Quantum Simulation
of
Geometry

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NATURE

MODEL

Exact Computation

Classical Simulation

Quantum Computer

NATURE
QM MODEL ≈ Quantum Simulation QM'
Quantum Simulation as an intelligent window to QC

Quantum Computer
- General purpose quantum computation
- Shor’s factorization algorithm
- Oracle problems, NAND trees, …

Quantum Simulator
- Efficient analysis of specific quantum problems
- Explore physics on unphysical regions
- Simulate unsolved problems
- Invent!!!

Classical Computer
- Tensor Networks strategies: PEPS, MERA
- Monte Carlo
TENSOR NETWORKS

Contraction of PEPS in d=3
Tensor Networks

Most efficient representation of entanglement on a classical computer

Represent large series of numbers as products of few numbers

\[ 6, 9, 14, 21, 24, 36, 56, 84 = \left( \frac{2}{3} \right) \times \left( \frac{1}{4} \right) \times \left( \frac{3}{7} \right) \]

\[ 2^n \]  Represented as \( n \) products of \( 2n \) numbers

Potential exponential compression
\[ |\Psi\rangle = \sum_{i_1=1}^{d} \ldots \sum_{i_n=1}^{d} c_{i_1\ldots i_n} |i_1\ldots i_n\rangle \]

Matrix Product State representation (MPS)

\[ c_{i_1\ldots i_n} = \sum_{\alpha_1\ldots \alpha_{n-1}} A^{[1]}_{\alpha_1 i_1} A^{[2]}_{\alpha_1 \alpha_2 i_2} A^{[3]}_{\alpha_2 \alpha_3 i_3} \ldots A^{[n]}_{\alpha_{n-1} i_n} \]

\# parameters \( \approx nd\chi^2 \ll d^n \)

\[ |\psi\rangle \]

\( \alpha = 1, \ldots, \chi \)

\( i = 1, \ldots, d \)
Efficient computation of scalar products

Efficient minimization and evolution

+ recast into MPS structure
Entanglement support of Matrix Product States

\[ S = -\sum_{\alpha=1}^{\chi} \lambda_\alpha \log \lambda_\alpha \]
\[ \lambda_\alpha = ct = \frac{1}{\chi} \]
\[ S \leq S_{MPS,\text{max}} = \log \chi \]

For CFT
\[ S_{MPS} = \frac{c}{6} \log \chi^\kappa \]

\[ \kappa = \frac{6}{c\sqrt{\frac{12}{c}} + 1} \]

Tagliacozzo, de Oliveira, Iblisdir, JIL Pollmann, Mukerjee, Turner, Moore
Is this enough for d=1 critical systems?

\[ S_{\text{critical}} = \frac{c}{3} \log L \quad S_{\text{MPS}} = \frac{c}{6} \log \chi^\kappa \]

\[ S_{\text{non-critical}} = \frac{c}{6} \log \xi \]

\[ \chi \approx \text{poly}(L) \]

Hence, success of DMRG

Srednicki
Callan, Wilczek
Vidal, Rico, Kitaev, JIL
Calabrese, Cardy
An alternative Tensor Network: MERA (Vidal)

Wilsonian Renormalization Group

\[ H_1 \otimes H_2 \rightarrow H_{eff} \]

\[ H_1 \otimes H_2 \text{ exact} \]

Preprocess with a disentangler! (Vidal)
All entanglement on one line

All entanglement distributed on scales
For \( d=2 \) **PEPS** are natural tensor networks (Verstraete, Cirac)

Entropy is proportional to the boundary

Contour \( A = L \)

\[
S_A = S_B \approx \kappa L
\]

Finite size PEPS support “Area law”

The computational problem shows up when performing contractions
Bad news: **Contraction of PEPS is \#P**

An exact Tensor Network can be constructed that describes the solution of a 3-SAT problem

The contraction of the network counts the solution of 3-SAT

\[ Q_{i,\alpha} = 0,1 \quad \forall i \]

\[ C_{\alpha\beta\gamma} = 1 \quad \text{if} \quad \text{clause}_{\alpha\beta\gamma} \ \text{ok} \]

\[ |\Psi\rangle = Tr \left( C^1 \ldots C^l Q_1 \ldots Q_l \right) \]

\[ \langle \Psi | \Psi \rangle = \# \text{ solutions} \]

**Contraction complexity = \#P**

García-Sáez, JIL
Good news: **Approximate contractions**

**RG contraction in d=3**

- Exact RG contraction
- Schmidt decompositions
- Isometries truncation

**Graphs**

- Graph 1: Circle with labels $E^{(m,n)}$ and $E^{(m+1,n)}$.
- Graph 2: Triangle with labels $\chi^2$ and $\chi^4$.

