ are shown in figure 4. It is important to point out that homology is not the same as homotopy.

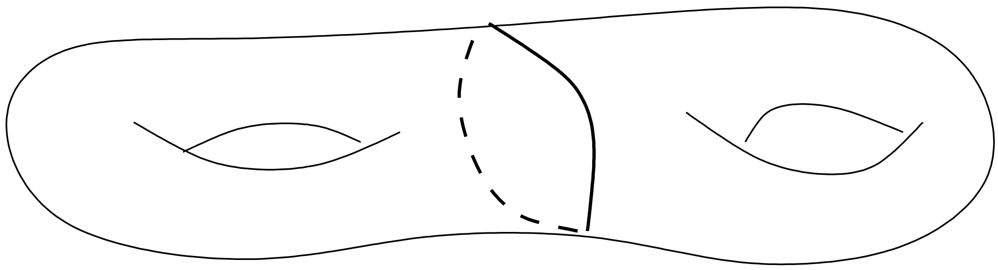

Figure 6: A homologically trivial 1-cycles which is not homotopically trivial.

In particular, homotopically trivial cycles (contractible cycles) are always homologically trivial (boundaries), but homologically trivial cycles may not be homotopically trivial. One example is shown in figure 6.

1.6 de Rahm duality

We can notice a close analogy between the construction of cohomology and homology groups, as follows

closed form	cycle
exact form	trivial chain
d	∂
$H^p(M > \mathbf{R})$	$H_p(M, \mathbf{R})$

Indeed this is not accidental. There is a duality between the vector spaces $H^p(M, \mathbf{R})$ and $H_p(M, \mathbf{R})$ which explains the analogies in their construction. The duality is obtained via the operation of integration of forms over chains.

We define the integral of a p-form $A_{(p)}$ over a p-dimensional submanifold N of M, by splitting A into pieces A^α in the U_α, and integrating the components of A over the U_α in the usual calculus sense

$$\int_N A_{(p)} = \sum_\alpha \int_{U_\alpha} A^\alpha_{i_1 \dots i_p} dx^1 \, \dots dx^n \tag{32}$$

In fact, we should define this more carefully so as to make sure that we do not overcount the points of M, because of overlapping of the patches U_α. Each point in M should count only once in the integral. This can be done by using partitions of unity (see e.g. [3], but we will not enter into this detail, hoping the idea is clear. Note that on the overlaps it does not matter which coordinates we use, since the integrand is invariant under coordinate transformations (the change of the form component is an inverse jacobien which cancels agains the change of the differential calculus measure).

One can now define the integral of a p-form $A_{(p)}$ over a p-chain $a_{(p)} = \sum_k c_k N_k$ by

$$\int_{a_{(p)}} A_{(p)} = \sum_k c_k \int_{N_k} A_{(p)} \tag{33}$$

An important property is Stokes theorem, which states that for any $(p-1)$-form $B_{(p-1)}$ and p-chain $a_{(p)}$,

$$\int_{a_{(p)}} dB_{(p-1)} = \int_{\partial a_{(p)}} B_{p-1} \tag{34}$$

A simple example is provided by 0-forms (functions) and the 1-chain $[0,1]$ (or other similar chains of closed sets in $\mathbf{R}$)

$$\int_{[0,1]} df \stackrel{\text{def}}{=} \int_{[0,1]} \frac{\partial f}{\partial x} dx = f(x)|_{x=0}^{x=1} = f(0) - f(1) = \int_{\partial[0,1]} f \tag{35}$$

(since the natural definition of an integral of a 0-form f over a 0d space (point) is simply evaluation of f at the point; the sign is due to opposite induced orientations).

Very interestingly, the integral of a closed p-form $A_{(p)}$ over a p-cycle $a_{(p)}$ depends only of their cohomology and homology classes, $[A]$ and $[a]$, respectively. Namely, the integral is unchanged if we take a different closed p-form $A'_{(p)}$ and a different p-cycle $a'_{(p)}$ in the same class $A'_(p) = A_{(p)} + dB_{(p-1)}$, $a'_{(p)} = a_{(p)} + \partial b_{(p+1)}$.

$$\begin{aligned} \int_a A' &= \int_a A + \int_a dB = \int_a A + \int_{\partial a} B = \int_a A \\ \int_{a'} A &= \int_a A + \int_{\partial b} A = \int_a A + \int_b dB = \int_a A \end{aligned} \qquad (36)$$

This is often called the period of $[A]$ over $[a]$.

This implies that integration is well defined for cohomology and homology classes, since it does not depend on the particular representatives chosen. Thus integration define a linear mapping $H^p(M, \mathbf{R}) \times \mathbf{H_p}(\mathbf{M}, \mathbf{R}) \to \mathbf{R}$. Equivalently, this shows that $H^p(M, \mathbf{R})$ is the vector space dual to $H_p(M, \mathbf{R})$, and vice versa. Namely, a p-cohomology class $[A_{(p)}]$ can be regarded as a linear mapping

$$\begin{aligned} [A_{(p)}] \,:\; H_p(M, \mathbf{R}) &\longrightarrow \mathbf{R} \\ [a_{(p)}] &\longmapsto \int_{a_{(p)}} A_{(p)} \end{aligned} \qquad (37)$$

This implies the promised result that the dimensions of the p^{th} cohomology and homology groups are the same.

Notice that the duality implies that it is always possible to choose basis of cycles $\{a_i\}$ and forms $\{A_j\}$ such that $\int_{[a_i]} [A_j] = \delta_{ij}$. An example in $\mathbf{T^2}$ is given by the 1-forms dx, dy on the two independent circles, and the nontrivial 1-cycles.

1.7 Hodge structures

Now consider that M is a Riemannian manifold, i.e. it is endowed with a metric g of euclidean signature. The previous structures are topological and independent of the metric (they were constructed without any metric at all). In the presence of a metric, we can define some additional structures which are important, but not topologically invariant.

We define the Hodge operation $*$ as the map between p-forms and $(n-p)$-forms defined by the action on the basis

$$*(dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \;=\; \frac{1}{(n-p)!}\sqrt{\det g}g^{i_1 j_1} \ldots g^{i_p j_p}\epsilon_{j_1 \ldots j_p j_{p+1} \ldots j_n}\, dx^{j_{p+1}} \wedge \ldots \wedge dx^{j_n} \quad (38)$$

It has the property that for a p-form $A_{(p)}$, $* * A_{(p)} = (-1)^{p(n-p)} A_{(p)}$.

The Hodge operator defines an positive-definite inner product betwen p-forms

$$(A_{(p)}, B_{(p)}) = \int_M A_{(p)} \wedge *B_{(p)} \quad (39)$$

Notice that this is not topological (however it is very important in physics, since it corresponds to

$$(A_{(p)}, B_{(p)}) = \int_M \sqrt{\det g} A_{i_1 \ldots i_p} B^{i_1 \ldots i_p} dx^1 \ldots dx^p \quad (40)$$

which is used to define the kinetic term of $(p-1)$-form gauge fields $C_{(p-1)}$ by taking $A_{(p)} = B_{(p)} = dC_{(p-1)}$ the gauge invariant field strength).

It is natural to define the adjoint $d^\dagger$ of d with respect to this inner product, i.e. it is defined by

$$(A_{(p)}, dB_{(p-1)}) = (d^\dagger A_{(p)}, B_{(p-1)}) \quad (41)$$

Hence $d^\dagger$ maps p-forms to $(p-1)$-forms. One can check that $d^\dagger = *d*$ for n even and $d^\dagger = (-1)^p * d*$ for n odd.

There is a theorem that ensures that any p-form $A_{(p)}$ has a *unique* decomponsition (known as Hodge decomposition) as

$$A_{(p)} = B_{(p)} + dC_{(p-1)} + d^\dagger D_{(p+1)} \quad (42)$$

with $B_{(p)}$ a harmonic form, namely obeys $dB_{(p)} = 0$, $d^+ B_{(p)} = 0$.

For closed p-forms, $dA_{(p)} = 0$ implies $dd^\dagger D_{(p+1)} = 0$. Taking the inner product with $D_{(p+1)}$,

$$(D_{(p+1)}, dd^\dagger D_{(p+1)}) = 0 \to (d^\dagger D_{(p+1)}, d^\dagger D_{(p+1)}) = 0 \tag{43}$$

the positive definiteness of the product implies $d^\dagger D_{(p+1)} = 0$. Then

$$A_{(p)} = B_{(p)} + dC_{(p-1)} \tag{44}$$

Thus in the cohomology class $[A]$ there is a unique harmonic p-form representative.