**Equation**

$$h_c = 5.29 \quad (h_c = 5.14)$$

A. García-Sáez, JIL
Conclusion of part I

Taming entanglement with Tensor Networks

MPS, MERA, PEPS, Trees, Random Networks,....
D=2,3 minimization and contraction schemes

Building up libraries

Non-trivial problems ahead
QUANTUM SIMULATION

Geometry & Dimensions
The many faces of Quantum Simulation

How?

• Q simulation via **exact** quantum circuits
• Q simulation via **analogue** systems
• Q simulation via **adiabatic** theorem

What?

• Exact Q simulation of relevant dynamics
• Exact Q simulation of relevant states
• Q simulation of models beyond classical simulation (better than TNS/MC)
• Q simulation of critical systems
• Q simulation of gauge theories, geometry, …
• Q simulation of exotic effects

Where?

• Ion traps
• Cold gases
• Molecules, solids, graphene, …
Exact Quantum Simulation

Problem: Exact Q Simulation of Quantum Ising model

$$|\psi\rangle_{free} \xrightarrow{U} |\psi_{GS}\rangle_{ISING}$$

$$|\psi_{GS}\rangle_{ISING} = U |\psi\rangle_{free}$$

Frank Verstraete, Ignacio Cirac, JIL
Can we find the exact disentangler of the Q Ising Hamiltonian?

\[ H_{\text{ISING}} = U H_{\text{free}} U^+ \]

\[ H_{\text{free}} = \sum_i \varepsilon_i \sigma_i^z \]

If so,

• We can prepare any state in a lab

• We can produce time evolution without time

\[ e^{-itH_{\text{ISING}}} = U e^{-itH_{\text{free}}} U^+ \]

• We can produce finite temperature at zero temperature

Thermal states

\[ e^{-\beta H_{\text{ISING}}} = U e^{-\beta H_{\text{free}}} U^+ \]
Quantum circuit for 4-qubit

Bogoliubov

Fast Fourier transform

\[
F\alpha = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \sqrt{2} & 0 \\
0 & 1 & \sqrt{2} & 0 \\
0 & 0 & 0 & -e^{-i\alpha}
\end{pmatrix}
\]

\[
U(B) = \begin{pmatrix}
\cos(\vartheta(\lambda)) & 0 & 0 & i\sin(\vartheta(\lambda)) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i\sin(\vartheta(\lambda)) & 0 & 0 & \cos(\vartheta(\lambda))
\end{pmatrix}
\]

\[
\vartheta(\lambda) = \text{ArcTan}\left(\lambda - \sqrt{1 + \lambda^2}\right)
\]
Quantum Simulation of gravitational backgrounds

Carriers in graphene are described by the Dirac equation

Graphene acts a quantum simulator for Dirac fermions

\[
\left( \gamma^0 \partial_t + \gamma^x \partial_x + \gamma^y \partial_y \right) \psi = 0
\]

\[
\left\{ \gamma^a, \gamma^b \right\} = 2\eta^{ab}
\]

Can we obtain the Dirac equation on optical lattices?

Can we simulate curved spaces?
Dirac Hamiltonian in 2+1 dimensions

\[ i \partial_t \psi = H \psi = -i \gamma_0 (\gamma_1 \partial_x + \gamma_2 \partial_y) \psi = 0 \]

\[ \gamma_0 \gamma_1 = -\gamma_2 = \sigma_x \quad \gamma_0 \gamma_2 = \gamma_1 = \sigma_y \quad \gamma_1 \gamma_2 = \gamma_0 = i \sigma_z \]

\[ H \psi = -i (\sigma_x \partial_x + \sigma_y \partial_y) \psi = 0 \]

\[ H = \int dx dy \; \psi^+ H \psi \]