Namely, for each p-cohomology class, there exists a unique harmonic representative. Namely the p^{th} Betti number b_p is the number of independent harmonic p-forms on M. These are interesting statements: although the metric determines which particular p-form in the class is the harmonic one, the statement that there exists a unique one is independent of the metric. This is one simple example of a result which is topological invariant, but which is reached using additional non-topolocial structures, like a metric (there is no paradox, the result is independent of the metric chosen). Later on we will find more involved topological invariants which are easily defined using additional structures, althogh they are independent of the particular choices of these additional structures.

Harmonic p-forms will be quite useful in the study of KK compactification on curved spaces. Namely, the harmonic forms will provide the internal part of wavefunctions of the zero modes in the KK reduction of 10d p-form gauge fields. See lecture on Calabi-Yau compactification.

Another useful property due to Hodge operation is Poincare duality. The Hodge operator induces a homomorphism between $H^p(M, \mathbf{R})$ and $H^{n-p}(M, \mathbf{R})$. This can be seen by starting with a p-cohomology class, taking its harmonic

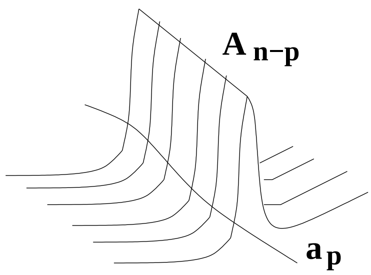

Figure 7: The Poincare dual form of a cycle can be though of as a delta function (bump form) with support on the cycle.

representative, taking its Hodge dual (which is also harmonic) and finally taking the corresponding $(n-p)$-cohomology class.

This implies in particular $b_p = b_{n-p}$. Again this is an statement which we reach by using a metric, but is a topological statement.

Another consequence is that for any p-homology class $[a_{(p)}]$ we can define the Poincare dual $(n-p)$-cohomology class $[A_{(n-p)}]$, such that for any p-form $B_{(p)}$

$$\int_{a_{(p)}} B_{(p)} = \int_M B_{(p)} \wedge A_{(n-p)} \tag{45}$$

Intuitively, $[A_{(n-p)}]$ can be considered as the class of a $(n-p)$-form 'delta function' with support on the volume of any p-cycle $a_{(p)}$ in the class $[a_{(p)}]$, see figure 7.

Finally for completeness we define the intersection numbers of a p-cycle and $a_{(p)}$ and an $(n-p)$-cycle $b_{(n-p)}$ to be

$$\#(a_{(p)}, b_{(n-p)}) = \int_M A_{(n-p)} \wedge B_{(p)} \tag{46}$$

where $A_{(n-p)}$, $B_{(p)}$ are the Poincare dual forms. Recalling the interpretation of Poincare dual forms as 'delta functions' localized on the corresponding cycles, the above number is an integer which counts the number of intersections (weighted with signs due to orientations) of the cycles $a_{(p)}$ and $b_{(n-p)}$.

2 Fiber bundles

Fiber bundles are a useful geometric concept in physics when studying fields that transform not only with respect to spacetime coordinate changes, but also have some particular behaviour under some internal gauge symmetries.

2.1 Fiber bundles

A vector bundle or fiber bundle E over a differential manifold M is a family of vector spaces V_P for each $P \in M$ (all isomorphic to an m-dimensional vector space V), which varies smoothly with P. V_P is called the fiber of E over the point P. The spaces M and V are referred to as the base and fiber of the bundle.

Equivalently, E can be defined with a set of charts $(U_\alpha \times V, (v_{(\alpha)}, x_{(\alpha)}))$, with $(U_\alpha, x_{(\alpha)})$ being charts describing M, and $v_{(\alpha)}$ being coordinates in V, such that on $U_\alpha \bigcap U_\beta$ coordinates on the base and fiber are related by

$$x_{(\beta)} \circ x_{(\alpha)})^{-1}$$

$$v_{(\beta)} = R_{(\alpha\beta)}(x_{(\alpha)}) \cdot v_\alpha \tag{47}$$

where $R_{(\alpha\beta)}$ are (point dependent) matrices in $GL(m, \mathbf{R})$, known as transition functions. Notice that coordinate indices in V are implicit here (α, β denote the patches).

Intuitively, a bundle is locally identical to $\mathbf{R^n} \times \mathbf{V}$, and different local patches are glued on the base, and on the fiber, up to a linear transformation on the fiber.

A bundle E is therefore specified by the set of patches $U_\alpha \times V$ and the transition functions for the base and fiber, the latter satisfying the consistency condition $R_{(\alpha\gamma)} R_{(\gamma\beta)} R_{(\beta\alpha)} = 1$.

The total bundle E has a natural projection π to the base M given by the map defined by 'forgetting the fiber'

$$\begin{aligned} \pi \quad : \quad & E \longrightarrow M \\ & (P, v) \longmapsto P \end{aligned} \tag{48}$$

The simplest example of bundle is a trivial bundle, which is simply a space of the form $M \times V$. All transition functions $R = 1$ in this case.

A less trivial example is given by a Moebius strip. Consider $M = \mathbf{S^1}$, and $V = \mathbf{R}$. To build the bundle, cover $\mathbf{S^1}$ with two patches U, U' with coordinates x, x', as in section 1.4 and put coordinates y, y' on $\mathbf{R}$, and use the glueing conditions

$$x' = x + 1 \; y' = y \quad \text{and} \quad x' = x - 1 \; y' = -y \tag{49}$$

on the two disconnected pieces of $U \bigcap U'$. The result is a non-trivial bundle. This is schematically shown in figure 8.

A richer example is provided by the bundle formed by all tangent spaces $T_P(M)$ to a manifold M. The base is M, the fiber over $P \in M$ is $T_P(M)$, and the transition functions on the fiber on $U_\alpha \bigcap U_\beta$ are

$$v^i_{(\beta)} = \frac{\partial x^i_{(\beta)}}{\partial x^j_{(\alpha)}} v^j_{(\alpha)} \tag{50}$$

Similarly one can define the cotangent bundle, the tensor bundles, the p-form bundle, etc...

A section σ of a bundle E is a mapping, such that $\pi \circ \sigma = 1$, i.e. of the form

$$\sigma \quad : \quad M \longrightarrow E$$

Figure 8: Construction of the Moebius strip as a nontrivial bundle with fiber $\mathbf{R}$ over $\mathbf{S^1}$.

$$P \longmapsto (P, \sigma(P)) \tag{51}$$

That is for each point $P \in M$ we pick a point (vector) in V_P.

A simple example is a vector field, which is a section of the tangent bundle: $V^i(x)\partial_i$ defines a tangent vector for each point x on M. Similarly the cotangent vector fields, tensor fields, p-form fields,... are sections of the corresponding bundles.

2.2 Principal bundles, associated bundles

It is useful to extend the notion of vector bundle to other possible fibers with some structure.

A principal G-bundle is a bundle where the fiber is a group G [2]. Namely, on the overlaps of the patches of the base $U_\alpha \bigcap U_\beta$, the fibers (which are isomorphic to G) are glued up to an (point dependent) tranformation in G. The elements of the fiber G in U_α and U_β, denoted $g_{(\alpha)}$ and $g_{(\beta)}$ are related by

$$g_{(\beta)} = f_{\alpha\beta}(x_{(\alpha)}) g_{(\alpha)} f_{\alpha\beta}(x_{(\alpha)})^{-1} \tag{52}$$

[2] We will center on compact Lie groups in this lecture.

This kind of bundle underlies the geometric description of gauge theories. For instance, a gauge transformation is nothing but a section of a principal G-bundle: $g(x)$ a group element for each point of M.

When we have a group G, we can consider its representations R and the representation vector spaces on which the group acts. Given a principal G-bundle we can define the associated fiber bundles, which are vector bundles with the fiber the represention space of a representation R of G, and transition functions on the fiber

$$v_{(\beta)} = R(f_{(\alpha\beta)}) \cdot v_{(\alpha)} \tag{53}$$

In a gauge theory, fields in a representation R of the gauge group are sections of the corresponding associated bundle. The fact that the transition funcions for different associated bundles are simply different representations of the same transitions function of the principal G-bundle reflects the fact that the gauge group is unique, and we only have different fields charged differently under it. With the above definitions, all the gauge transformation properties of fields charged under a gauge group are recovered.

Notice that a general vector bundle can be regarded as the associated bundle of a principal $GL(m, \mathbf{R})$-bundle (corresponding to the vector representation of $GL(m, \mathbf{R})$. (since the transition funcions are matrices, which represent the action of the group $GL(m, \mathbf{R})$ on vectors of V).

3 Connections

In physics, vector bundles usually come equipped with an additional structure, a connection. The main idea is that in a vector bundle there is in principle no canonical way to compare two basis of the fiber at different points. A connection is an additional structure which allows to do so.