Discretized Dirac Hamiltonian

\[ \partial_i \psi \rightarrow \frac{\psi(i + a) - \psi(i)}{a} \]

\[ H = \frac{1}{2a} \sum_{m,n} \left( \psi^+_{m+1,n} \gamma(i \sigma_x) \psi_{m,n} + \psi^+_{m,n+1} \gamma(i \sigma_y) \psi_{m,n} \right) + h.c. \]

SU(2) Fermi Hubbard model
Dirac equation in curved space-time

\[ D_\mu = \partial_\mu + \frac{1}{2} \omega^a_{\mu} \gamma_{ab} \]

\[ \gamma^\mu D_\mu \psi = 0 \]

If there exists a timelike Killing vector (time translation invariance in certain coordinates)

there exists \( H \) conserved and well defined

Sufficient condition \( \partial_t g_{\mu\nu} = 0 \)

\[ H = -i \gamma_t \left( \gamma^i \partial_i + \frac{1}{4} \gamma^i \omega^a_{\mu} \gamma_{ab} + \frac{1}{4} \gamma^i \omega^b_{\mu} \gamma_{ab} \right) \]

\[ H = \int \sqrt{-g} dx dy \ \psi^* \gamma_0 \gamma^d H \psi \]
Rindler space-time

\[ ds^2 = -(Cx)^2 dt^2 + dx^2 + dy^2 \]

\[ e^0 = |Cx| dt \quad e^1 = dx \quad e^2 = dy \]

Steady Rindler observer is an accelerated Minkowsky observer

acceleration \( \frac{1}{Cx} \) \hspace{1cm} \text{Unruh effect} \hspace{1cm} \text{temperature} \hspace{1cm} \frac{1}{Cx}

Rindler is the near horizon limit of Schwarzschild black hole
For any metric

\[ ds^2 = -e^{\Phi(x,y)} dt^2 + dx^2 + dy^2 \]

The lattice version turns out to be

\[
H = \frac{i}{2a} \sum_{m,n} J_{mn} \left( \psi_{m+1,n}^+ \sigma_x \psi_{m,n} + \psi_{m,n+1}^+ \sigma_y \psi_{m,n} \right) + h.c.
\]

\[ J_{mn} = e^{\Phi(am,an)} \]

Site dependent couplings!
Discretized Dirac equation in a Rindler space

\[ H = \frac{i}{2a} \sum_{m,n} cm \left( \psi^+_{m+1,n} \sigma_x \psi_{m,n} + \psi^+_{m,n+1} \sigma_y \psi_{m,n} \right) + h.c. \]

**Geometry** = Position dependent energy principle

Experimental options

- superlattice techniques
- laser waist
Quantum Simulation of Extra dimensions

Dimensionality = Connectivity

D dimensions can be simulated in D-1 dimensions by tuning appropriately the nearest neighbor couplings

O. Boada, A. Celi, M. Lewenstein, JIL
\[ H = -J \sum_{\tilde{q}} \sum_{j=1}^{D+1} a^+_{\tilde{q}+\tilde{u}_j} a_{\tilde{q}} + h.c. \]

\[ \tilde{q} = (\tilde{r}, \sigma) \]

\[ H = -J \sum_{\tilde{r}, \sigma} \sum_{j=1}^{D} \left( a_{\tilde{r}+\tilde{u}_j}^{(\sigma)+} a_{\tilde{r}}^{(\sigma)} + a_{\tilde{r}}^{(\sigma+1)+} a_{\tilde{r}}^{(\sigma)} \right) + h.c. \]
State dependent lattice / On site dressed lattice
Two 3D sub-lattices are connected via Raman transitions

\[
J_{\text{bilayer}} = \frac{\Omega}{2} \int d^2 x \ w^*(\vec{x})w(\vec{x} - \vec{r})
\]

Exponential decay of Wannier functions supresses undesired transitions

Boada, Celi, Lewenstein, JIL
Single particle observable

**Kaluza-Klein modes**
Single particle correlators take contributions from jumps back and forth to other dimensions in the form of exponential (KK) massive corrections

Many-body observable

**Shift of phase transition**
interpolates between dimensions
Conclusion of part II

Quantum Simulation beyond classical computation

Quantum Simulation of unphysical quantum systems
In progress

Q Simulation of Boundary Conditions

Klein Bottle on a nine-level system
Moëbius band on a six-level system
Scaling of Frustration

![Graph showing the scaling of frustration](image)
Conclusion

Quantum Computation

Quantum Simulation

Classical simulation

It is all about Entanglement