In a bundle with connection, in a patch where the point P has coordinates x^i, the canonical change of a basis $\{e^a\}$ of V_P as P changes in the direction i is given by

$$D_i e^a(x) = \partial_i e^a(x) + \omega_{i\,b}^{\,a}(x) e^b(x) \tag{54}$$

where ω is the connection. On overlaps $U_\alpha \bigcap U_\beta$ the connection transforms not just as a 1-form, but has the additional transformation

$$\omega_{i,(\beta)} = R_{(\alpha\beta)} \omega_i R^{-1}_{(\alpha\beta)} - (\partial_i R_{(\alpha\beta)}) R^{-1}_{(\alpha\beta)} \tag{55}$$

which ensures that for a section σ of E, its covariant derivative $D_i\sigma(x)$ transforms as a section of E as well.

There are two classes of physical theories where fiber bundles with connections appear. The first is the case of gauge theories, where charged fields are sections of bundles associated to a principal G-bundle, and carry connections given by the representation of the connection of the principal G-bundle. For a representation R of G, the associated bundle has connection $\omega_i^{\,a}{}_b = A_i^m (T_R)^a_b$, where A is the connection on the principal G-bundle, m runs over the generators of the Lie algebra, and T_R is the representation of a generator in the representation R.

The second situation is in theories of gravity. The introduction of a metric g in a manifold M can be described in terms of fiber bundles as follows. At each point $x \in M$ introduce a set of tangent vectors $\{e^a(x)\}$, orthonormal with respect to the metric g

$$g_{ij} e^{a,i} e^{b,j} = \delta^{ab} \tag{56}$$

which also implies $e_i^a e_{a,j} = g_{ij}$. All the information of the metric is encode in the 'tetrad' $\{e^a\}$.

The tetrad is however defined up to $SO(N)$ rotations at each point, so this behaves as a local gauge invariance of the system. Indeed, such local rotations are sections of a principal $SO(N)$-bundle, and the tangent bundle is an associated bundle (for the vector representation).

Clearly one can construct other associated bundles; one of the most interesting ones is the spinor bundle, whose associated connection (see below) is known as the spin connection.

The metric induces a preferred connection on the tangent bundle, namely the Christoffel connection on vectors. One can then obtain a connection in terms of the tetrad, from the condition

$$D_i e_j^a = \partial_i e_j^a - \Gamma_{ij}^k e_k^a + \omega_i{}^a{}_b e_j^b = 0 \tag{57}$$

which defines a connection in the principal $SO(N)$-bundle. The latter then defines connections in all associated bundles, like the spinor bundle. In fact the tetrad formalism was originally deviced to be able to define parallel transport of spinors.

Given a general connection on a fiber bundle, we define its curvature by

$$R_{ij}^a{}_b = \partial_i \omega_j{}^a{}_b - -\partial_j \omega_i{}^a{}_b + [\omega_i, \omega_j]^a_b \tag{58}$$

they behave as 2-form with respect to coordinate reparametrizations, and transform covariantly under gauge transformations.

In gauge theories, the curvature of the connection of the principal bundle are denoted $F_{ij} = F_{ij}^m t^m$, where t^m are the Lie algebra generators. In a vector bundle associated to a representation R, it is given by $F_{ij}^a{}_b = F_{ij}^m (T_R^m)^a_b$.

3.1 Holonomy of a connection

We start with a vector bundle E (with fiber V over a base manifold M) with connection. Consider a point $P \in M$, and consider the set of closed loops which start and end at P. It is a group under the operation of adjoining

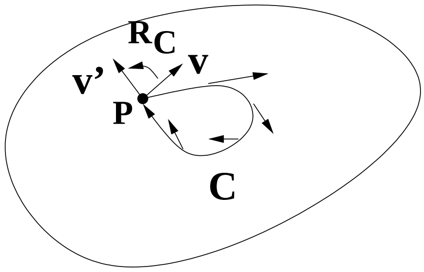

Figure 9: The holonomy group is given by the set of rotation R_C suffered by a vector under parallel transport around all possible closed loops in the manifold.

loops. Consider a vector v in the fiber V_P and parallel transport it along a loop C with the connection. It will come back to a vector v' in V_P, related to the original v by some $GL(m, \mathbf{R})$ rotation R_C. The set of such rotations for all closed loops is a group, known as the holonomy group of the connection. See figure 9.

For a connection induced from a metric, the holonomy of the connection is often referred to as the holonomy of the metric or of the Riemannian manifold.

For a metric connection, the norm of the tangent vector is preserved during parallel transport, hence the holonomy of the connection is necessarily a subgroup of $SO(n)$. For a principal G-bundle, and its associated bundles, like in gauge theories, the holonomy of a connection is necessarily a subgroup of G.

3.2 Characteristic classes

Our motivation is to construct topological quantities for fiber bundles, that characterize non-trivial bundles. In this section we see that there are certain quantities, which are computed using additional structures, like metrics or connections, but which at the end turn out to be independent of the particular metric or connection chosen. They are therefore topological. Before constructing them, it will be useful to give a simple example of a non-trivial fiber bundle.

The Wu-Yang magnetic monopole

Consider a $U(1)$ gauge theory on $M = \mathbf{S^2}$. The underlying geometry is a principal $U(1)$-bundle over the base $\mathbf{S^2}$. Let us classify all topologically inequivalent non-trivial gauge bundles. To do so, we cover $\mathbf{S^2}$ with two open sets, U_+ and U_-, which cover the North and South hemispheres, see figure 10. The bundle over each patch is trivial, so all the information about the bundle over $\mathbf{S^2}$ is encoded in the transition function in $U_+ \bigcap U_-$, which is an $\mathbf{S^1}$, the equator. For a principal $U(1)$-bundle, the transition function $g(\phi)$ takes values on $U(1)$ which is also a circle. Therefore the topologically different bundles are classified by topologically different maps from the equator $\mathbf{S^1}$ to the fiber $\mathbf{S^1}$. Such topologically different maps are classified by the homotopy group $\Pi_1(\mathbf{S^1}) = \mathbf{Z}$. Namely, there exist inequivalent classes of maps labeled by an integer. Simple representatives of these maps are

$$\begin{aligned} g_n \quad : \quad & \mathbf{S^1} \longrightarrow \mathbf{S^1} \\ & e^{i\phi} \longmapsto e^{in\phi} \end{aligned} \tag{59}$$

Namely, the label n corresponds to how many time one goes around the target $\mathbf{S^1}$ when going once around the origin $\mathbf{S^1}$.

This example is simple enough to be more explicit about the connections we can put on these bundles (that is, the gauge field configurations corresponding to these bundles). Here we describe a simple case.

Consider polar coordinates θ, ϕ, and introduce the $U(1)$ gauge potentials

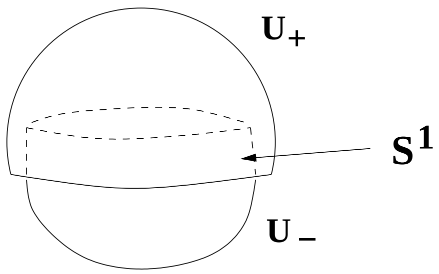

Figure 10: .

on $U_\pm$

$$A_\pm = \frac{1}{2}\frac{\pm 1 - \cos\theta}{\sin\theta}\, d\phi \tag{60}$$

On the intersection of $U_\pm$, namely at $\theta = \pi/2$ they differ by a gauge transformation

$$A_+ - A_- = d\phi \tag{61}$$

so they define a global connection for the bundle. The curvatures on $U_\pm$ agree on the intersection $F_+ = F_-$.

The above expression shows that the transition function for this bundle is the map

$$\begin{aligned} g_1 \quad : \quad & \mathbf{S^1} \longrightarrow \mathbf{S^1} \\ & e^{i\phi} \longmapsto e^{i\phi} \end{aligned} \tag{62}$$

So the bundle is non-trivial.

There is a nice general relation between the winding of the map $g(\phi)$ and the flux of F on $\mathbf{S^2}$. This provides a way of characterizing non-trivial bundles which we will generalize in next section. In a bundle defined by the transition function g_n, the gauge potentials $A_\pm$ satisfy $A_+ - A_- = nd\phi$ on the equator. Hence we have

$$\int_{\mathbf{S^2}} F = \int_{U_+} F_+ + \int_{U_-} F_- = \int_{U_+} dA_+ + \int_{U_-} dA_- = \int_{\mathbf{S^1}} A_+ - \int_{\mathbf{S^1}} A_- = \int_{\mathbf{S^1}} n\, d\phi = 2\pi n \quad (63)$$

This example is familiar in the study of magnetic monopoles: When the $\mathbf{S^2}$ is taken to describe the angular part of 3d space, the gauge configuration describes a magnetic monopole sitting at the origin or $\mathbf{R^3}$.

Since F is closed and its integral over $\mathbf{S^2}$ does not vanish, it defines a non-trivial cohomology class. Indeed, $\frac{F}{2\pi}$ defines an integer cohomoly class $[F/2\pi]$, which characterizes the bundle. Notice that although we used a connection to define this quantity, it finally depends only on the transition functions, and therefore is a topological invariant of the bundle. It is known as first Chern class of the bundle.

Another simple example of non-trivial bundle is obtained by considering a $U(1)$ gauge field configuration on $\mathbf{T^2}$, with a constant magnetic field; abusing of language, this can be described by a gauge potential $A = kxdy$.

A final example, familiar from nonabelian 4d gauge theories, is the classification of topological sectors of gauge configurations by the value of

$$k = \frac{1}{8\pi^2} \int_{4d} \mathrm{tr}\, F \wedge F \quad (64)$$

known as the instanton number of the configuration.

All these topological invariants are simple examples of characteristic classes. Let us generalize the $U(1)$ case for a general manifold M. To do that, on each U_α we introduce the local form of the connection A_α, such that on overlaps $U_\alpha \bigcap U_\beta$ we have

$$A_\beta = A_\alpha + d\phi_{(\alpha\beta)} \quad (65)$$

Then $F = dA_\alpha$ is globally defined, and satisfies $dF = 0$, hence defines a cohomology class $[F]$. We know show that his class is a topological invariant of the bundle. Namely, although to define it we have used a connection, the final class depends only on the transition functions of the bundle $\phi_{(\alpha\beta)}$, and is independent of the particular connection chosen.

To show that, consider a different connection defined by A'_α, still with the same transition functions

$$A'_\beta = A'_\alpha + d\phi_{(\alpha\beta)} \tag{66}$$

From (65) and (66), it follows that $A_\alpha - A'_\alpha = A_\beta - A'_\beta$ so the differences are patch independent and define a global 1-form B. Then $F - F' = dB$ globally, which implies that they define the same cohomology class $[F]$, as we wanted to show.

More sophisticated tools can be used to show that $[F/2\pi]$ is in fact an integer cohomology class, known as first Chern class of the $U(1)$ bundle.

The generalization to principal G-bundles with arbitrary group is analogous. One simply constructs polynomials in wedge products of the curvatures of the connection, tracing in the Lie algebra indices. The resulting form is closed and the corresponding cohomology class is a topological invariant of the bundle. These are known as characteristic classes.

We now give some examples appearing often for $SU(N)$ and $SO(N)$. Consider the closed $2k$-form

$$\Omega_{2k} = \sum_{m_1,\ldots,m_k} (F^{m_1} \wedge \ldots \wedge F^{m_k}) \mathrm{Str}(t^{m_1} \ldots t^{m_k}) \tag{67}$$

where Str denoted the symmetrized trace of the generators. This is usually written $\Omega_{2k} = \mathrm{tr}\, F^k$ (with wedge products implied). The corresponding cohomology class is a topological invariant of the corresponding bundle. For $U(N)$ it is known as the k^{th} Chern class, and has the generating function

$$ch(E) = \mathrm{tr}\,(e^{F/2\pi}) \tag{68}$$

known as the Chern character. For $SO(2N)$, Ω_{2k} automatically vanishes unless k is even $k = 2r$; the cohomology class is in this case known as r^{th} Pontryagin class. The Pontryagin classes also appear often in a generating function

$$\hat{A} = 1 + \frac{1}{8\pi^2}\mathrm{tr}\, R^2 + \dots \tag{69}$$

known as A-roof genus.

Characteristic classes are very useful in characterizing the topology of nontrivial bundles [3]. Clearly much more can be said about bundles and their characterization. However, this will be enough for our purposes and applications.

References

[1] P. Candelas, 'Lectures on complex manifolds', Trieste, proceedings Superstrings 87.

[2] M. B. Green, J.H. Schwarz, E. Witten. 'Superstring theory, vol 2: Loop amplitudes, anomalies and phenomenology', Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics.

[3] Bott and Tu, 'Differential forms in algebraci topology', Springer-Verlag.

[3] Although this characterization is not complete, different gauge bundles may still have all characteristic classes equal, and differ in some additional topological quantities. We may see some of this in the discussion of K-theory when discussing stable non-BPS branes.